BIOPERATIONS ON α-SEPARATIONS AXIOMS IN TOPOLOGICAL SPACES

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Abstract. In this paper, we consider the class of $\alpha_{[\gamma,\gamma']}$-generalized closed set in topological spaces and investigate some of their properties. We also present and study new separation axioms by using the notions of α-open and α-bioperations. Also, we analyze the relations with some well known separation axioms.

1. Introduction

The study of α-open sets was initiated by Njåstad [3]. Maheshwari [8] and Maki [9] introduced and studied a new separation axiom called α-separation axiom. Kasahara [2] defined the concept of an operation on topological spaces and introduced α-closed graphs of an operation. Ogata [4] called the operation α as γ operation and introduced the notion of γ-open sets and used it to investigate some new separation axioms. For two operations on τ some bioperation-separation axioms were defined [7], [5]. Moreover, Hariwan [6] defined the concept of an operation on $\alpha O(X,\tau)$ and introduced α_γ-open sets and α_γ-T_i ($i = 0, \frac{1}{2}, 1, 2$) in topological spaces. In this paper, In Section 3, we introduce the concept of $\alpha_{[\gamma,\gamma']}$-generalized closed sets and investigate some of its important properties. The notion of new bioperation α-separation axioms is introduced in section 4. We compare these separation axioms with the separation axioms in [10], [4], [6], [7] and [5].

2. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let (X,τ) be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X,τ) is said to be α-open [3] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open (resp. α-closed) sets in a topological space (X,τ) is denoted by $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$). An operation γ [2] on a topology τ is a mapping from τ in to

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power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open [4] if for each $x \in A$, there exists an open set U such that $x \in U$ and $U^\gamma \subseteq A$. An operation $\gamma : \alpha O(X,\tau) \to P(X)$ [6] is a mapping satisfying the following property, $V \subseteq V^\gamma$ for each $V \in \alpha O(X,\tau)$. We call the mapping γ an operation on $\alpha O(X,\tau)$. A subset A of X is called an α_γ-open set [6] if for each point $x \in A$, there exists an α-open set U of X containing x such that $U^\gamma \subseteq A$. We denote the set of all α_γ-open sets of (X,τ) by $\alpha O(X,\tau)_\gamma$. An operation γ on $\alpha O(X,\tau)$ is said to be α-regular [6] if for every α-open sets U and V containing $x \in X$, there exists an α-open set W of X containing x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. An operation γ on $\alpha O(X,\tau)$ is said to be α-open [6] if for every α-open set U of each $x \in X$, there exists an α_γ-open set V such that $x \in V$ and $V \subseteq U^\gamma$. A subset A of X is said to be $\alpha_{[\gamma,\gamma']}$-open [1] if for each $x \in A$ there exist α-open sets U and V of X containing x such that $U^\gamma \cap V^{\gamma'} \subseteq A$. The set of all $\alpha_{[\gamma,\gamma']}$-open sets of (X,τ) is denoted by $\alpha O(X,\tau)_{[\gamma,\gamma']}$. A subset F of (X,τ) is said to be $\alpha_{[\gamma,\gamma']}$-closed if its complement $X \setminus F$ is $\alpha_{[\gamma,\gamma']}$-open. The intersection of all $\alpha_{[\gamma,\gamma']}$-closed sets containing A is called the $\alpha_{[\gamma,\gamma']}$-closure of A and denoted by $\alpha_{[\gamma,\gamma']}$-$Cl(A)$. The union of all $\alpha_{[\gamma,\gamma']}$-open sets contained in A is called the $\alpha_{[\gamma,\gamma']}$-interior of A and denoted by $\alpha_{[\gamma,\gamma']}$-$Int(A)$.

In the remainder of this section all the definitions and results are from [1].

Proposition 2.1. *Let A be any subset of a topological space (X,τ). Then, $X \setminus \alpha_{[\gamma,\gamma']}$-$Int(A) = \alpha_{[\gamma,\gamma']}$-$Cl(X \setminus A)$.*

Theorem 2.2. *If γ and γ' are α-open operations and A a subset of (X,τ). Then, we have $\alpha Cl_{[\gamma,\gamma']}(\alpha Cl_{[\gamma,\gamma']}(A)) = \alpha Cl_{[\gamma,\gamma']}(A)$.*

Proposition 2.3. *Let A be any subset of a topological space (X,τ). If A is $[\gamma,\gamma']$-open* [5]*, then A is $\alpha_{[\gamma,\gamma']}$-open.*

Remark 2.4. If γ and γ' are α-regular operations, then $\alpha O(X,\tau)_{[\gamma,\gamma']}$ form a topology on X.

Proposition 2.5. *Let A and B be any subsets of a topological space (X,τ). If A is α_γ-open and B is $\alpha_{\gamma'}$-open, then $A \cap B$ is $\alpha_{[\gamma,\gamma']}$-open.*

Definition 2.6. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous if for each point $x \in X$ and each α-open sets W and S of Y containing $f(x)$ there exist α-open sets U and V of X containing x such that $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'}$.

Definition 2.7. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed if for $\alpha_{[\gamma,\gamma']}$-closed set A of X, $f(A)$ is $\alpha_{[\beta,\beta']}$-closed in Y.

3. $\alpha_{[\gamma,\gamma']}$-g.Closed Sets

In this section, we define and study some properties of $\alpha_{[\gamma,\gamma']}$-g.closed sets.

Definition 3.1. A subset A of X is said to be an $\alpha_{[\gamma,\gamma']}$-generalized closed (briefly, $\alpha_{[\gamma,\gamma']}$-g.closed) set if $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an $\alpha_{[\gamma,\gamma']}$-open set in (X,τ).

Remark 3.2. It is clear that every $\alpha_{[\gamma,\gamma']}$-closed set is $\alpha_{[\gamma,\gamma']}$-g.closed. But the converse is not true in general as it is shown in the following example.

Example 3.3. Let $X = \{a,b,c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$. For each $A \in \alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } A = \{b\} \text{ or } \{a,c\}, \\ X & \text{otherwise.} \end{cases}$$

Now, if we let $A = \{a\}$, since the only $\alpha_{[\gamma,\gamma']}$-open supersets of A are $\{a,c\}$ and X, then A is $\alpha_{[\gamma,\gamma']}$-g.closed. But it is easy to see that A is not $\alpha_{[\gamma,\gamma']}$-closed.

