# $^{**}g\alpha$ - CLOSED AND $^{**}g\alpha$ - OPEN SETS IN THE DIGITAL PLANE

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## ABSTRACT

Digital topology was first studied in the late 1970's by the computer analysis researcher Azriel Rosenfeld [15]. In this paper we derive some of the properties of $^{**}g\alpha$-open and $^{**}g\alpha$-closed sets in the digital plane. Moreover, we show that the Khalimsky line $(Z^2, K^2)$ is not an ${}_{\alpha}T_{1/2}{}^{***}$ space. Also we prove that the family of all $^{**}g\alpha$-open sets of $(Z^2, K^2)$, say $^{**}G\alpha O(Z^2, K^2)$, forms an alternative topology of $Z^2$ and the topological space $(Z^2, {}^{**}G\alpha O(Z^2, K^2))$ is a $T_{1/2}$ space. Moreover, we derive the properties of $*g\alpha$-closed and $*g\alpha$-open sets in the digital plane via the singleton's points.

**Keywords.** $^{**}g\alpha$-closed sets, $^{**}g\alpha$-open sets, digital plane

**AMS Subject Classification.** 54A05, 68U05, 68U10

## 1. INTRODUCTION

Digital topology was first studied by the computer image analysis researcher Azriel Rosenfeld [15]. The digital line, the digital plane and the three-dimensional digital spaces are of great importance in the study of applications of point set topology to computer graphics. Digital topology consists in providing algorithmic tools for Pattern Recognition, Image Analysis and Image processing using a discrete formalism for geometrical objects. It is applied in image processing.

The problems that might arise are finding connected components, set boundaries or any other operations which are needed in image processing. The well-known digital Jordan curve theorem is proved by using topological approach [8]. This theorem is an important result in the theory of computer graphics.

K. D. Khalimsky [8] and T. Y. Kong et. al [9] introduced the concepts of digital topology. K. Nono et al. [14] studied the explicit forms of GO-kernel and Kernel, also obtained some of the basic properties of $g^{\#}\alpha$-closed sets in the digital plane. S. I. Nada [12] studied the properties of semi-open and semi-closed sets and also studied the separation axioms, namely semi-$T_{1/2}$ semi-$T_1$ and semi-$T_2$ spaces in the Khalimsky n-space. Recently, Chatyrko et al. [1] investigated basic properties of the separation axiom semi-T½ not only in digital topology but also in domain theory.

R. Devi et al. [4] proved that the digital plane $(Z^2, K^2)$ is a $T_\xi$ - space and investigated characterizations of $\xi$-closed sets and $\xi^{**}$ - closed sets in the digital plane. R. Devi et al. [2] investigated the topological properties of wg$\rho$-closed sets in the digital plane where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$. R. Devi et al. [3] studied the concepts of $g^{\#}\alpha$-closed and $g^{\#}\alpha$-open sets in the digital plane $(Z^2, K^2)$. Also they proved that the family of all $g^{\#}\alpha$-open sets of $(Z^2, K^2)$, say $G^{\#}\alpha O$, forms an alternative topology of $Z^2$. Moreover they proved that the digital plane $(Z^2, K^2)$ is neither $T_{1/2}$ nor $T_{3/4}$ space.

R. Devi and M. Vigneshwaran [5] obtained some of the properties of $^*g\alpha$-closed and $^*g\alpha$-open sets in the digital plane. Moreover, they investigated the explicit forms of $^*G\alpha O$-kernel in the digital plane and had shown that the digital plane is an ${}_\alpha T_{1/2}{}^{**}$ space. R. Devi and M. Vigneshwaran [6] established the relationship between closed and $^*g\alpha$-closed set and open and $^*g\alpha$-open set in the digital plane. Also they derived some of the properties $^*g\alpha$-closed and $^*g\alpha$-open sets in the digital plane.

