# DISTRIBUTION OF BOUNDARY POINTS OF EXPANSION AND APPLICATION TO THE LONELY RUNNER CONJECTURE

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Abstract. In this paper we study the distribution of boundary points of expansion. As an application, we say something about the lonely runner problem. We show that given $k$ runners $\mathcal{S}_i$ round a unit circular track with the condition that at some time $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $i = 1, 2 \ldots, k-2$, then at that time we have
$$||\mathcal{S}_{i+1} - \mathcal{S}_i|| > \frac{\mathcal{D}(n)\pi}{k-1}$$
for all $i = 1, \ldots, k-1$ and where $\mathcal{D}(n) > 0$ is a constant depending on the degree of a certain polynomial of degree $n$. In particular, we show that given at most eight $\mathcal{S}_i$ $(i = 1, 2, \ldots, 8)$ runners running round a unit circular track with distinct constant speed and the additional condition $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $1 \leq i \leq 6$ at some time $s > 1$, then at that time their mutual distance must satisfy the lower bound
$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \frac{\pi}{7C\sqrt{3}}$$
for some constant $C > 0$ for all $1 \leq i \leq 7$.

## 1. Introduction

The lonely runner conjecture is the assertion that given $n$ runners round a unit circle with constant distinct speed and starting at a common time and place, there must exist a time for which their mutual distances should be at least $\frac{1}{n}$. The conjecture has been verified for many special cases. For instance in [2], it has been shown that the conjecture hold for **six** runners. It is also shown in [1] for at most **seven** runners. In this paper, by studying the distribution of boundary points of an expansion, we verify this conjecture in it's crude form with an extra conditioning for at most **eight** runners. We obtain a conditional result of this conjecture by showing that:

**Theorem 1.1.** *Given $k$ runners $\mathcal{S}_i$ round a unit circular track with the condition that at some time $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $i = 1, 2 \ldots, k-2$, then at that time we have*
$$||\mathcal{S}_{i+1} - \mathcal{S}_i|| > \frac{\mathcal{D}(n)\pi}{k-1}$$
*for all $i = 1, \ldots, k-1$ and where $\mathcal{D}(n) > 0$ is a constant depending on the degree of a certain polynomial of degree $n$.*

In particular, we show that

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**Theorem 1.2.** *Let $\mathcal{S}_i$ $(i = 1, 2, \ldots, 8)$ be runners running round the unit circular track. Under the condition $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $1 \leq i \leq 6$ at some time $s > 1$, then*

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \frac{\pi}{7D\sqrt{3}}$$

*for some constant $D > 0$ for all $1 \leq i \leq 7$.*

## 2. Definitions and background

**Definition 2.1.** Let $\mathcal{S} = (f_1, f_2, \ldots, f_n)$ such that each $f_i \in \mathbb{R}[x]$. By the derivative of $\mathcal{S}$ denoted $\nabla(\mathcal{S})$, we mean

$$\nabla(\mathcal{S}) = \bigg(\frac{df_1}{dx}, \frac{df_2}{dx}, \ldots, \frac{df_n}{dx}\bigg).$$

We denote the derivative of this tuple at a point $a \in \mathbb{R}$ to be

$$\nabla_a(\mathcal{S}) = \bigg(\frac{df_1(a)}{dx}, \frac{df_2(a)}{dx}, \ldots, \frac{df_n(a)}{dx}\bigg).$$

**Definition 2.2.** Let $\mathcal{S} = (f_1, f_2, \ldots, f_n)$ such that each $f_i \in \mathbb{R}[x]$. By the integral of $\mathcal{S}$ denoted $\Delta(\mathcal{S})$, we mean

$$\Delta(\mathcal{S}) = \bigg(\int f_1(x)dx, \ldots, \int f_n(x)dx\bigg).$$

The corresponding integral between the points $\mathcal{S}_a = (a_1, \ldots, a_n)$ and $\mathcal{S}_b = (b_1, \ldots, b_n)$, denoted $\Delta_{\mathcal{S}_a, \mathcal{S}_b}(\mathcal{S})$ is given by

$$\Delta_{\mathcal{S}_a, \mathcal{S}_b}(\mathcal{S}) = \bigg(\int_{a_1}^{b_1} f_1(x)dx, \ldots, \int_{a_n}^{b_n} f_n(x)dx\bigg).$$

