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ON CERTAIN TYPES OF NOTIONS VIA PREOPEN SETS

Dedicated to the memories of Hamid and Mahmoud Jafari

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Abstract. In this paper, we deal with the new class of pre-regular p-open sets in which the notion of preopen set is involved. We characterize these sets and study some of their fundamental properties. We also present two other notions called extremally p-discreteness and locally p-indiscreteness by utilizing the notions of preopen and preclosed sets by which we obtain some equivalence relations for pre-regular p-open sets. Moreover, we define the notion of regular p-open sets by utilizing the notion of pre-regular p-open sets. We investigate some of the main properties of these sets and study their relations to pre-regular p-open sets.

1. Introduction

In 1964, Corson and Michael [3] introduced the notion of locally dense sets, also called preopen sets by Mashhour et al. [8]. The class of preopen sets properly contains the class of open sets. As the intersection of two preopen sets may fail to be preopen, the class of preopen sets does not always form a topology. In a submaximal space, i.e. a topological space X in which every dense subset is open, collection of all preopen sets form a topology. Indeed, many notions in Topology are (can) be defined in terms of preopen sets (see [2], [4], [6], [7], [9] and [10]). We also offer some new notions by utilizing preopen sets and investigate some of their properties.

Throughout this paper, (X, τ) (or X) is always a topological space. A set A in a space X is called preopen [8] if $A \subset Int(Cl(A))$. The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing a subset A is called the preclosure [4] of A and is denoted by $pCl(A)$. The preinterior of a subset A of a topological space (X, τ) is the union of all preopen sets of X contained in A and is denoted by $pInt(A)$. The family of all preopen sets of X will be denoted by $PO(X)$. For a point $x \in X$, we set $PO(X, \tau) = \{U \mid x \in U \in PO(X)\}$.

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2. Pre-regular p-open sets

Definition 1. A preopen set A of a space (X, τ) is said to be pre-regular p-open [7] if $A = pInt(pCl(A))$. The complement of a pre-regular p-open set is called pre-regular p-closed set, equivalently $pCl(pInt(A)) = A$. The family of all pre-regular p-open (resp. pre-regular p-closed) sets of a space (X, τ) will be denoted by $PRO(X)$ (resp. $PRC(X)$).

Recall that a set A of a space (X, τ) is called p-clopen if it is preopen and preclosed. Clearly, X and $\emptyset$ are pre-regular p-open and also that every p-clopen set is pre-regular p-open. Now consider the following:

Example 2.1. Take the usual space of reals and the open interval $A = (0, 1)$. Then $pCl(A) = [0, 1]$ and $pInt(pCl(A)) = A$. This means that A is pre-regular p-open but A is not preclosed.

Moreover, the intersection of two pre-regular p-open sets is not pre-regular p-open in general as the following example shows:

Example 2.2. Let $X = \{a, b, c\}$ with topology $\tau = \{X, \emptyset, \{b, c\}\}$ and $PO(X, \tau) = \{X, \emptyset, \{b\}, \{a, b\}, \{c\}, \{b, c\}, \{a, c\}\}$. Thus $A = \{a, b\}$ and $B = \{a, c\}$ are both pre-regular p-open but $A \cap B = \{a\}$ is not since it is not even preopen. Notice that the set $\{b\} \in PRO(X, \tau)$ and $\{c\} \in PRO(X, \tau)$ but $\{b, c\} \notin PRO(X, \tau)$. Take the usual space of reals and let $A = (\frac{1}{2}, \ 1)$ and $B = (1, \frac{3}{2})$. It is obvious that both A and B are pre-regular p-open but $A \cup B$ is not since $pInt(pCl(A \cup B)) = (\frac{1}{2}, \frac{3}{2})$.

The notions of pre-regular p-open and open are independent of each other. Take (X, τ) as in Example 2.2. Then $\{b, c\}$ is open but not pre-regular p-open. The set $\{a, c\}$ is pre-regular p-open but not open.

Theorem 2.3. *Let (X, τ) be a space and A any preopen subset of X. Then the following hold:*

(1) *If $A \subseteq B$, then $pInt(pCl(A)) \subseteq pInt(pCl(B))$.*
(2) *If $A \in PO(X, \tau)$, then $A \subset pInt(pCl(A))$.*
(3) *For every $A \in PO(X, \tau)$, we have $pInt(pCl(pInt(A))) = pInt(pCl(A))$.*
(4) *If A and B are disjoint preopen sets, then $pInt(pCl(A))$ and $pInt(pCl(B))$ are disjoint.*

Proof.