M. Vigneshwaran and R. Devi [16] studied the properties of $^*g\alpha$-closed sets in the digital plane and also characterized the definite forms of GαO-kernel in the digital plane.

Digital pictures contain pixel or collection of pixels in general. As far as this digital plane is concern discrete points are called pixels.

## METHDOLOGY

The digital line or the so-called Khalimsky line is the set of the integers Z, equipped with the topology k having $\{\{2n+1,2n,2n-1\}/n\in Z\}$ as a subbase. This is denoted by $(Z, K)$. Thus, a subset U is open in $(Z, K)$ if and only if whenever $x\in U$ is an even integer, then $x-1,x+1\in U$. Let $(Z^2, K^2)$ be the topological product of two digital lines $(Z, K)$, where $Z^2 = Z\times Z$ and $K^2 = K\times K$. This space is called the digital plane in the present paper (cf.[3],[4],[5],[6],[8],[9]). We note that for each point $x\in Z^2$ there exists the smallest open set containing x, say U(x). For the case of $x = (2n+1,2m+1)$, $U(x)=\{2n+1\}\times \{2m+1\}$; for the case of $x = (2n,2m)$, $U(x) = \{2n-1,2n,2n+1\}\times \{2m-1,2m,2m+1\}$; for the case of $x = (2n,2m+1)$, $U(x) = \{2n-1,2n,2n+1\}\times \{2m+1\}$; for the case of $x = (2n+1,2m)$, $U(x) = \{2n+1\}\times \{2m-1,2m,2m+1\}$, where $n,m\in Z$. For a subset E of $(Z^2, K^2)$, we define the following three subsets as follows: $E_F = \{x\in E/\ x \text{ is closed in } (Z^2, K^2)\}$; $E_{K^2} = \{x\in E/\ x \text{ is open in } (Z^2, K^2)\}$; $E_{mix} = E\setminus( E_F \cup E_{k2})$. Then it is shown that $E_F = \{(2n, 2m)\in E/\ n,m \in Z\}$, $E_{K^2} = \{(2n+1, 2m+1)\in E/\ n,m\in Z\}$ and $E_{mix}=\{(2n, 2m+1) \in E/\ n,m \in Z\}\cup \{(2n+1, 2m) \in E/\ n,m \in Z\}$. In the digital plane if the corner points of a digital picture are even then it is called closed set. If the corner points of a digital picture are odd then it is called open set in the digital plane.

## 2. PRELIMINARIES

**Definition 2.1**

A subset A of a space (X, τ) is called an α-open set [13] if A $\subseteq$ int(cl(int((A))) and an α-closed set if cl(int(cl(A))) $\subseteq$ A.

**Definition 2.2**

A subset A of a space (X, τ) is called

1. a generalized α-closed (briefly gα-closed) set [11] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is α-open in (X, τ),
2. a gα-open set [11] if U $\subseteq$ αint(A) whenever U $\subseteq$ A and U is α-closed in (X, τ),
3. a $^{*}$gα-closed set [16] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is gα-open in (X, τ),
4. a $^{*}$gα-open set [16] if U $\subseteq$ int(A) whenever U $\subseteq$ A and U is gα-closed in (X, τ),
5. a $^{**}$gα-closed set [17] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $^{*}$gα-open in (X, τ),
6. a $^{**}$gα-open set [17] if U $\subseteq$ int(A) whenever U $\subseteq$ A and U is $^{*}$gα-closed in (X, τ).

**Definition 2.3**

A subset A of a space (X, τ) is called a ${}_{\alpha}T_{1/2}^{***}$ space [17] if every $^{**}$gα-closed set is closed.

## 3. $^{**}$gα-CLOSED SETS AND $^{**}$gα-OPEN SETS IN THE DIGITAL PLANE

**Lemma 3.1[6]**

Let ($Z^2$, $K^2$ ) be a digital plane. Then the following properties holds:

(i) If m is an even point, that is m = (2n,2m), then cl({2n,2m}) = {2n,2m}.

