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A NOTE ON PROPERTIES OF HYPERMETRIC SPACES

MOHSEN ALIMOHAMMADY[1], SAEID JAFARI , SEITHUTI P. MOSHOKOA AND
MORTEZA KOOZEHGAR KALLEJI[1]*

Abstract. The note studies further properties and results of analysis in the setting of hypermetric spaces. Among others, we present some results concerning the hyper uniform limit of a sequence of continuous functions, the hypermetric identification theorem and the metrization problem for hypermetric space.

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1. Introduction

Metric spaces play an important role in the analysis as well as in the applications of mathematics in a variety of fields: fixed point theory and dynamical systems are just a few areas of such applications [5, 7, 9]. This note studies some fundamental properties of the class of hypermetric spaces, which properly contains the class of metric spaces. Let $\mathcal{P}(\mathbb{R})$ denote the family of all subsets of $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers equipped with usual metric and $\mathbb{R}_+ = [0, \infty)$. We recall [2, 8] that a hypermetric space is a pair (X, D), where X is a nonempty set and D is a set-valued map $D : X \times X \to \mathcal{P}(\mathbb{R}_+)$ such that for all x, y, z in X we have:

(a) $D(x, y) = \{0\}$ if and only if $x = y$.

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∗Address correspondence to ; E-mail: m.kalleji@yahoo.com

(b) $D(x, y) = D(y, x)$.

(c) $D(x, y) \subseteq D(x, z) + D(z, y)$, where for A, B in $\mathcal{P}(\mathbb{R}_+)$, we have $A + B = \{a + b : a \in A, b \in B\}$.

Examples of hypermetric spaces are provided in [2]. Furthermore, it is shown that every metric space (X, d) induces a hypermetric space (X, D_d). In particular, given a metric space (X, d) for the purposes of this paper we shall define $D_d(x, y) = \{0\}$ if $x = y$ and $D_d(x, y) = (-d(x, y), d(x, y))$ for $x \neq y$. Note that a topology $\tau(d)$ induced by the metric d on X is the same as the topology $\tau(D_d)$ on X induced by hypermetric D_d. In particular, let $X = \mathbb{R}$ with d being the usual metric on X. Then $\tau(D_d)$ is the usual topology on X. We now present the following example:

Let $X = \mathbb{R}$. Define $D(x, y) = \{0\}$, if $x = y$, and $D(x, y) = \{0, 1\}$ if $x \neq y$. Then (X, D) is the hypermetric space. In fact, the topology $\tau(D)$ on X is **discrete**. However, there is no metric d on X such that $\tau(D) = \tau(D_d)$.

Definition 1.1. [4](Kuratowski Convergence) Let (X, d) be a metric space and $\{A_n\}$ be a sequence of subsets of X. Then

(i) the **Upper Limit** or **Outer limit** of the sequence $\{A_n\}$ is a subset of X given by

$$\limsup_{n\to\infty} A_n = \left\{ x \in X \ ; \ \liminf_{n\to\infty} dist(x, A_n) = 0 \right\}$$

(ii) the **Lower limit** or **Inner limit** of the sequence $\{A_n\}$ is a subset of X given by

$$\liminf_{n\to\infty} A_n = \left\{ x \in X \ ; \ \limsup_{n\to\infty} dist(x, A_n) = 0 \right\}.$$

If $\limsup\limits_{n\to\infty} A_n = \liminf\limits_{n\to\infty} A_n$, then we say that the limit of $\{A_n\}_{n\in\mathbb{N}}$ exists and

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n.$$

In what follows the convergence of sequences will be important. We therefore recall some basic facts concerning sequences in hyper metric spaces. From Definition 1.1, a sequence $\{x_n\}$ in a hypermetric space (X, D) is said to converge to a point x in X if for any $\epsilon > 0$ there exists a natural number N such that for every $n \geq N$

$$D(x_n, x) \subset (-\epsilon, \epsilon),$$

then we shall write $\lim\limits_{n\to\infty} D(x_n, x) = \{0\}$. The sequence $\{x_n\}$ is said to be Cauchy if $\lim_{m,n} D(x_n, x_m) = \{0\}$. We shall say that a sequence $\{x_n\}$ has a cluster point x if there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ that converges to x. We easily observe that a sequence in a metric space (X, d) is Cauchy if and only if it is Cauchy in (X, D_d).