Proposition 3.4. *If A is γ-open and $\alpha_{[\gamma,\gamma']}$-g.closed then A is $\alpha_{[\gamma,\gamma']}$-closed.*

Proof. Suppose that A is γ-open and $\alpha_{[\gamma,\gamma']}$-g.closed. As every γ-open is $\alpha_{[\gamma,\gamma']}$-open and $A \subseteq A$, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq A$, also $A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A)$, therefore $\alpha_{[\gamma,\gamma']}$-$Cl(A) = A$. That is A is $\alpha_{[\gamma,\gamma']}$-closed. □

Remark 3.5. If A is $\alpha_{[\gamma,\gamma']}$-open and $\alpha_{[\gamma,\gamma']}$-g.closed then A is $\alpha_{[\gamma,\gamma']}$-closed.

Proposition 3.6. *The intersection of an $\alpha_{[\gamma,\gamma']}$-g.closed set and an $\alpha_{[\gamma,\gamma']}$-closed set is always $\alpha_{[\gamma,\gamma']}$-g.closed.*

Proof. Let A be $\alpha_{[\gamma,\gamma']}$-g.closed and F be $\alpha_{[\gamma,\gamma']}$-closed. Assume that U is $\alpha_{[\gamma,\gamma']}$-open set such that $A \cap F \subseteq U$, set $G = X \setminus F$. Then $A \subseteq U \cup G$, since G is $\alpha_{[\gamma,\gamma']}$-open, then $U \cup G$ is $\alpha_{[\gamma,\gamma']}$-open and since A is $\alpha_{[\gamma,\gamma']}$-g.closed, then $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U \cup G$. Now, $\alpha_{[\gamma,\gamma']}$-$Cl(A \cap F) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap \alpha_{[\gamma,\gamma']}$-$Cl(F) = \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap F \subseteq (U \cup G) \cap F = (U \cap F) \cup (G \cap F) = (U \cap F) \cup \phi \subseteq U$. □

Remark 3.7. The intersection of two $\alpha_{[\gamma,\gamma']}$-g.closed sets need not be $\alpha_{[\gamma,\gamma']}$-g.closed in general. It is shown by the following example.

Example 3.8. Let $X = \{a,b,c\}$ and τ be a discrete topology on X. For each $A \in \alpha O(X)$ we define two operations γ and γ', respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } A = \{a\}, \\ X & \text{if } A \neq \{a\}. \end{cases}$$

Set $A = \{a,b\}$ and $B = \{a,c\}$. Clearly, A and B are $\alpha_{[\gamma,\gamma']}$-g.closed sets, since X is their only $\alpha_{[\gamma,\gamma']}$-open superset. But $C = \{a\} = A \cap B$ is not $\alpha_{[\gamma,\gamma']}$-g.closed, since $C \subseteq \{a\} \in \alpha O(X,\tau)_{[\gamma,\gamma']}$ and $\alpha_{[\gamma,\gamma']}$-$Cl(C) = X \nsubseteq \{a\}$.

Proposition 3.9. *If γ ana γ' are α-regular operations on $\alpha O(X)$. Then the finite union of $\alpha_{[\gamma,\gamma']}$-g.closed sets is always an $\alpha_{[\gamma,\gamma']}$-g.closed set.*

Proof. Let A and B be two $\alpha_{[\gamma,\gamma']}$-g.closed sets, and let $A \cup B \subseteq U$, where U is $\alpha_{[\gamma,\gamma']}$-open. Since A and B are $\alpha_{[\gamma,\gamma']}$-g.closed sets, therefore $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$ and $\alpha_{[\gamma,\gamma']}$-$Cl(B) \subseteq U$ implies $\alpha_{[\gamma,\gamma']}$-$Cl(A) \cup \alpha_{[\gamma,\gamma']}$-$Cl(B) \subseteq U$. But, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) \cup \alpha_{[\gamma,\gamma']}$-$Cl(B) = \alpha_{[\gamma,\gamma']}$-$Cl(A \cup B)$. Therefore $\alpha_{[\gamma,\gamma']}$-$Cl(A \cup B) \subseteq U$. Hence $A \cup B$ is an $\alpha_{[\gamma,\gamma']}$-g.closed set. □

Remark 3.10. The union of two $\alpha_{[\gamma,\gamma']}$-g.closed sets need not be $\alpha_{[\gamma,\gamma']}$-g.closed in general. It is shown by the following example.

Example 3.11. Let $X = \{a, b, c\}$ and τ be a discrete topology on X. For each $A \in \alpha O(X)$ we define two operations γ and γ', respectively, by

$$A^{\gamma} = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\}, \\ X & \text{otherwise,} \end{cases}$$

and $A^{\gamma'} = X$. Let $A = \{a\}$ and $B = \{b\}$. Here A and B are $\alpha_{[\gamma,\gamma']}$-g.closed but $A \cup B = \{a, b\}$ is not $\alpha_{[\gamma,\gamma']}$-g.closed, since $\{a, b\}$ is $\alpha_{[\gamma,\gamma']}$-open and $\alpha_{[\gamma,\gamma']}$-$Cl(\{a, b\}) = X$.

Proposition 3.12. *If a subset A of X is $\alpha_{[\gamma,\gamma']}$-g.closed and $A \subseteq B \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A)$, then B is an $\alpha_{[\gamma,\gamma']}$-g.closed set in X.*

Proof. Let A be an $\alpha_{[\gamma,\gamma']}$-g.closed set such that $A \subseteq B \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A)$. Let U be an $\alpha_{[\gamma,\gamma']}$-open set of X such that $B \subseteq U$. Since A is $\alpha_{[\gamma,\gamma']}$-g.closed, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$. Now $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(B) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl[\alpha_{[\gamma,\gamma']}$-$Cl(A)] = \alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$. That is $\alpha_{[\gamma,\gamma']}$-$Cl(B) \subseteq U$, where U is $\alpha_{[\gamma,\gamma']}$-open. Therefore B is an $\alpha_{[\gamma,\gamma']}$-g.closed set in X. □

Proposition 3.13. *For each $x \in X$, $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed or $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-g.closed in (X, τ).*

Proof. Suppose that $\{x\}$ is not $\alpha_{[\gamma,\gamma']}$-closed, then $X \setminus \{x\}$ is not $\alpha_{[\gamma,\gamma']}$-open. Let U be any $\alpha_{[\gamma,\gamma']}$-open set such that $X \setminus \{x\} \subseteq U$, implies $U = X$. Therefore $\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-g.closed. □