(1) Suppose that $A \subseteq B$. It readily follows that $pInt(pCl(A)) \subseteq pInt(pCl(A))$.
(2) Suppose that $A \in PO(X, \tau)$. Since $A \subseteq pCl(A)$, then $A \subset pInt(pCl(A))$.
(3) It is obvious that $pInt(pCl(A)) \in PO(X, \tau)$, so by (2) we have $pInt(pCl(A)) \subset pInt(pCl(pInt(pCl(A))))$. On the other hand, we have $pInt(pCl(A)) \subseteq pCl(A)$. Therefore $pCl(pInt(pCl(A))) \subseteq pCl(pCl(A)) = pCl(A)$. Hence $pInt(pCl(pInt(pCl(A)))) = pInt(pCl(A))$.

(4) Since A and B are disjoint preopen sets, we have $A \cap pCl(B) = \emptyset$ and hence $A \cap pInt(pCl(B)) = \emptyset$. Since $pInt(pCl(B))$ is preopen, $pCl(A) \cap pInt(pCl(B)) = \emptyset$ and therefore $pInt(pCl(A)) \cap pInt(pCl(B)) = \emptyset$.

Remark 2.4. Notice that in a partition topology every preopen set is pre-regular p-open. The finite intersection of pre-regular p-open sets need not be pre-regular p-open.

Recall that a space (X, τ) is called submaximal if every dense subset of X is open.

Lemma 2.5.([11, Corollary 3]) *If a space (X, τ) is submaximal, then any finite intersection of preopen sets is preopen.*

Theorem 2.6. *If a space (X, τ) is submaximal, then any finite intersection of pre-regular p-open sets is pre-regular p-open.*

Proof. Let $(O_i)_{i \in I}$ be a finite family of pre-regular p-open sets. Since the space (X, τ) is submaximal, then by Lemma 2.5 we have $\cap_{i \in I}(O_i) \in PO(X, \tau)$. Therefore $\cap_{i \in I}(O_i) \subseteq pInt(pCl(\cap_{i \in I}(O_i)))$. For each $i \in I$, we have $\cap_{i \in I}(O_i) \subseteq O_i$ and thus $pInt(pCl(\cap_{i \in I}(O_i))) \subseteq pInt(pCl(O_i))$. Since $pInt(pCl(O_i)) = O_i$, then $pInt(pCl(\cap_{i \in I} (O_i))) \subseteq \cap_{i \in I}(O_i)$.

Remark 2.7. It should be noted that an arbitrary union of pre-regular p-open sets is pre-regular p-open. But the intersection of two pre-regular p-closed sets fails to be pre-regular p-closed: Let (X, τ) be as in Example 2.2. So it is easy to see that A and B are pre-regular p-closed but their intersection is not pre-regular p-closed.

The following hold for a subset A of a space (X, τ):

(1) If A is preclosed, then $pInt(A)$ is pre-regular p-open.
(2) If $A = pInt(A)$, then $pCl(A)$ is pre-regular p-closed.
(3) If A and B are pre-regular p-closed sets, then $A \subset B$ if and only if $pInt(A) \subset pInt(B)$.
(4) If A and B are pre-regular p-open sets, then $A \subset B$ if and only if $pCl(A) \subset pCl(B)$.

The following notions are due to Dontchev et al. [5]: A point $x \in X$ is said to be a pre-θ-accumulation point of a subset A of a space (X, τ) if $pCl(U) \cap A \neq \emptyset$ for every $U \in PO(X, x)$. The set of all pre-θ-accumulation points of A is called the pre-θ-closure of A and is denoted by $pCl_\theta(A)$.

Dontchev et al. ([5, Proposition 4.4]) have shown that if $A \in PO(X, \tau)$, then $pCl(A) = pCl_\theta(A)$.

Theorem 2.8. *If A is a pre-regular p-open subset of a space (X, τ), then $A = pInt(pCl_\theta(A))$.*

Proof. it is an immediate consequence of the result of Dontchev et al. ([5, Proposition 4.4]).

Lemma 2.9. *In any space (X, τ) the empty set is the only subset which is nowhere dense and pre-regular p-open.*

Proof. Suppose that A is nowhere dense and $A \in PRO(X, \tau)$. Then by [1, Theorem 3], $A = pInt(pCl(A)) = pCl(A) \cap Int(Cl(A))$. Since A is nowhere dense we have $IntCl(A)) = \emptyset$. Therefore $A = \emptyset$.

Recall that a rare set is a set with no interior points.

Lemma 2.10. *If $A \in PRC(X, \tau)$, then every rare set is preopen.*

Proof. By hypothesis, we have $A = pCl(pInt(A)) = pInt(A) \cup Cl(Int(A))$. Since A is is a rare set, then $A = pInt(A)$. This shows that A is preopen.

Definition 2. A space (X, τ) is called extremally p-disconnected if the preclosure of every preopen subset of X is preopen.