2. Main results

The following results shows that for any $x, y \in X$ the set $D(x, y)$ is nonempty set in $\mathcal{P}(\mathbb{R}_+)$.

Theorem 2.1. *Let (X, D) be a hypermetric space, then $\{0\} \subseteq D(x, y)$ for all $x, y \in X$.*

Proof. Let $x_0, y_0 \in X$. By definition of hypermetric spaces we have

$$\{0\} = D(x_0, y_0) \subseteq D(x_0, y_0) + D(y_0, x_0).$$

Since $D(x_0, y_0) \subseteq \mathbb{R}_+$, we infer that $0 \in D(x_0, y_0)$. $\square$

We recall that for a hypermetric space (X, D) and $x \in X$ then $N_r^h(x) = \{y \in X : supD(x, y) < r \wedge infD(x, y) > -r\}$ is called a hyper open ball with center x and radius r for each $r > 0$. Now let $\mathbb{Q}$ denote the set of rational numbers, then for each $x \in X$, the set $\{N^h_{r\in\mathbb{Q}}(x)\}$ is countable. From this and results in [2, 9] we deduce the following:

Theorem 2.2. *Every hypermetric space is first countable.*

Corollary 2.3. *Every metric space is first countable.*

Theorem 2.4. *Let (X, D) be a hypermetric space such that every Cauchy sequence has a hyper convergent subsequence. Then (X, D) is hyper complete.*

Proof. Let $\{x_n\}$ be a hyper Cauchy sequence in (X, D) and $\{x_{n(k)}\}$ be the subsequence of $\{x_n\}$. Let $\epsilon > 0$. We note that there exists a natural number N such that if $n, m \geq N$ then $D(x_m, x_n) \subset (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. Furthermore, there exists $x \in X$ such that $D(x_{n(k)}, x) \subset (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ for all $n(k) > N$. It follows that for $n \geq N$, we have $D(x_n, x) \subseteq D(x_n, x_{n(k)}) + D(x_{n(k)}, x_m)$. Hence, for $n > N$, we get $D(x_n, x) \subset (-\epsilon, \epsilon)$. $\square$

Theorem 2.5. *Let (X, D) be a complete hypermetric space. Let A be a subset of X. Then A is hyper complete if and only if it is closed.*

Definition 2.6. Let X be a set and let (Y, D) be a hypermetric space. Let $f : X \to Y$ be a function and for each $n \in \mathbb{N}$, $f_n : X \to Y$, be a function. Then the sequence $\{f_n\}$ hyper converges uniformly to f provided that for each $\epsilon > 0$, there is a natural number N such that if $n \geq N$, then $D(f_n(x), f(x)) \subset (-\epsilon, \epsilon)$ for all $x \in X$.

Theorem 2.7. *Let (X, τ) be a topological space and (Y, D) be a hypermetric space. Let $f : (X, \tau) \to (Y, D)$ be a function and for each $n \geq 1$ the function $f_n : (X, \tau) \to (Y, D)$ be continuous. If the sequence $\{f_n\}$ is hyper converging uniformly to f, then f is continuous.*

Proof. Let $x_0 \in X$ and let $V \in \tau(D)$ be such that V contains $f(x_0)$. Then there exists $\epsilon > 0$ such that $N^h_\epsilon(f(x_0)) \subseteq V$. Since $\{f_n\}$ hyper converges uniformly to f, there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $x \in X$, then $D(f_n(x), f(x_0)) \subset (-\frac{\epsilon}{4}, \frac{\epsilon}{4})$. Since f_N is continuous, there exists $U \in \tau$ containing x_0 such that $f_N(U) \subseteq N^h_{\frac{\epsilon}{2}}(f_N(x_0))$.

We shall show that $D(f(x), f(x_0)) \subset (-\epsilon, \epsilon)$ for $x \in U$. Now, let us consider $x \in U$, then $D(f(x), f_N(x)) \subset (-\frac{\epsilon}{4}, \frac{\epsilon}{4}), D(f_N(x), f_N(x_0)) \subset (-\frac{\epsilon}{4}, \frac{\epsilon}{4})$, and $D(f_N(x_0), f(x_0)) \subset (-\frac{\epsilon}{4}, \frac{\epsilon}{4})$. Hence, $D(f(x), f(x_0)) \subset (-\epsilon, \epsilon)$. This completes our proof. □

Since every metric space admits a compatible hypermetric space, the proof of the following is immediate.