Proposition 3.14. *A subset A of X is $\alpha_{[\gamma,\gamma']}$-g.closed if and only if $\alpha_{[\gamma,\gamma']}$-$Cl(\{x\}) \cap A \neq \phi$, holds for every $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$.*

Proof. Let U be an $\alpha_{[\gamma,\gamma']}$-open set such that $A \subseteq U$ and let $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$. By assumption, there exists a point $z \in \alpha_{[\gamma,\gamma']}$-$Cl(\{x\})$ and $z \in A \subseteq U$. It follows that $U \cap \{x\} \neq \phi$, hence $x \in U$, this implies $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$. Therefore A is $\alpha_{[\gamma,\gamma']}$-g.closed.
Conversely, suppose that $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$ such that $\alpha_{[\gamma,\gamma']}$-$Cl(\{x\}) \cap A = \phi$. Since, $\alpha_{[\gamma,\gamma']}$-$Cl(\{x\})$ is $\alpha_{[\gamma,\gamma']}$-closed. Therefore, $X \setminus \alpha_{[\gamma,\gamma']}$-$Cl(\{x\})$ is an

$\alpha_{[\gamma,\gamma']}$-open set in X. Since $A \subseteq X \setminus (\alpha_{[\gamma,\gamma']}\text{-}Cl(\{x\}))$ and A is $\alpha_{[\gamma,\gamma']}$-g.closed implies that $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \subseteq X \setminus \alpha_{[\gamma,\gamma']}\text{-}Cl(\{x\})$ holds, and hence $x \notin \alpha_{[\gamma,\gamma']}\text{-}Cl(A)$. This is a contradiction. Therefore $\alpha_{[\gamma,\gamma']}\text{-}Cl(\{x\}) \cap A \neq \phi$. □

Proposition 3.15. *A set A of a space X is $\alpha_{[\gamma,\gamma']}$-g.closed if and only if $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A$ does not contain any non-empty $\alpha_{[\gamma,\gamma']}$-closed set.*

Proof. **Necessity.** Suppose that A is an $\alpha_{[\gamma,\gamma']}$-g.closed set in X. We prove the result by contradiction. Let F be an $\alpha_{[\gamma,\gamma']}$-closed set such that $F \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A$ and $F \neq \phi$. Then $F \subseteq X \setminus A$ which implies $A \subseteq X \setminus F$. Since A is $\alpha_{[\gamma,\gamma']}$-g.closed and $X \setminus F$ is $\alpha_{[\gamma,\gamma']}$-open, therefore $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \subseteq X \setminus F$, that is $F \subseteq X \setminus \alpha_{[\gamma,\gamma']}\text{-}Cl(A)$. Hence $F \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap (X \setminus \alpha_{[\gamma,\gamma']}\text{-}Cl(A)) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A$ does not contain any non-empty $\alpha_{[\gamma,\gamma']}$-closed set in X.
Sufficiency. Let $A \subseteq U$, where U is $\alpha_{[\gamma,\gamma']}$-open in X. If $\alpha_{[\gamma,\gamma']}\text{-}Cl(A)$ is not contained in U, then $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap X \setminus U \neq \phi$. Now, since $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap X \setminus U \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A$ and $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap X \setminus U$ is a non-empty $\alpha_{[\gamma,\gamma']}$-closed set, then we obtain a contradication and therefore A is $\alpha_{[\gamma,\gamma']}$-g.closed. □

Proposition 3.16. *If A is an $\alpha_{[\gamma,\gamma']}$-g.closed set of a space X, then the following are equivalent:*

1. *A is $\alpha_{[\gamma,\gamma']}$-closed.*
2. *$\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A$ is $\alpha_{[\gamma,\gamma']}$-closed.*

Proof. (1) $\Rightarrow$ (2). If A is an $\alpha_{[\gamma,\gamma']}$-g.closed set which is also $\alpha_{[\gamma,\gamma']}$-closed, then by Proposition 3.15, $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A = \phi$, which is $\alpha_{[\gamma,\gamma']}$-closed.
(2) $\Rightarrow$ (1). Let $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A$ be an $\alpha_{[\gamma,\gamma']}$-closed set and A be $\alpha_{[\gamma,\gamma']}$-g.closed. Then by Proposition 3.15, $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A$ does not contain any non-empty $\alpha_{[\gamma,\gamma']}$-closed subset. Since $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A$ is $\alpha_{[\gamma,\gamma']}$-closed and $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \setminus A = \phi$, this shows that A is $\alpha_{[\gamma,\gamma']}$-closed. □

Proposition 3.17. *For a space (X, τ), the following are equivalent:*

1. *Every subset of X is $\alpha_{[\gamma,\gamma']}$-g.closed.*
2. $\alpha O(X,\tau)_{[\gamma,\gamma']} = \alpha C(X,\tau)_{[\gamma,\gamma']}$.

Proof. (1) $\Rightarrow$ (2). Let $U \in \alpha O(X,\tau)_{[\gamma,\gamma']}$. Then by hypothesis, U is $\alpha_{[\gamma,\gamma']}$-g.closed which implies that $\alpha_{[\gamma,\gamma']}\text{-}Cl(U) \subseteq U$, so, $\alpha_{[\gamma,\gamma']}\text{-}Cl(U) = U$, therefore $U \in \alpha C(X,\tau)_{[\gamma,\gamma']}$. Also let $V \in \alpha C(X,\tau)_{[\gamma,\gamma']}$. Then $X \setminus V \in \alpha O(X,\tau)_{[\gamma,\gamma']}$, hence by hypothesis $X \setminus V$ is $\alpha_{[\gamma,\gamma']}$-g.closed and then $X \setminus V \in \alpha C(X,\tau)_{[\gamma,\gamma']}$, thus $V \in \alpha O(X,\tau)_{[\gamma,\gamma']}$ according to the above we have $\alpha O(X,\tau)_{[\gamma,\gamma']} = \alpha C(X,\tau)_{[\gamma,\gamma']}$.
(2) $\Rightarrow$ (1). If A is a subset of a space X such that $A \subseteq U$ where $U \in$

$\alpha O(X,\tau)_{[\gamma,\gamma']}$, then $U \in \alpha C(X,\tau)_{[\gamma,\gamma']}$ and therefore $\alpha_{[\gamma,\gamma']}$-$Cl(U) \subseteq U$ which shows that A is $\alpha_{[\gamma,\gamma']}$-g.closed. □

Definition 3.18. A subset A of X is $\alpha_{[\gamma,\gamma']}$-g.open if its complement $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-g.closed in X.