**Definition 2.3.** Let $\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Then by an expansion on $\{\mathcal{S}_i\}_{i=1}^{\infty}$, we mean the composite map

$$\gamma^{-1} \circ \beta \circ \gamma \circ \nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty},$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

**Definition 2.4.** Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. By the boundary points of the $n$th expansion, denoted $\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j)]$, we mean the set

$$\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j)] := \big\{(a_1, a_2, \ldots, a_n) : \mathrm{Id}_i[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n_{a_i}(\mathcal{S}_j)] = 0\big\}.$$

## 3. Distribution of boundary points of expansion

In this section we study the distribution of the boundary points of any phase of expansion. We first introduce the notion of integration of polynomials along the boundaries of various phases of expansion, which we then use as a main tool. We launch the following definition in that regard.

**Definition 3.1.** Let $f(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ be a polynomial of degree $n$, then we call the tuple

$$\begin{aligned}\mathcal{S}_f &= (c_nx^n, c_{n-1}x^{n-1}, \ldots, c_1x + c_0)\\ &= (g_1(x), g_2(x), \ldots, g_n(x))\end{aligned}$$

the tuple representation of $f$. By the integral of $f(x)$ along the boundary of the $m$th phase expansion, we mean the formal integral

$$\int\limits_{\substack{\mathcal{B}^m(\mathcal{S}_f)\\ m<n}} f(t)dt := \sum_{i=1}^{\#\mathcal{B}^m(\mathcal{S}_f)-1} \sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{B}^m(\mathcal{S}_f)\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}} \overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)} \cdot \overrightarrow{O\mathcal{S}_e}$$

where

$$\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f) = \left(\int\limits_{a_1}^{b_1} g_1(x)dx, \int\limits_{a_2}^{b_2} g_2(x)dx, \ldots, \int\limits_{a_n}^{b_n} g_n(x)dx\right)$$

and where $\mathcal{S}_e = (1, 1, \ldots, 1)$ is the unit tuple, and $\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}}$ and $\overrightarrow{O\mathcal{S}_e}$ are the position vectors of $\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}$ and $\mathcal{S}_e$, respectively, with $\mathcal{S}_i = (a_1, a_2, \ldots, a_n)$ and $\mathcal{S}_{i+1} = (b_1, b_2, \ldots, b_n)$.

*Remark* 3.2. It is in practice very difficult to ascertain the local distribution of boundary points of expansion. However, we can show that if we shrink the space bounded by the boundary of an expansion, then points on the boundary should be closely packed in some sense. We use the notion of integration along boundaries as a black box.

**Theorem 3.3.** *Let $f(x) := c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ be a polynomial of degree $n$. Then*

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m(\mathcal{S}_f)\\ m<n}} f(t)dt\right|\right| > \epsilon$$

*for some $\epsilon > 0$ if and only if $||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \delta$ for some $\delta > 0$ for some*

$$\mathcal{S}_i \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)]$$

*with $1 \leq i \leq \#\mathcal{B}^m(\mathcal{S}_f) - 1$ and $||\mathcal{S}_i - \mathcal{S}_{i+1}|| < ||\mathcal{S}_i - \mathcal{S}_j||$ for all $j \neq i + 1$.*

*Proof.* Let $f(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \in \mathbb{R}[x]$ be a polynomial of degree $n$ and suppose

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m(\mathcal{S}_f)\\ m<n}} f(t)dt\right|\right| > \epsilon$$

for some $\epsilon > 0$. By a repeated application of the triangle inequality, we find that

$$\begin{aligned}\left|\left|\int\limits_{\substack{\mathcal{B}^m(\mathcal{S}_f)\\ m<n}} f(t)dt\right|\right| &\leq \sum_{i=1}^{\#\mathcal{B}^m(\mathcal{S}_f)-1} \sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{B}^m(\mathcal{S}_f)\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}} ||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}||||\overrightarrow{O\mathcal{S}_e}||\\ &= \sqrt{n}\sum_{i=1}^{\#\mathcal{B}^m(\mathcal{S}_f)-1} \sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{B}^m(\mathcal{S}_f)\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}} ||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}||\\ &\leq (\#\mathcal{B}^m(\mathcal{S}_f)-1)\sqrt{n}\max\left\{||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}||\right\}_{\substack{i=1\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}}^{\#\mathcal{B}^m(\mathcal{S}_f)-1}.\end{aligned}$$