Corollary 2.8. *Let (X, τ) be a topological space and (Y, d) be a metric space. Let $f : (X, \tau) \to (Y, d)$ be a function and for each $n \geq 1$ the function $f_n : (X, \tau) \to (Y, d)$ be continuous. If the sequence $\{f_n\}$ is converging uniformly to f, then f is continuous.*

Definition 2.9. Let X be a nonempty set. The set-valued function $D : X \times X \to \mathcal{P}(\mathbb{R}_+)$ such that for all x, y, z in X we have:

(a) $D(x, x) = \{0\}$.

(b) $D(x, y) = D(y, x)$.

(c) $D(x, y) \subseteq D(x, z) + D(y, z)$,

will be called a hyper pseudo metric. The pair (X, D) is called a hyper pseudo metric space.

For a hyper pseudo metric space, denote $R_D = \{(x, y) \in X \times X : D(x, y) = \{0\}\}$ and for a pseudo metric space (X, d), set $R_d = \{(x, y) : d(x, y) = 0\}$. Note that if (X, d) is a pseudo metric space then $R_{D_d} = R_d$.

Theorem 2.10. *Let (X, D) be a hyper pseudo metric space. Then R_D is an equivalence relation on X.*

Proof. That R_D is reflexive and symmetric is clear. We shall show that R_D is transitive. Suppose that $(x, y) \in R_D$ and $(y, z) \in R_D$. We need to show that $(x, z) \in R_D$. From $D(x, z) \subseteq D(x, y) + D(y, z)$, it follows from Theorem 2.1 that $\{0\} \subseteq D(x, z) \subset \{0\} + \{0\} = \{0\}$. Hence $(x, z) \in R_D$. □

We are now ready to show that every hyper pseudo metric space admits a hypermetric identification.

Theorem 2.11. *Let (X, D) be a hyper pseudo metric space. Then there exists a hypermetric space $(X/R_D, D^*)$.*

Proof. Let $D^* : X/R_D \times X/R_D \to \mathcal{P}(\mathbb{R}_+)$ be defined by $D^*(f(x), f(y)) = D(x, y)$, where $f : X \to X/R_D$ is a natural map. Then D^* is a hypermetric on X/R_D. □

The hypermetric space $(X/R_D, D^*)$ in Theorem 2.11 will be called the hypermetric identification associated with the hyper pseudo metric space (X, D). Observe that if the hyper pseudo metric space (X, D) is hyper complete, then $(X/R_D, D^*)$ is hyper complete.

Theorem 2.12. *Let $(X/R_D, D^*)$ be the metric identification of the hyper pseudo metric space (X, D). Then the topology for the quotient space X/R_D is the topology generated by the hypermetric D^*.*

We say that a topological space (X, τ) admits a compatible hypermetric if there is a hypermetric D on X such that $\tau = \tau(D)$. By definition of hypermetric space and basic results in [2] we see that every metrizable topological space admits a compatible hypermetric space. Conversely, we shall show that every hypermetric space is metrizable.

A classical result on the theory of metrizable topological spaces due to Kelly[8] is the following:

Lemma 2.13. *A T_1 topological space (X, τ) is metrizable if and only if it admits a compatible uniformity with a countable base.*

The following lemma can be deduced from [2, 8]:

Lemma 2.14. *Let (X, D) be a hypermetric space. Then $\tau(D)$ is a Hausdorff topology and for each $x \in X$, $\{N^h_{\frac{1}{n}}(x)\}$ is a local base at point of x for topological space $(X, \tau(D))$.*

We recall [6] that a uniform space is a set with uniform structure. In general, There are three equivalent definitions for a uniform space.

They all consist of a space equipped with a uniform structure. Uniform spaces were introduced in 1938 by A. Weil [10].

Definition 2.15. A nonempty collection $\mathcal{F}$ of subsets $U \subseteq X \times X$ is a uniform structure if it satisfies the following axioms:

1) If $U \in \mathcal{F}$ then $\Delta \subseteq U$ where, $\Delta = \{(x, x)\ ;\ x \in X\}$ is the diagonal on X.