Definition 3. A point $x \in X$ is said to be a pre-limit point of a subset A of a space (X, τ) if $pCl(U) \cap A \neq \emptyset$ for every open set U of X containing x. We denote the set of pre-limit points of A by $pd(A)$.

Theorem 2.11. *For a space (X, τ) the following are equivalent:*

(1) *(X, τ) is extremally p-disconnected.*
(2) *Every pre-regular p-open subset is p-clopen.*
(3) *$pd(A) \subset pInt(A)$ for every pre-regular p-closed subset A of X.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is pre-regular p-open. Then $A = pInt(pCl(A))) = pCl(A)$. Combined with (1), this means that A is p-clopen.
(2)$\Rightarrow$(3): By hypothesis, A is p-clopen and therefore $pd(A) \subset pCl(A) = pInt(A)$.
(3)$\Rightarrow$ (1): Suppose that A is preopen. Then $pCl(A)$ is pre-regular p-closed. By hypothesis, we have $pd(A) \subset pd(pCl(A)) \subset pInt(pCl(A))$. Hence $pCl(A) = A \cup pd(A) \subset pInt(pCl(A))$. This shows that $pCl(A)$ is preopen.

Now it is clear that extremally p-disconnected is equivalent with extremally disconnected.

Definition 4. A space (X, τ) is called locally p-indiscrete if every preopen subset of X is preclosed or if every preclosed subset is preopen.

Theorem 2.12. *For a space (X, τ) the following are equivalent:*

(1) *(X, τ) is locally p-indiscrete.*
(2) *Every preopen subset is p-clopen.*
(3) *Every preopen subset is pre-regularly p-open.*

(4) *$pCl(\{x\})$ is a pre-neighborhood of x for every $x \in X$.*
(5) *$pInt(pCl(\{x\})) \neq \emptyset$ for every $x \in X$.*
(6) *The empty set is the only nowhere dense subset of X.*
(7) *$pd(A) \subset A$ for every $A \in PO(X)$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $A \in PO(X,\tau)$. Then $pInt(A)$ is preclosed. Therefore $pCl(A) = pCl(pInt(pCl(A))) = pInt(pCl(A))$. Hence $A = pInt(pCl(A)) = pCl(A)$ is p-clopen.
(2) $\Rightarrow$ (3): Obvious.
(3) $\Rightarrow$ (4): Suppose that $\{x\} \cap pInt(pCl(\{x\}) = \emptyset$. It follows that $pInt(pCl(\{x\})) = pCl(\{x\}) \cap pInt(pCl(\{x\})) = \emptyset$. By (3), we have $pCl(\{x\}) = pCl(pInt(pCl(\{x\})))$. This means that $pInt(pCl(\{x\})) \neq \emptyset$. It follows that $x \in pInt(pCl(\{x\}))$ for every $x \in X$.
(4) $\Rightarrow$ (5): Obvious.
(5) $\Rightarrow$ (6): It is an immediate consequence of Lemma 2.9.
(6) $\Rightarrow$ (1): Suppose that A is a preclosed set. Then $A - pInt(A)$ is nowhere dense by (6). Therefore $A = pInt(A)$ and hence preopen.
(1) $\Leftrightarrow$ (7): $pd(A) \subset A$ if and only if A is preclosed.

Remark 2.13. Observe that it follows from the above theorem that locally p-indiscrete is equivalent with locally indiscrete.

3. Regular p-Open Sets

Definition 5. A subset A of a space (X,τ) is said to be regular p-open if there exists a pre-regular p-open set U such that $U \subset A \subset pCl(U)$.

Proposition 3.1. *Every pre-regular p-open set is regular p-open.*

Proof. Suppose that A is pre-regular p-open set. Take A, then we have $A \subset A \subset pCl(A)$.

Example 3.2. Let $X = \{a,b,c\}$, $\tau = \{X, \emptyset, \{b,c\}\}$. Then $\{b\}$ is pre-regular p-closed and also regular p-open. But let $Y = \{a,b,c\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a,b\}\}$. Then $\{a,c\}$ and $\{b,c\}$ are regular p-open but not pre-regular p-open.

Remark 3.3. The intersection of two regular p-open sets is not regular p-open. Take $\{a,c\}$ and $\{b,c\}$ in the above example. Then $\{a,c\} \cap \{b,c\} = \{c\}$ is not regular p-open. Notice also that the set of pre-regular p-open sets of a space (X,τ) in Example 3.2 does not establish a topology since $\{a,b\} \cup \{a,c\} = \{a,b,c\}$ which is not a pre-regular p-open set.