Since the inequality

$$\begin{aligned}||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}|| &= \sqrt{|\int\limits_{a_1}^{b_1} g_1(x)dx|^2 + \cdots + |\int\limits_{a_n}^{b_n} g_n(x)dx|^2}\\ &\leq M\sqrt{|a_1-b_1|^2 + \cdots + |a_n-b_n|^2}\end{aligned}$$

is valid for some $M > 0$, it follows that there exist some $\mathcal{S}_i, \mathcal{S}_{i+1} \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]$ with $||\mathcal{S}_i - \mathcal{S}_{i+1}|| < ||\mathcal{S}_i - \mathcal{S}_j||$ for all $j \neq i+1$. It follows that for some closest pair of boundary points, the inequality

$$\frac{\epsilon}{(\#\mathcal{B}^m(\mathcal{S}_f)-1)M\sqrt{n}} < \sqrt{|a_1-b_1|^2 + \cdots + |a_n-b_n|^2}$$

is valid, and thus it must be that $||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \delta$ by choosing

$$\delta = \frac{\epsilon}{(\#\mathcal{B}^m(\mathcal{S}_f)-1)M\sqrt{n}}.$$

Conversely, suppose there exist some closest boundary point $\mathcal{S}_i, \mathcal{S}_{i+1} \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]$ such that

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \delta$$

for some $\delta := \delta(n) > 0$. Then it follows that $\sqrt{|a_1-b_1|^2 + \cdots + |a_n-b_n|^2} > \delta$. By choosing $R = \min\{|g_i(x)| : x \in [a_i, b_i]\}_{i=1}^n$, we find that

$$\begin{aligned}||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}|| &= \sqrt{|\int\limits_{a_1}^{b_1} g_1(x)dx|^2 + \cdots + |\int\limits_{a_n}^{b_n} g_n(x)dx|^2}\\ &\geq R\sqrt{|a_1-b_1|^2 + \cdots + |a_n-b_n|^2}\\ &= \delta R.\end{aligned}$$

It follows that

$$\sum_{i=1}^{\#\mathcal{B}^m(\mathcal{S}_f)-1} \sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{B}^m(\mathcal{S}_f)\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}} \overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)} \cdot \overrightarrow{O\mathcal{S}_e} > \sum_{i=1}^{\#\mathcal{B}^m(\mathcal{S}_f)-1} \sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{B}^m(\mathcal{S}_f)\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}} \delta R||\overrightarrow{O\mathcal{S}_e}||\cos\alpha$$

$$= \delta(\#\mathcal{B}^m(\mathcal{S}_f)-1)R\sqrt{n}\cos\alpha$$

where $\alpha$ is the angle between the vectors $\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}$ and $\overrightarrow{O\mathcal{S}_e}$. It follows that

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m(\mathcal{S}_f)\\ m<n}} f(t)dt\right|\right| > \delta C(\#\mathcal{B}^m(\mathcal{S}_f)-1)R\sqrt{n}|\cos\alpha|.$$

for some constant $C = C(n) > 0$. The result follows by taking

$$\delta := \frac{\epsilon}{(\#\mathcal{B}^m(\mathcal{S}_f)-1)CR\sqrt{n}|\cos\alpha|}.$$

□

*Remark* 3.4. Theorem 3.3 in the affirmative tells us that we can use the area as a yardstick to determine the distribution of points on the boundary of any phase of expansion.

## 4. Rotation of the boundary of expansion

In this section we introduce the concept of rotation of the boundary of an expansion.

**Definition 4.1.** Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_j)$ be an expansion with corresponding boundary $\mathcal{B}^m(\mathcal{S}_j)$. Then we say the map $\vee$ is a rotation of the boundary $\mathcal{B}^m(\mathcal{S}_j)$ if

$$\vee : \mathcal{B}^m(\mathcal{S}_j) \longrightarrow \mathcal{B}^m(\mathcal{S}_j).$$

We say an expansion admits a rotation if there exist such a map. In other words, we say the map $\vee$ induces a rotation on the expansion. We say the boundary is stable under rotation if $||\vee(\mathcal{S}_a)|| \approx ||\mathcal{S}_a||$ for $\mathcal{S}_a \in \mathcal{B}^m(\mathcal{S}_j)$. Otherwise we say it is unstable.