2) If $U \in \mathcal{F}$ and $U \subseteq V$ for $V \subseteq X \times X$ then $V \in \mathcal{F}$.

3) if $U \in \mathcal{F}$ and $V \in \mathcal{F}$ then $U \cap V \in \mathcal{F}$.

4) if $U \in \mathcal{F}$ the there is $V \in \mathcal{F}$, $VoV \subset U$ where VoV denotes the composite of V with itself. The composite of two subset V and U of $X \times X$ is defined by

$$VoU := \Bigg\{(x, z) \in X \times X\ ;\ \exists y \in X\ ;\ (x, y) \in U \wedge (y, z) \in \Bigg\}.$$

5) If $U \in \mathcal{F}$ then $U^{-1} \in \mathcal{F}$ where, $U^{-1} := \Bigg\{(y, x) \in X \times X\ ;\ (x, y) \in U\Bigg\}$ is the inverse of relation $U \subset X \times X$.

Every uniform space X becomes a topological space by defining a subset O of X to be open if and only if for every x in O there exists an element V such that V_x is a subset of O. In this topology the neighborhood of point x is $\{V_x\ :\ V \in \mathcal{F}\}$. For simplicity, we call uniformity a set X with such a uniform structure.

Theorem 2.16. *Let (X, D) be hypermetric space. Then for $n \geq 1$ and $x \in X$, the set $\{N^h_{\frac{1}{n}}(x)\}$ is a countable base for a uniformity $\mathcal{U}$, where*

$$N^h_{\frac{1}{n}}(x) = \{y \in X\ :\ supD(x, y) < \frac{1}{n} \wedge infD(x, y) > -\frac{1}{n}\}.$$

Proof. For each $n \in \mathbb{N}$, define

$$U_n = \{(x, y) \in X \times X : D(x, y) \subset (-\frac{1}{n}, \frac{1}{n})\}.$$

Clearly, the set $\{U_n : n \in \mathbb{N}\}$ is countable. Next, we shall show that $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}$ on X. First note that for each $n \in \mathbb{N}, \{(x, x) : x \in X\} \subseteq U_n$, $U_{n+1} \subseteq U_n$ and $U_n = U_n^{-1}$. Also, for each $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $U_m \circ U_m \subseteq U_n$. Without loss of generality, let $m > 2n$. Suppose that $(x, y) \in U_m \circ U_m$, then there is $z \in X$ such that $(x, z) \in U_m$ and $(z, y) \in U_m$. It follows that

$D(x,y) \subseteq (-\frac{2}{m}, \frac{2}{m}) \subset (-\frac{1}{n}, \frac{1}{n})$. Hence, $(x,y) \in U_n$ Thus $U_m \circ U_m \subseteq U_n$. Thus $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}$ on X. □

Theorem 2.17. *Let (X, D) be a hypermetric space, then (X, D) is metrizable.*

Proof. For each $x \in X$ and each $n \in \mathbb{N}$, we have $U_n(x) = \{y \in X : D(x,y) \subset (-\frac{1}{n}, \frac{1}{n})\} = N^h_{\frac{1}{n}}(x)$. By Lemma 2.14 we deduce that the topology induced by the uniformity $\mathcal{U}$ is the same as the topology induced by the hypermetric D on X. Therefore by Lemma 2.13 $(X, \tau(D))$ is a metrizable topological space □

Corollary 2.18. *A topological space is metrizable if and only if it admits a compatible hypermetric.*

Proof. First suppose that (X, τ) is a metrizable topological space. Then by definition of metrizable space $\tau = \tau(d)$. Using Corollary 1.2 in [2], X admits a compatible hypermetric D_d. Now, suppose that X admits a compatible hypermetric, then by Theorem 2.17, Lemma 2.14 and Lemma 2.13, the topological space X is a metrizable. □

Let us recall that a metrizable topological space (X, τ) is said to be completely metrizable if it admits a complete metric. On the other hand, a hypermetric space is said to be complete if every Cauchy sequence in X hyper converges to a point in X with respect to D. If (X, D) is hyper complete, we say that D is a hyper complete hypermetric on X.