Theorem 3.4. *If A is a regular p-open set in a space X, then $pCl(X - A)$ is pre-regular p-closed.*

Proof. Suppose that U is a pre-regular p-open set such that $U \subset A \subset pCl(U)$. We have

$$X - U = pCl(X - U) \supset pCl(X - A) \supset pCl(X - pCl(U)) = pCl(pInt(X - U)) = X - U.$$

Corollary 3.5. *If A is regular p-open, then $pInt(A)$ is pre-regular p-open.*

Proof. Let A be a regular p-open set. Then by Theorem 3.4, we have $pCl(X - A)$ is pre-regular p-closed. Now, we have $X - (pCl(X - A)) = pInt(A)$ which is pre-regular p-open.

Corollary 3.6. *Let (X, τ) be submaximal. If A and B are regular p-open, then $pInt(A \cap B)$ is also regular p-open.*

Proof. Let A and B be regular p-open sets. Then by Corollary 3.5, $pInt(A)$ and $pInt(B)$ are pre-regular p-open. Hence $pInt(A \cap B) = pInt(A) \cap pInt(B)$ is pre-regular p-open.

Theorem 3.7. *If A is regular p-open and $A \subset B \subset pCl(A)$, then B is regular p-open.*

Proof. Suppose that U is pre-regular p-open set such that $U \subset A \subset pCl(U)$. By setting $pCl(A) = pCl(U)$, we have $U \subset A \subset B \subset pCl(A) = pCl(U)$.

Remark 3.8. The intersection of an open (resp. preopen) set and a regular p-open set is not regular p-open in general. Take (X, τ) from Example 3.2. The set $\{b, c\}$ is an open set (even preopen) and X is a regular p-open set but $\{b, c\} \cap X = \{b, c\}$ is not regular p-open.

Theorem 3.9. *A set is pre-regular p-open if and only if it is preopen and regular p-open.*

Proof. "Necessity". It follows from Proposition 3.1.
"Sufficiency". Let A be preopen and regular p-open. Then since A is preopen, we have $A = pInt(A)$. Therefore by Corollary 3.5, A is pre-regular p-open.

Theorem 3.10. *If A and B are regular p-open sets of the spaces X and Y, respectively, then $A \times B$ is a regular p-open set of $X \times Y$.*

Proof. Let U be a pre-regular p-open set in X and V is a pre-regular p-open set in Y such that $U \subset A \subset pCl(U)$ and $V \subset B \subset pCl(V)$. Then $U \times V \subset A \times B \subset pCl(U) \times pCl(V) = pCl(U \times V)$. Now, we have $pCl(pInt(U \times V)) = pInt(pCl(U) \times pCl(V)) = pInt(pCl(U)) \times pInt(pCl(V)) = U \times V$. This means that $U \times V$ is pre-regular p-open in $X \times Y$. Therefore $A \times B$ is a regular p-open set in $X \times Y$.

Definition 6. A subset A of a space X is said to be predense in X if $pCl(A) = X$.

Theorem 3.11. *Let Y be a predense subspace of a space X and $A \subset Y$. If A is pre-regular p-open in X, then it is pre-regular p-open in Y.*

Proof. If (Y, τ_Y) is a predense subspace of (X, τ) and $A \subset Y$, then $pInt_Y(pCl_Y(A)) = pInt(pCl(A)) \cap Y$.

Theorem 3.12. *Let Y be a predense subspace of a space X and $A \subset Y$. If A is regular p-open in X, then it is regular p-open in Y.*

Proof. Let U be a regular p-open set of X such that $U \subset A \subset pCl(U)$. Then $U \cap Y \subset A \cap Y \subset pCl(U) \cap Y$. Hence $U \subset A \subset pCl_Y(U)$ which by Theorem 3.11 means that U is a pre-regular p-open set and thus regular p-open in Y.

It is shown by Professor Miguel Caldas in a private conversation that if A is a preopen subset of a space (X, τ), then $A \subset pInt(pCl(pInt(A)))$ is equivalent to the preopenness of A.

Observe that every pre-regular p-open set is preopen but the converse need not be true. Take (X, τ) be as in Example 3.2. Then $PO(X, \tau) = \{X, \emptyset, \{b\}, \{a, b\}, \{c\}, \{b, c\}, \{a, c\}\}$. It is easy to verify that $\{b, c\}$ is preopen but not pre-regular p-open .

Definition 7. A space X is called PR-door if every subset of X is either pre-regular p-open or pre-regular p-closed.

Remark 3.13. A discrete space X is PR-door, then every preopen set in X is pre-regular p-open.

Theorem 3.14. *If a space (X, τ) is PR-door, then every preopen set in the space is pre-regular p-open.*

Proof. Let A be a preopen set of X. If A is pre-regular p-closed, we have $A = pCl(pInt(A))$ and $pInt(A) = pInt(pCl(pInt(A)))$. Since A is preopen, then $A \subset pInt(pCl(pInt(A)))$ and therefore $A = pInt(A)$. This shows that A is p-clopen. Hence A is pre-regular p-open.

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