*Remark* 4.2. Next we prove a result that indicates that boundary points of an expansion whose boundary occupies a small enough region must be stable.

**Proposition 4.1.** *Let $f(x) := c_nx^n+\cdots+c_1x+c_0 \in \mathbb{R}[x]$, a polynomial with degree $n\geq 3$. Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)$ be an expansion with corresponding boundary $\mathcal{B}^m(\mathcal{S}_f)$ admits a rotation $\vee$. If*

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m(\mathcal{S}_f)\\ m<n}} f(t)dt\right|\right| < 1$$

*then the boundary $\mathcal{B}^m(\mathcal{S}_f)$ is stable.*

*Proof.* Let $f(x) := c_nx^n + \cdots + c_1x + c_0 \in \mathbb{R}[x]$, a polynomial with degree $n \geq 3$. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)$ be an expansion with corresponding boundary $\mathcal{B}^m(\mathcal{S}_f)$ admits a rotation $\vee$. Suppose also that

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m(\mathcal{S}_f)\\ m<n}} f(t)dt\right|\right| < 1$$

then it follows from Theorem 3.3 that $||\mathcal{S}_i|| \approx ||\mathcal{S}_{i+1}||$ for all $1 \leq i \leq \#\mathcal{B}^m(\mathcal{S}_f) - 1$ with $\mathcal{S}_i, \mathcal{S}_{i+1} \in \mathcal{B}^m(\mathcal{S}_f)$. It follows that for the rotation $\vee : \mathcal{B}^m(\mathcal{S}_f) \longrightarrow \mathcal{B}^m(\mathcal{S}_f)$, we have that for any $\mathcal{S}_i \in \mathcal{B}^m(\mathcal{S}_f)$, then

$$\vee(\mathcal{S}_i) = \mathcal{S}_k$$

for some $\mathcal{S}_k \in \mathcal{B}^m$. It follows that $|| \vee (\mathcal{S}_i)|| = ||\mathcal{S}_k|| \approx ||\mathcal{S}_i||$, thereby ending the proof. □

**Definition 4.3.** Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_j)$ be an expansion with corresponding boundary $\mathcal{B}^m(\mathcal{S}_j)$. Then we say the map $\vee$ is a rotation of the boundary $\mathcal{B}^m(\mathcal{S}_j)$ with frequency $s$ if

$$\vee^s : \mathcal{B}^m(\mathcal{S}_j) \longrightarrow \mathcal{B}^m(\mathcal{S}_j),$$

where $\vee^s = \vee \circ \vee \circ \cdots \circ \vee$ is the $s$-fold rotation on the boundary of expansion.

*Remark* 4.4. It is important to recognize that rotation with frequency $s$ is the time for which points on the boundary of expansion are allowed to be in motion by an induced rotation.

**Proposition 4.2.** *Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_j)$ be an expansion with corresponding boundary $\mathcal{B}^m(\mathcal{S}_j)$. Then any permutation*

$$\sigma : \mathcal{B}^m(\mathcal{S}_j) \longrightarrow \mathcal{B}^m(\mathcal{S}_j)$$

*where $\sigma(\mathcal{S}_i) = \mathcal{S}_{\sigma(i)}$ for $1 \leq i \leq \#\mathcal{B}^m(\mathcal{S}_j)$ for $\mathcal{S}_i \in \mathcal{B}^m(\mathcal{S}_j)$ is a rotation of the boundary of expansion.*

## 5. Spherical defoliation of the boundary of expansion

**Definition 5.1.** Let $\mathcal{B}^m(\mathcal{S}_f)$ and $\mathbb{S}^{k-1}$ be the boundary of the $m$th expansion and the $k$ dimensional unit sphere, respectively. Then by the spherical defoliation of the boundary of expansion, we mean the map

$$\Lambda : \mathcal{B}^m(\mathcal{S}_f) \longrightarrow \mathbb{S}^{k-1}$$

such that for any $\mathcal{S}_a \in \mathcal{B}^m(\mathcal{S}_f)$, then we have

$$\Lambda(\mathcal{S}_a) = \frac{\mathcal{S}_a}{||\mathcal{S}_a||}.$$

## 6. Application to the Lonely runner conjecture

**Theorem 6.1.** *Given $k$ runners $\mathcal{S}_i$ round a unit circular track with the condition that at some time $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $i = 1, 2 \ldots, k-2$, then at that time we have*

$$||\mathcal{S}_{i+1} - \mathcal{S}_i|| > \frac{\mathcal{D}(n)\pi}{k-1}$$

*for all $i = 1, \ldots, k-1$ and where $\mathcal{D}(n) > 0$ is a constant depending on the degree of a certain polynomial of degree $n$.*