Theorem 2.19. *Let (X, D) be a complete hypermetric space. Then $(X, \tau(D))$ is completely metrizable.*

Proof. From the proof of Theorem 2.16 and Corollary 2.18, the set $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}$ on X compatible with the topology $\tau(D)$, where

$$U_n = \{(x,y) \in X \times X : D(x,y) \subset (-\frac{1}{n}, \frac{1}{n})\}$$

for every $n \in \mathbb{N}$. Then there exists a metric d on X whose induced uniformity coincides with $\mathcal{U}$. We want to show that D is hyper complete on X. Let $\{x_n\}$ be a Cauchy sequence in (X, d). It easily follows that $\{x_n\}$ is hyper Cauchy in (X, D_d), hence it is hyper convergent. Thus that the Cauchy sequence $\{x_n\}$ is convergent with respect to $\tau(D_d)$. Hence d is complete. We conclude that $(X, \tau(D))$ is completely metrizable. □

Corollary 2.20. *A topological space is completely metrizable if and only if it admits a compatible complete hypermetric space.*

Proof. Suppose that the topological space (X, τ) is completely metrizable. Let d be a complete metric on X compatible with τ. Clearly, the hypermetric D_d induced by d is hyper complete, and is compatible with τ. Conversely, if (X, D) is a complete hypermetric space then by Theorem 2.19, the $(X, \tau(D))$ is completely metrizable. $\square$

Theorem 2.21. *Let (X, D) be a separable hypermetric space. Then (X, D) is second countable.*

Proof. Let (X, D) be a separable hypermetric space. By Theorem 2.16 $(X, \tau(D))$ is a separable metrizable space. So, it is second countable [3]. $\square$

Since every completely metrizable space is Baire [1], we deduce from Theorem 2.19 that the following holds:

Corollary 2.22. *Every complete hypermetric space is a Baire space.*

Definition 2.23. A hypermetric space (X, D) is said to be pre-compact if for each $r > 0$, there is a finite subset A of X, such that

$$X = \cup_{a \in A} N_r^h(a).$$

In this case we say that D is a pre-compact hypermetric on X.

A hypermetric space (X, D) is said to be compact if (X, τ_D) is a compact topological space.

Lemma 2.24. *A hypermetric space (X, D) is pre-compact if and only if every sequence has a Cauchy subsequence.*

Proof. Let the hypermetric space (X, D) be pre-compact and $\{x_n\}$ be a sequence in X. We shall construct a subsequence $\{x_{n(n)}\}$ of $\{x_n\}$ and show that $\{x_{n(n)}\}$ is a Cauchy sequence in (X, D). By pre-compactness of (X, D) there is a finite subset A_m of X such that $X = \cup_{a \in A_m} N_{\frac{1}{m}}^h(a)$ for each $m \in \mathbb{N}$. Now for $m = 1$, there exists an A_1 and a subsequence $\{x_{1(n)}\}$ of $\{x_n\}$ such that

$$x_{1(n)} \in N_1^h(a_1)$$

for every $n \in \mathbb{N}$.

Similarly, there exists an $a_2 \in A_2$ and a subsequence $\{x_{1(n)}\}$ such that

$$x_{2(n)} \in N_{\frac{1}{2}}^h(a_2)$$

for every $n \in \mathbb{N}$. Continuing with this process, for $m > 1$, there

$$x_{m(n)} \in N^h_{\frac{1}{m}}(a_m)$$

for every $n \in \mathbb{N}$. Now consider the subsequence $\{x_{n(n)}\}$ of $\{x_n\}$. Given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{2}{n_0} < \epsilon$ and for all $k, m \geq n_0$, we have

$$D(x_{k(k)}, x_{m(m)}) \subseteq D(x_{k(k)}, a_{n_0}) + D(a_{n_0}, x_{m(m)}).$$

Similar to proof of Theorem 2.16 we can consider

$$U_{n_0} = \left\{(a_{n_0}, y) \in X \times X\ ;\ D(a_{n_0}, y) \subset (-\frac{1}{n_0}, \frac{1}{n_0})\right\}$$

as a countable base for uniformity of $\{N^h_{\frac{1}{n_0}}(a_{n_0})\}$. Then

$$D(x_{k(k)}, a_{n_0}) + D(x_{m(m)}, a_{n_0}) \subset (-\frac{1}{n_0}, \frac{1}{n_0}) + (-\frac{1}{n_0}, \frac{1}{n_0}) \subset (-\frac{2}{n_0}, \frac{2}{n_0}).$$