Remark 3.19. It is clear that every $\alpha_{[\gamma,\gamma']}$-open set is $\alpha_{[\gamma,\gamma']}$-g.open. But the converse is not true in general as it is shown in the following example.

Example 3.20. Consider Example 3.3, if $A = \{b, c\}$ then A is $\alpha_{[\gamma,\gamma']}$-g.open but not $\alpha_{[\gamma,\gamma']}$-open.

Proposition 3.21. *A subset A of X is $\alpha_{[\gamma,\gamma']}$-g.open if and only if $F \subseteq \alpha_{[\gamma,\gamma']}$-$Int(A)$ whenever $F \subseteq A$ and F is $\alpha_{[\gamma,\gamma']}$-closed in (X,τ).*

Proof. Let A be $\alpha_{[\gamma,\gamma']}$-g.open and $F \subseteq A$ where F is $\alpha_{[\gamma,\gamma']}$-closed. Since $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-g.closed and $X \setminus F$ is an $\alpha_{[\gamma,\gamma']}$-open set containing $X \setminus A$ implies $\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus A) \subseteq X \setminus F$. By Proposition 2.1, $X \setminus \alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq X \setminus F$. That is $F \subseteq \alpha_{[\gamma,\gamma']}$-$Int(A)$.
Conversely, suppose that F is $\alpha_{[\gamma,\gamma']}$-closed and $F \subseteq A$ implies $F \subseteq \alpha_{[\gamma,\gamma']}$-$Int(A)$. Let $X \setminus A \subseteq U$ where U is $\alpha_{[\gamma,\gamma']}$-open. Then $X \setminus U \subseteq A$ where $X \setminus U$ is $\alpha_{[\gamma,\gamma']}$-closed. By hypothesis $X \setminus U \subseteq \alpha_{[\gamma,\gamma']}$-$Int(A)$. That is $X \setminus \alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq U$. By Proposition 2.1, $\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus A) \subseteq U$. This implies $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-g.closed and A is $\alpha_{[\gamma,\gamma']}$-g.open. □

Remark 3.22. The union of two $\alpha_{[\gamma,\gamma']}$-g.open sets need not be $\alpha_{[\gamma,\gamma']}$-g.open in general. It is shown by the following example.

Example 3.23. Consider Example 3.8, if $A = \{b\}$ and $B = \{c\}$ then A and B are $\alpha_{[\gamma,\gamma']}$-g.open sets in X, but $A \cup B = \{b, c\}$ is not an $\alpha_{[\gamma,\gamma']}$-g.open set in X.

Proposition 3.24. *Let γ and γ' be an α-regular operations on $\alpha O(X)$, and let A and B be two $\alpha_{[\gamma,\gamma']}$-g.open sets in a space X. Then $A \cap B$ is also $\alpha_{[\gamma,\gamma']}$-g.open.*

Proof. If A and B are $\alpha_{[\gamma,\gamma']}$-g.open sets in a space X, then $X \setminus A$ and $X \setminus B$ are $\alpha_{[\gamma,\gamma']}$-g.closed sets in X. By Proposition 3.9, $X \setminus A \cup X \setminus B$ is also an $\alpha_{[\gamma,\gamma']}$-g.closed set in X. That is $X \setminus A \cup X \setminus B = X \setminus (A \cap B)$ is an $\alpha_{[\gamma,\gamma']}$-g.closed set in X. Therefore $A \cap B$ is an $\alpha_{[\gamma,\gamma']}$-g.open set in X. □

Proposition 3.25. *Every singleton point set in a space X is either $\alpha_{[\gamma,\gamma']}$-g.open or $\alpha_{[\gamma,\gamma']}$-closed.*

Proof. Suppose that $\{x\}$ is not $\alpha_{[\gamma,\gamma']}$-g.open, then by definition $X \setminus \{x\}$ is not $\alpha_{[\gamma,\gamma']}$-g.closed. This implies that by Proposition 3.13, the set $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed. □

Proposition 3.26. *If* $\alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq B \subseteq A$ *and* A *is* $\alpha_{[\gamma,\gamma']}$*-g.open, then* B *is* $\alpha_{[\gamma,\gamma']}$*-g.open.*

Proof. $\alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq B \subseteq A$ implies $X \setminus A \subseteq X \setminus B \subseteq X \setminus \alpha_{[\gamma,\gamma']}$-$Int(A)$. That is, $X \setminus A \subseteq X \setminus B \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(X \setminus A)$ by Proposition 2.1. Since $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-g.closed, by Proposition 3.12, $X \setminus B$ is $\alpha_{[\gamma,\gamma']}$-g.closed and B is $\alpha_{[\gamma,\gamma']}$-g.open. □

4. $\alpha_{[\gamma,\gamma']}$-Separations Spaces

In this section we introduce $\alpha_{[\gamma,\gamma']}$-$T_i$ spaces ($i = 0, \frac{1}{2}, 1, 2$) and investigate relations among these spaces.

Definition 4.1. A topological space (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$ if every $\alpha_{[\gamma,\gamma']}$-g.closed set is $\alpha_{[\gamma,\gamma']}$-closed.

Remark 4.2. It follows from Remark 3.2 that (X, τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$ if and only if the $\alpha_{[\gamma,\gamma']}$-g.closedness coincides with the $\alpha_{[\gamma,\gamma']}$-closedness.

Definition 4.3. A topological space (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-$T_0$ if for each pair of distinct points x, y in X, there exist an α-open sets U and V such that $x \in U \cap V$ and $y \notin U^{\gamma} \cap V^{\gamma'}$, or $y \in U \cap V$ and $x \notin U^{\gamma} \cap V^{\gamma'}$.

Definition 4.4. A topological space (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-$T_1$ if for each pair of distinct points x, y in X, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $y \notin U^{\gamma} \cap V^{\gamma'}$ and $x \notin W^{\gamma} \cap S^{\gamma'}$.

Definition 4.5. A topological space (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-$T_2$ if for each pair of distinct points x, y in X, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $(U^{\gamma} \cap V^{\gamma'}) \cap (W^{\gamma} \cap S^{\gamma'}) = \phi$.