*Proof.* First assume any polynomial $g(x) := b_1 x^n + \cdots + b_1 x + b_0$ for some choice of $n$ so that the size of the boundary of expansion $\#\mathcal{B}^m(\mathcal{S}_g) = k$ for some $m < n$. The under the condition $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $i = 1, 2 \ldots, k-2$ at some time, we set

$$\left|\left|\int\limits_{\mathcal{B}^m(\mathcal{S}_g)} g(t)dt\right|\right| = \pi$$

and apply the $s$-fold rotation $\vee^s = \vee \circ \vee \circ \cdots \circ \vee$ on the boundary $\mathcal{B}^m(\mathcal{S}_g)$. Then by Proposition 4.1 points on this boundary are now unstable for time $s > 1$ with each moving at distinct speeds and satisfying the lower bound

$$||\mathcal{S}_{i+1} - \mathcal{S}_i|| > \frac{\mathcal{C}(n)\pi}{k-1}$$

for all $i = 1, \ldots, k-1$ and where $\mathcal{D}(n) > 0$ is a constant depending on the degree of a certain polynomial of degree $n$. Since some point on the boundary of expansion may not be a point on the unit circle, we apply the spherical defoliation $\Lambda : \mathcal{B}^m(\mathcal{S}_g) \longrightarrow \mathbb{S}^k$, and we obtain

$$||\mathcal{S}_{i+1} - \mathcal{S}_i|| > \frac{\mathcal{D}(n)\pi}{k-1}$$

for all $i = 1, \ldots, k-1$ and where $\mathcal{D}(n) > 0$ is a constant depending on the degree of a certain polynomial of degree $n$. □

**Lemma 6.2.** *Let $f(x) := c_3 x^3 + c_2 x^2 + c_1 x + c_0$ be a polynomial of degree 3 and suppose $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $1 \leq i \leq 6$. Then*

$$\left|\left|\int\limits_{\mathcal{B}^1(\mathcal{S}_f)} f(t)dt\right|\right| = \pi$$

*if and only if*

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \frac{\pi}{7C\sqrt{3}}$$

*for some constant $C > 0$ for all $1 \leq i \leq 7$.*

*Proof.* The result follows by taking $n = 3$ in Theorem 3.3. □

**Theorem 6.3.** *Let $\mathcal{S}_i$ $(i = 1, 2, \ldots, 8)$ be runners running round the unit circular track. Under the condition $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $1 \leq i \leq 6$ at some time $s > 1$ then*

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \frac{\pi}{7D\sqrt{3}}$$

*for some constant $D > 0$ for all $1 \leq i \leq 7$.*

*Proof.* Let $f(x) := c_3x^3 + c_2x^2 + c_1x + c_0$ be a polynomial of degree 3 and suppose $||\mathcal{S}_i - \mathcal{S}_{i+1}|| = ||\mathcal{S}_{i+1} - \mathcal{S}_{i+2}||$ for all $1 \leq i \leq 6$. Then we set

$$\left|\left|\int\limits_{\mathcal{B}^1(\mathcal{S}_f)} f(t)dt\right|\right| = \pi$$

and apply a rotation $\vee^s$ with frequency $s > 1$ on the boundary $\mathcal{B}^1(\mathcal{S}_f)$. Then by Proposition 4.1, boundary points of expansion are now unstable for time $s > 1$, with each moving at a different speed. Then by applying Lemma 6.2, it follows that

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \frac{\pi}{7C\sqrt{3}}$$

for some constant $C > 0$ for all $1 \leq i \leq 7$. By applying the defoliation $\Lambda : \mathcal{B}^1(\mathcal{S}_f) \longrightarrow \mathbb{S}^3$,we obtain

$$||\mathcal{S}_k - \mathcal{S}_{k+1}|| > \frac{\pi}{7D\sqrt{3}}$$

for some $D > 0$ for all $1 \leq k \leq 7$. □

[1].

## References


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