Hence $D(x_{k(k)}, x_{m(m)}) \subseteq (-\frac{2}{n_0}, \frac{2}{n_0}) \subset (-\epsilon, \epsilon)$ for all $k, m \geq n_0$. This shows that every subsequence of a sequence in (X, D) is a Cauchy sequence. Conversely, suppose that the hypermetric space (X, D) is not pre-compact. Then there exists $r > 0$ such that for each finite subset A of X , we have $X \neq \cup_{a \in A} N^h_r(a)$. Fix $x_1 \in X$. There is $x_2 \in X$ which does not belong to $N^h_r(x_1)$. Similarly, we can find $x_3 \in X - \cup_{k=1}^2 N^h_r(x_k)$. We continue with this process to construct a sequence of distinct points in X, such that $x_{n+1} \in X - \cup_{k=1}^n N^h_r(x_k)$ for all $n \in \mathbb{N}$. Clearly, the sequence $\{x_n\}$ does not have a Cauchy subsequence. If we suppose that x_{n_i} is a Cauchy subsequence in (X, D) then for any $\epsilon > 0$ there exists a n_0 such that for all $m, j > n_0$, $x_m, x_j \in N^h_{\frac{1}{n_0}}(x_{n_0})$. Now consider $\epsilon = r$ and finite subset $A = \{x_1, ..., x_{n_0+1}, x_{n_0+2}\}$. Since X is not pre-compact so $x_{n_0+1} \notin N^h_r(x_{n_0})$ while $x_{n_0+1} \in N^h_r(x_{n_0})$ from definition of Cauchy sequence but this makes a contradiction. Hence $\{x_n\}$ does not have any Cauchy subsequence. This is complete our proof. $\square$

Theorem 2.25. *A hypermetric space (X, D) is separable if and only if (X, τ_D) admits a compatible pre-compact hypermetric.*

Proof. Let the hypermetric space (X, D) be separable. Since $(X, \tau(D))$ is metrizable, so we see that $(X, \tau(D))$ is separable metrizable topological space and so it admits a compatible pre-compact metric space (X, d). It remains to show that the hypermetric space (X, D_d) induced by the metric d is pre-compact. Since (X, D) is separable thus by Theorem

2.21 (X,D) is second countable space. Moreover, from Theorem 2.19 and Corollary 2.20 (X,D) is compatible complete hypermetric space. Hence, every sequence has a Cauchy subsequence. Thus (X,D) is a pre-compact space. Indeed, let $\{x_n\}$ be a sequence in (X,D). Since (X,D) is pre-compact then the sequence $\{x_n\}$ has a Cauchy subsequence $\{x_{k(n)}\}$ with respect to d. That is $lim_{k\to\infty} d(x_{k(n)}, x_{k(m)}) = 0$. Therefore by definition of hypermeter D_d which is defined as $D_d(x,y) = \{0\}$ if $x = y$ and $D_d(x,y) = (-d(x,y), d(x,y))$ if $x \neq y$, it follows that $lim_k D_d(x_{k(n)}, x_{k(m)}) = \{0\}$. Hence $\{x_{k(n)}\}$ is Cauchy with respect to D_d. By Lemma 2.24 we conclude that the hypermetric space (X,D_d) is pre-compact.
Conversely, suppose that $(X,\tau(D))$ admits a compatible pre-compact hypermetric L. Then, for each $n\in\mathbb{N}$ there is a finite subset A_n of X such that $X = \cup_{a\in A_n} N^h_{\frac{1}{n}}(a)$, where $N^h_{\frac{1}{n}}(a) = \{y \in X : supL(x,y) < \frac{1}{n} \wedge infL(x,y) > -\frac{1}{n}\}$. Put $A = \cup_{n=1}^{\infty} A_n$. Then A is countable. We shall show that A is dense in X. Indeed let $x\in X$ and $N^h_{\frac{1}{m}}(x)$ be a basic neighborhood of x. Then there exists $a \in A_m$ such that $x \in N^h_{\frac{1}{m}}(a)$. Thus, A is dense in X. We conclude that hypermetric space (X,L) is separable. That is, $(X,\tau(D))$ is a separable topological space. □

We observe that if $\{x_n\}$ is Cauchy sequence in (X,D) and the subsequence $\{x_{k(n)}\}$ of $\{x_n\}$ converges to x in X then it follows that $limD(x_n,x) = \{0\}$.