Remark 4.6. For given two distinct points x and y, the $\alpha_{[\gamma,\gamma']}$-$T_0$-axiom requires that there exist α-open sets U, V, W and S satisfying one of conditions $(1), (2), (3)$ and (4):

(1) $x \in U \cap V$, $y \in W \cap S$, $y \notin U^{\gamma} \cap V^{\gamma'}$ and $x \notin W^{\gamma} \cap S^{\gamma'}$.
(2) $x \in U \cap V$, $x \in W \cap S$, $y \notin U^{\gamma} \cap V^{\gamma'}$ and $y \notin W^{\gamma} \cap S^{\gamma'}$.
(3) $y \in U \cap V$, $y \in W \cap S$, $x \notin U^{\gamma} \cap V^{\gamma'}$ and $x \notin W^{\gamma} \cap S^{\gamma'}$.
(4) $y \in U \cap V$, $x \in W \cap S$, $x \notin U^{\gamma} \cap V^{\gamma'}$ and $y \notin W^{\gamma} \cap S^{\gamma'}$.

Remark 4.7. A space X is $\alpha_{[\gamma,\gamma']}$-$T_0$ if and only if for each pair of distinct points x, y in X, there exists an α-open sets W such that $x \in W$ and $y \notin W^{\gamma} \cap W^{\gamma'}$, or $y \in W$ and $x \notin W^{\gamma} \cap W^{\gamma'}$.

Proposition 4.8. *A topological space* (X, τ) *is* $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$ *if and only if for each* $x \in X$, $\{x\}$ *is either* $\alpha_{[\gamma,\gamma']}$*-closed or* $\alpha_{[\gamma,\gamma']}$*-open.*

Proof. **Necessity.** Suppose $\{x\}$ is not $\alpha_{[\gamma,\gamma']}$-closed. Then by Proposition 3.13, $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-g.closed. Since (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$, $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed, that is $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-open.
Sufficiency. Let A be any $\alpha_{[\gamma,\gamma']}$-g.closed set in (X,τ) and $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$. It suffices to prove it for the following two cases:

Case 1. Suppose that $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed, then $x \notin A$ will imply $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A$, which is not possible by Proposition 3.15. Hence $x \in A$. Therefore, $\alpha_{[\gamma,\gamma']}$-$Cl(A) = A$, that is A is $\alpha_{[\gamma,\gamma']}$-closed.

Case 2. Suppose that $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-open then as $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$, $\{x\} \cap A \neq \phi$. Hence $x \in A$ and A is $\alpha_{[\gamma,\gamma']}$-closed. So, (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$. □

Proposition 4.9. *Let γ and $\gamma^{'}$ be α-open operations. Then, a topological space (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_0$ if and only if for each pair of distinct points x, y of X, $\alpha Cl_{[\gamma,\gamma']}(\{x\}) \neq \alpha Cl_{[\gamma,\gamma']}(\{y\})$.*

Proof. **Necessity.** Let (X,τ) be an $\alpha_{[\gamma,\gamma']}$-$T_0$ space and x, y be any two distinct points of X, then there exist an α-open sets U and V such that $x \in U \cap V$ and $y \notin U^{\gamma} \cap V^{\gamma^{'}}$. Then $(U^{\gamma} \cap V^{\gamma'}) \cap \{y\} = \phi$ this implies that $x \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$. Consequently $\alpha Cl_{[\gamma,\gamma']}(\{x\}) \neq \alpha Cl_{[\gamma,\gamma']}(\{y\})$.
Sufficiency. Suppose that $x, y \in X$, $x \neq y$ and $\alpha Cl_{[\gamma,\gamma']}(\{x\}) \neq \alpha Cl_{[\gamma,\gamma']}(\{y\})$. Let z be a point of X such that $z \in \alpha Cl_{[\gamma,\gamma']}(\{x\})$ but $z \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$. We claim that $x \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$. For, if $x \in \alpha Cl_{[\gamma,\gamma']}(\{y\})$ then $\alpha Cl_{[\gamma,\gamma']}(\{x\}) \subseteq \alpha Cl_{[\gamma,\gamma']}(\{y\})$ by Theorem 2.2. This contradicts the fact that $z \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$. Consequently $x \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$, then there exist an α-open sets U and V such that $x \in U \cap V$ and $(U^{\gamma} \cap V^{\gamma'}) \cap \{y\} = \phi$, this implies that $y \notin U^{\gamma} \cap V^{\gamma^{'}}$. Therefore, (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_0$.

□

Proposition 4.10. *A topological space (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_1$ if and only if for each $x \in X$, $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed.*

Proof. Let (X,τ) be $\alpha_{[\gamma,\gamma']}$-$T_1$ and x any point of X. Suppose $y \in X \setminus \{x\}$, then $x \neq y$ and so there exist α-open sets W and S containing y and $x \notin W^{\gamma} \cap S^{\gamma^{'}}$. Consequently $y \in W^{\gamma} \cap S^{\gamma^{'}} \subseteq X \setminus \{x\}$, that is $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-open.
Conversely, suppose $\{p\}$ is $\alpha_{[\gamma,\gamma']}$-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X \setminus \{x\}$ and $x \in X \setminus \{y\}$. Hence $X \setminus \{y\}$ is an $\alpha_{[\gamma,\gamma']}$-open set contains x, so there exist α-open sets U and V containing x such that $U^{\gamma} \cap V^{\gamma^{'}} \subseteq X \setminus \{y\}$. Similarly $X \setminus \{x\}$ is an $\alpha_{[\gamma,\gamma']}$-open set contains y, so there exist α-open sets W and S containing y such that $W^{\gamma} \cap S^{\gamma^{'}} \subseteq X \setminus \{x\}$. Accordingly X is an $\alpha_{[\gamma,\gamma']}$-$T_1$ space. □

Proposition 4.11. *The following statements are equivalent for a topological space (X,τ) with an operations γ and $\gamma^{'}$ on $\alpha O(X)$:*

1. *X is $\alpha_{[\gamma,\gamma']}$-$T_2$.*
2. *Let $x \in X$. For each $y \neq x$, there exist an α-open sets U and V containing x such that $y \notin \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma^{'}})$.*
3. *For each $x \in X$, $\cap\{\alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma^{'}}) : U, V \in \alpha O(X)$ and $x \in U \cap V\} = \{x\}$.*

Proof. (1) $\Rightarrow$ (2). Since X is $\alpha_{[\gamma,\gamma']}$-$T_2$, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $(U^{\gamma} \cap V^{\gamma^{'}}) \cap (W^{\gamma} \cap S^{\gamma^{'}}) = \phi$, implies that $y \notin \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma^{'}})$.
(2) $\Rightarrow$ (3). If possible for some $y \neq x$, we have $y \in \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma^{'}})$ for every α-open sets U and V containing x, which then contradicts (2).
(3) $\Rightarrow$ (1). Let $x, y \in X$ and $x \neq y$. Then there exist α-open sets U and V containing x such that $y \notin \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma^{'}})$, implies that $(U^{\gamma} \cap V^{\gamma^{'}}) \cap (W^{\gamma} \cap S^{\gamma^{'}}) = \phi$ for some α-open sets W and S containing y. □

Proposition 4.12. (1) *If (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_2$, then it is $\alpha_{[\gamma,\gamma']}$-$T_1$.*
(2) *If (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_1$, then it is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$.*
(3) *If (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$, then it is $\alpha_{[\gamma,\gamma']}$-$T_0$.*

Proof. (1) The proof is straightforward from the Definitions 4.4 and 4.5.
(2) The proof is obvious by Proposition 4.10.
(3) Let x and y be any two distinct points of X. By Proposition 4.8, the singleton set $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed or $\alpha_{[\gamma,\gamma']}$-open.
   (a) If $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed, then $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-open containing y and there exist α-open sets W and S containing y such that $W^{\gamma} \cap S^{\gamma^{'}} \subseteq X \setminus \{x\}$, implies that $y \in W \cap S$ and $x \notin W^{\gamma} \cap S^{\gamma^{'}}$.
   (b) If $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-open, then there exist α-open sets U and V containing x such that $U^{\gamma} \cap V^{\gamma^{'}} \subseteq \{x\}$, implies that $x \in U \cap V$ and $y \notin U^{\gamma} \cap V^{\gamma^{'}}$. Therefore, we have X is $\alpha_{[\gamma,\gamma']}$-$T_0$.

□

Remark 4.13. The following series of examples show that all converses of Proposition 4.12 can not be reserved.

Example 4.14. Let (X,τ), γ and $\gamma^{'}$ be the same space and the same operations as in Example 3.11. Then, it is shown directly that each singleton is $\alpha_{[\gamma,\gamma']}$-closed in (X,τ). By Proposition 4.10, (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_1$. But, we can show that $(U^{\gamma} \cap V^{\gamma^{'}}) \cap (W^{\gamma} \cap S^{\gamma^{'}}) \neq \phi$ holds for any α-open sets U, V, W and S. This implies (X,τ) is not $\alpha_{[\gamma,\gamma']}$-$T_2$

Example 4.15. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ be a topology on X. For each $A \in \alpha O(X)$ we define two operations γ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = A$. Then, it is shown directly that each singleton is $\alpha_{[\gamma,\gamma']}$-closed or $\alpha_{[\gamma,\gamma']}$-open in (X, τ). By Proposition 4.8, (X, τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$. However, by Proposition 4.10, (X, τ) is not $\alpha_{[\gamma,\gamma']}$-$T_1$, in fact, a singleton $\{a\}$ is not $\alpha_{[\gamma,\gamma']}$-closed.

Example 4.16. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ be a topology on X. For each $A \in \alpha O(X)$ we define two operations γ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = A$. Then, (X, τ) is not $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$ because a singleton $\{b\}$ is neither $\alpha_{[\gamma,\gamma']}$-open nor $\alpha_{[\gamma,\gamma']}$-closed. It is shown directly that (X, τ) is $\alpha_{[\gamma,\gamma']}$-$T_0$.

Remark 4.17. From Proposition 4.12 and Examples 4.14, 4.15 and 4.16, the following implications hold and none of the implications is reversible:

$$\alpha_{[\gamma,\gamma']}\text{-}T_2 \longrightarrow \alpha_{[\gamma,\gamma']}\text{-}T_1 \longrightarrow \alpha_{[\gamma,\gamma']}\text{-}T_{\frac{1}{2}} \longrightarrow \alpha_{[\gamma,\gamma']}\text{-}T_0,$$

where $A \to B$ represents that A implies B.

Proposition 4.18. *If (X, τ) is $\alpha_{[\gamma,\gamma']}$-$T_i$, then it is α-T_i, where $i = 0, \frac{1}{2}, 1, 2$.*

Proof. The proofs for $i = 0, 2$ follow from definitions.
The proof for $i = 1$ (resp. $i = \frac{1}{2}$) follows from Proposition 4.10 (resp. Proposition 4.8). □

Remark 4.19. The following of example show that all converses of Proposition 4.18 can not be reserved.

Example 4.20. Let $X = \{a, b, c\}$ and τ be a discrete topology on X. For each $A \in \alpha O(X)$ we define two operations γ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = X$. Then, (X, τ) is α-T_i but it is not $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.

Proposition 4.21. *If (X, τ) is α_{γ}-T_i, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.*

Proof. The proofs for $i = 0, 1, 2$ follow from Definitions 4.3, 4.4, 4.5 and [[6]; Definition 3.6].
The proof for $i = \frac{1}{2}$ is obtained as follows: Let $x \in X$. Then,$\{x\}$ is α_{γ}-open or α_{γ}-closed by [[6]; Theorem 3.2]. So, $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-open or $\alpha_{[\gamma,\gamma']}$-closed because every α_{γ}-open is $\alpha_{[\gamma,\gamma']}$-open. The proof is completed from Proposition 4.8. □

Remark 4.22. The following series of examples show that all converses of Proposition 4.21 can not be reserved.

Example 4.23. Let $X = \{a, b, c\}$ and τ be a discrete topology on X. For each $A \in \alpha O(X)$ we define two operations γ and $\gamma^{'}$, respectively, by

$$A^{\gamma} = A^{\gamma^{'}} = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\}, \\ X & \text{otherwise.} \end{cases}$$

Then, (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_2$ but not α_γ-T_2.

Example 4.24. Let $X=\{a,b,c\}$ and τ be a discrete topology on X. For each $A\in\alpha O(X)$ we define two operations γ and $\gamma^{'}$, respectively, by

$$A^\gamma=\begin{cases} A & \text{if } A=\{a,b\} \text{ or } \{a,c\},\\ X & \text{otherwise,}\end{cases}$$

and

$$A^{\gamma^{'}}=\begin{cases} A & \text{if } A=\{b,c\},\\ X & \text{otherwise.}\end{cases}$$

Then, (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_i$ but not α_γ-T_i, where $i=\frac{1}{2},1$.

Example 4.25. Let $X=\{a,b,c\}$ and τ be a discrete topology on X. For each $A\in\alpha O(X)$ we define two operations γ and $\gamma^{'}$, respectively, by

$$A^\gamma=\begin{cases} A & \text{if } A=\{a\},\\ X & \text{otherwise,}\end{cases}$$

and

$$A^{\gamma^{'}}=\begin{cases} A & \text{if } A=\{b\},\\ X & \text{otherwise.}\end{cases}$$

Then, (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_0$ but not α_γ-T_0.