Lemma 2.26. *Let (X,D) be a hypermetric space. If a Cauchy sequence clusters to a point x in X, then the sequence converges to x.*

Let us recall that a metric space (X,d) is said to be sequentially compact if every sequence has a convergent subsequence. In a similar way we shall say that a hypermetric space (X,D) is sequentially compact if every sequence has convergent subsequence.

Theorem 2.27. *A hypermetric space (X,D) is compact if and only if it is pre-compact and complete.*

Proof. Suppose the hypermetric space (X,D) is compact. Then $\{N^h_r(x) : x\in X\}$ is an open cover for X hence $\{N^h_r(x) : x \in X\}$ has a finite subcover. This shows that (X,D) is pre-compact. Now we shall show that (X,D) is complete. Let $\{x_n\}$ be a Cauchy sequence by compactness there exists a cluster point y for $\{x_n\}$ and by Lemma 2.26, we have $lim_n D(x_n,x) = \{0\}$. Therefore (X,D) is complete.

Conversely, suppose that (X, D) is pre-compact and complete. We shall show that $(X, \tau(D))$ is compact. Let $\{x_n\}$ be a sequence in (X, D) and $\{x_{k(n)}\}$ be subsequence of $\{x_n\}$. By pre-compactness (X, D) the sequence $\{x_{k(n)}\}$ is Cauchy and by completeness of (X, D) there exists $y \in X$ such that $lim_k D(x_{k(n)}, y) = \{0\}$. Thus (X, D) is sequentially compact. Now $(X, \tau(D))$ is metrizable by Theorem 2.17 and every sequentially compact metrizable space is compact we conclude that the hypermetric space (X, D) is compact. □

Theorem 2.28. *A metrizable topological space is compact if and only if every compatible hypermetric is pre-compact.*

Proof. Suppose that the topological space (X, τ) is metrizable and compact and let (X, D) be a compatible hypermetric space. Then (X, D) is pre-compact by Theorem 2.27.

Conversely, let the metrizable topological space (X, τ) be pre-compact and (X, D) be a compatible hypermetric on X. We shall show that (X, τ) is compact. Clearly, we have $\tau = \tau(d) = \tau(D)$, where d is a metric on X. Now consider the hypermetric D_d on X induced by d.Then (X, d_d) is pre-compact. Let $\{x_n\}$ be a sequence in (X, D_d). Then by pre-compactness $\{x_n\}$ has a Cauchy sequence subsequence $\{x_{k(n)}\}$ in (X, D_d). Hence, $\{x_{k(n)}\}$ is a Cauchy sequence in (X, d). Therefore, every metric d on X which is compatible with τ is pre-compact. We conclude, therefore that (X, τ) is compact. □

Theorem 2.29. *A metrizable topological space is compact if every compatible hypermetric is complete.*

Proof. Suppose that (X, τ) is a compact metrizable space. By Theorem 2.27, every compatible hypermetric is complete.

Conversely,let d be any metric on X compatible with τ. Consider the hyper metric D_d induced by d. By hypothesis (X, D_d) is complete so (X, d) is also complete. Hence by Niemytzki-Tychonoff theorem we deduce that (X, τ) is compact. □

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M. Alimohammady
Department of Mathematics, University of Mazandaran, P.O.Box 47416-1468, Babolsar, Iran
Email: amohsen@umz.ac.ir

S. Jafari
Department of Statistics, College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark
Email: jafaripersia@gmail.com

S. Moshokoa
Department of Statistics, University of South Africa, P. O . Box 392 Pretoria, South Africa
Email: moshosp@gmail.com

M. Koozehgar Kalleji
Department of Mathematics, University of Mazandaran, P.O.Box 47416-1468, Babolsar, Iran
Email: m.kalleji@yahoo.com