Proposition 4.26. *If (X,τ) is $[\gamma,\gamma^{'}]$-T_i, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i=0,\frac{1}{2},1,2$.*

Proof. The proofs for $i=0,2$ follow from Definitions 4.3, 4.5 and [[5]; Definitions 5.2, 5.4].
The proof for $i=1$ (resp. $i=\frac{1}{2}$) follows from [[5]; Proposition 5.8] (resp. [5]; Proposition 5.7]) and Proposition 2.3. □

Remark 4.27. The following example show that the converses of Proposition 4.26 can not be reserved, for $i=0,\frac{1}{2}$.

Example 4.28. Let $X=\{a,b,c\}$ and $\tau=\{\phi,X,\{a\}\}$ be a topology on X. For each $A\in\alpha O(X)$ we define two operations γ and $\gamma^{'}$, respectively, by $A^\gamma=A^{\gamma^{'}}=A$. Then, (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_i$ but not $[\gamma,\gamma^{'}]$-T_i, where $i=0,\frac{1}{2}$.

Proposition 4.29. *If (X,τ) is $(\gamma,\gamma^{'})$-T_i, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i=0,\frac{1}{2},1,2$.*

Proof. The proofs follow from [[5]; Proposition 6.12] and Proposition 4.26. □

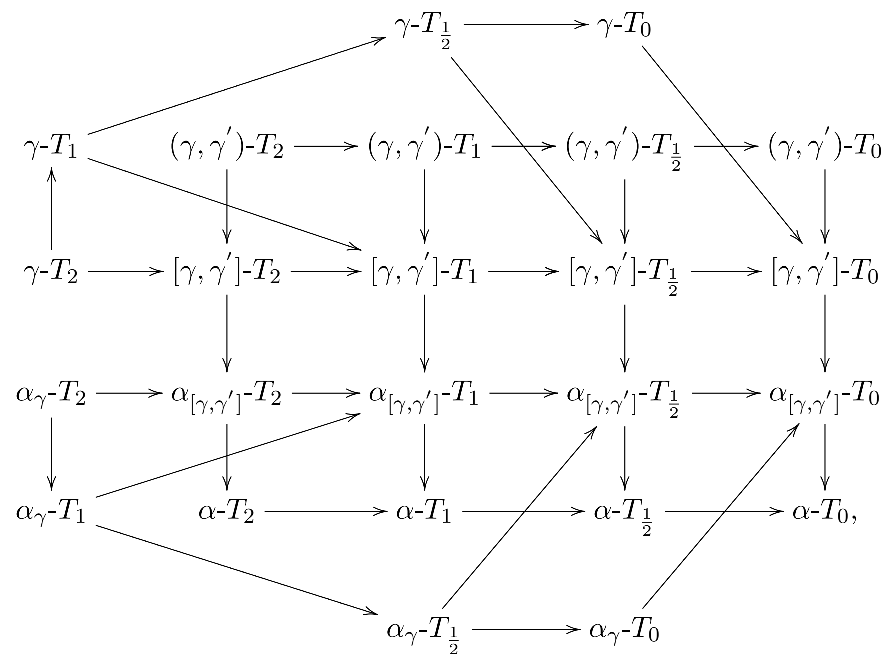
Remark 4.30. The converse of Proposition 4.29 can not reversible by [[5]; Remark 6.13, Examples 6.14 and 6.15] and Proposition 4.26.

Proposition 4.31. *If (X,τ) is γ-T_i, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i=0,\frac{1}{2},1,2$.*

Proof. The proofs follow from [[5]; Proposition 6.1] and Proposition 4.26. □

Remark 4.32. The converse of Proposition 4.31 can not reversible by [[5]; Remark 6.2, Examples 6.3, 6.4 and 6.6] and Proposition 4.26.

Remark 4.33. From Propositions 4.12, 4.18, 4.21, 4.26, 4.29, 4.31, [[10]; Remark 2.1], and [[4]; p.180], for distinct operations γ and $\gamma^{'}$ we have the following diagram. We note that none of the implications in the following diagram is reversible by Remarks 4.13, 4.19, 4.22, 4.30 and 4.32:



where $A \to B$ represents that A implies B.

Remark 4.34. We propose the following two questions since we could not find counter examples :

Are the spaces $\alpha_{[\gamma,\gamma']}$-$T_1$ and $[\gamma,\gamma^{'}]$-T_1 equivalent or not? What about $\alpha_{[\gamma,\gamma']}$-$T_2$ and $[\gamma,\gamma^{'}]$-T_2?

Proposition 4.35. *Suppose that γ and $\gamma^{'}$ are α-regular operations on $\alpha O(X)$. A space (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_i$ if and only if an associated space $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is T_i, where $i = 1, 1/2$.*

Proof. It follows from Remark 2.4 that a subset A is $\alpha_{[\gamma,\gamma']}$-open in (X,τ) if and only if A is open in $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$. Therefore, the proof for $i = \frac{1}{2}$ (resp. $i = 1$) follows from Propositions 4.8 (resp. Proposition 4.10). □

Proposition 4.36. *If γ and $\gamma^{'}$ are α-regular operations on $\alpha O(X)$ and $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is T_2, then (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_2$*

Proof. This follows from the Hausdorffness of $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ and definition of $\alpha_{[\gamma,\gamma']}$-open and Definition 4.5. □

Proposition 4.37. *If γ and $\gamma^{'}$ are α-regular and α-open and (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_2$, then $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is T_2.*

Proof. Let x and y be distinct points of X. By assumptions there exist α_γ-open sets U, W and $\alpha_{\gamma'}$-open sets V, S such that $x \in U \cap V$, $y \in W \cap S$ and $(U \cap V) \cap (W \cap S) = \phi$. It follows from Proposition 2.5 that $U \cap V \in \alpha O(X,\tau)_{[\gamma,\gamma']}$ and $W \cap S \in \alpha O(X,\tau)_{[\gamma,\gamma']}$. This implies that $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is T_2. □

Proposition 4.38. *If γ and $\gamma^{'}$ are α-regular and α-open and (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_0$, then $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is T_0.*

Proof. This follows from the Definition 4.3, and Propositon 2.5. □

Proposition 4.39. *If $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous and $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed, then*

1. *$f(A)$ is $\alpha_{[\beta,\beta']}$-g.closed for every $\alpha_{[\gamma,\gamma']}$-g.closed set A of (X,τ).*
2. *$f^{-1}(B)$ is $\alpha_{[\gamma,\gamma']}$-g.closed for every $\alpha_{[\beta,\beta']}$-g.closed set B of (Y,σ).*

Proof. (1) Let V be an $\alpha_{[\beta,\beta']}$-open set containing $f(A)$. Then, $f^{-1}(V)$ is an $\alpha_{[\gamma,\gamma']}$-open set containing A and so $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq f^{-1}(V)$. It follows that $f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A))$ is an $\alpha_{[\beta,\beta']}$-closed set and hence $\alpha_{[\beta,\beta']}$-$Cl(f(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A))) = f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A)) \subseteq V$. This implies that $f(A)$ is $\alpha_{[\beta,\beta']}$-g.closed.

(2) Let U be any $\alpha_{[\gamma,\gamma']}$-open set such that $f^{-1}(B) \subseteq U$. Let $F = \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(B)) \cap (X \setminus U)$, then F is $\alpha_{[\gamma,\gamma']}$-closed in (X,τ). This implies $f(F)$ is $\alpha_{[\beta,\beta']}$-closed set in (Y,σ). Since $f(F) = f(\alpha_{[\gamma,\gamma']}\text{-}Cl((f^{-1}(B)) \cap (X \setminus U))) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(B) \cap f(X \setminus U) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(B) \cap (Y \setminus B)$. Therefore, $\alpha_{[\beta,\beta']}\text{-}Cl(B) \setminus B$ contains an $\alpha_{[\beta,\beta']}$-closed set $f(F)$. It follows from Proposition 3.15 that $f(F) = \phi$ and hence $F = \phi$. Therefore $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(B)) \subseteq U$. This shows that $f^{-1}(B)$ is $\alpha_{[\gamma,\gamma']}$-g.closed.

□

Theorem 4.40. *Suppose that $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous and $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed, then*

1. *If f is injective and (Y,σ) is $\alpha_{[\beta,\beta']}$-$T_{\frac{1}{2}}$, then (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$.*
2. *If f is surjective and (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$, then (Y,σ) is $\alpha_{[\beta,\beta']}$-$T_{\frac{1}{2}}$.*

Proof. (1) Let A be an $\alpha_{[\gamma,\gamma']}$-g.closed set of (X,τ). Now to prove that A is $\alpha_{[\gamma,\gamma']}$-closed. By Propostion 4.39 (1), $f(A)$ is $\alpha_{[\beta,\beta']}$-g.closed. Since (Y,σ) is $\alpha_{[\beta,\beta']}$-$T_{\frac{1}{2}}$, this implies that $f(A)$ is $\alpha_{[\beta,\beta']}$-closed. Since f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous and injective, then, we have $A = f^{-1}(f(A))$ is $\alpha_{[\gamma,\gamma']}$-closed. Hence (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$.

(2) Let B be an $\alpha_{[\beta,\beta']}$-g.closed set in (Y,σ). By Propostion 4.39 (2), $f^{-1}(B)$ is $\alpha_{[\gamma,\gamma']}$-g.closed, since (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$ space, this implies that $f^{-1}(B)$ is $\alpha_{[\gamma,\gamma']}$-closed. Since f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed and surjective, then we have $B = f(f^{-1}(B))$ is $\alpha_{[\gamma,\gamma']}$-closed. Hence (Y,σ) is $\alpha_{[\beta,\beta']}$-$T_{\frac{1}{2}}$.

□

Theorem 4.41. *If $f : (X,\tau) \to (Y,\sigma)$ is an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous injection and if (Y,σ) is $\alpha_{[\beta,\beta']}$-T_i, then (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, 1, 2$.*

Proof. The proof for $i = 1$ is as follows: Let $x \in X$. Then, by Proposition 4.10, $\{f(x)\}$ is $\alpha_{[\beta,\beta']}$-closed in (Y,σ). By $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous and Proposition 4.10, $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed and hence (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_1$. The proofs for $i = 0, 2$ follow from Definitions 4.3, 4.5 and 2.6. □

Definition 4.42. A function $f : (X,\tau) \to (Y,\sigma)$ is called an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-homeomorphism if f is an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous bijection and $f^{-1} : (Y,\sigma) \to (X,\tau)$ is $(\alpha_{[\beta,\beta']}, \alpha_{[\gamma,\gamma']})$-continuous.

Theorem 4.43. *Suppose that $f : (X,\tau) \to (Y,\sigma)$ is an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-homeomorphism. Then, (X,τ) is $\alpha_{[\gamma,\gamma']}$-$T_i$ if and only if (Y,σ) is $\alpha_{[\beta,\beta']}$-T_i, where $i = 0, \frac{1}{2}, 1, 2$.*

Proof. This follows from Theorems 4.40, 4.41 and Definition 4.42. □

References

[1] Alias B. Khalaf, Saeid Jafari and Hariwan Z. Ibrahim, Bioperations on α-open sets in topological spaces, (Submitted).

[2] S. Kasahara, Operation-compact spaces, *Math. Japonica*, 24 (1979), 97-105.

[3] O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.* 15 (1965), 961-970.

[4] H. Ogata, Operation on topological spaces and associated topology, *Math. Japonica*, 36 (1) (1991), 175-184.

[5] H. Maki and T. Noiri, Bioperations and some separation axioms, *Scientiae Mathematicae Japonicae Online*, 4 (2001), 165-180.

[6] H. Z. Ibrahim, On a class of α_γ-open sets in a topological space, *Acta Scientiarum. Technology*, 35 (3) (2013), 539-545.

[7] J. Umehara, H. Maki and T. Noiri, Bioperations on topological spaces and some separation axioms, *Mem. Fac. Sci. Kochi Univ. Ser. A (Math.)* 13 (1992), 45-59.

[8] S. N. Maheshwari and S. S. Thakur, On α-irresolute mappings, *Tamkang J. Math.* 11 (1980), 209-214.

[9] H. Maki, R. Devi and K. Balachandran, Generalized α-closed sets in topology, *Bull. Fukuoka Univ. Ed. Part III*, 42 (1993), 13-21.

[10] M. Caldas, D. N. Georgiou and S. Jafari, Characterizations of low separation axioms via α-open sets and α-closure operator, *Bol. Soc. Paran. Mat.* 21 (2003), 1-14.

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