(ii) If m is an odd point, that is m = (2n+1,2m+1),
then cl({2n+1,2m+1}) = {2n,2n+1,2n+2}×{2m,2m+1,2m+2}

(iii) If m is a mixed point, that is
either m = (2n,2m+1), then cl({2n,2m+1}) = {2n}× {2m,2m+1,2m+2}
or m = (2n+1,2m), then cl({2n+1,2m}) = {2n,2n+1,2n+2}× {2m}.

**Theorem 3.2**

If a set A is closed in **(**$Z^2$, $K^2$), then it is a $^{**}g\alpha$-closed set in ($Z^2$, $K^2$).

**Proof**

**Case(i):** A contains all even points ($E_F \subseteq A$)

Let A = (2n,2m) and U = {2n,2n $\pm$ 1}$\times$ {2m,2m $\pm$ 1} for any m, n.

Assume that A $\subseteq$ U and U is $^{*}g\alpha$-open in ($Z^2$, $k^2$). Then cl(A) = A $\subseteq$ U, since A is closed in ($Z^2$, $k^2$). We have cl(A) $\subseteq$ U. Therefore, A is $^{**}g\alpha$-closed in ($Z^2$, $k^2$).

**Case(ii):** A contains all even, odd and mixed points ($E_{mix} \cup E_F \cup E_{k2} \subseteq A$)

Let A = {2n, 2n $\pm$ 1, 2n $\pm$ 2,…,2n $\pm$ q}$\times${2m, 2m $\pm$ 1, 2m $\pm$ 2,…,2m $\pm$ q}for any n,m and q is an even integer.

Let U = {2n, 2n $\pm$ 1, 2n $\pm$ 2,…,2n $\pm$ q, 2n $\pm$ q $\pm$ 1}$\times${2m, 2m $\pm$ 1, 2m $\pm$ 2,…,2m $\pm$ q, 2m $\pm$ q $\pm$ 1}for any n,m and q is an even integer.

The proof is analogous to case (i).

**Case(iii):** A contains all even and mixed points ($E_F \cup E_{mix} \subseteq A$)

Let A = {2n }$\times${2m, 2m $\pm$ 1, 2m $\pm$ 2,…,2m $\pm$ q}for any n,m and q is an even integer.

Let U = {2n, 2n $\pm$ 1, 2n $\pm$ 2,…,2n $\pm$ q, 2n $\pm$ q $\pm$ 1}$\times${2m, 2m $\pm$ 1, 2m $\pm$ 2,…,2m $\pm$ q, 2m $\pm$ q $\pm$ 1}for any n,m and q is an even integer.

The proof is analogous to case (i).

Similarly we prove for A = {2n, 2n $\pm$ 1, 2n $\pm$ 2,…,2n $\pm$ q}$\times${2m} for any n,m and q is an even integer.

**Remark 3.3**

The converse of Theorem 3.2 need not be true. It can be seen by the following example.

**Example 3.4**

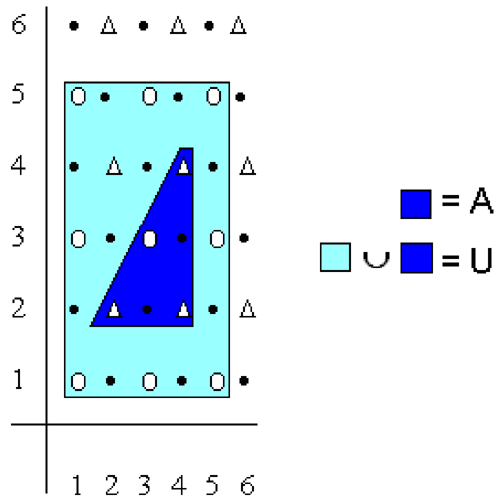
Let A = {(2,2),(3,2),(3,3),(4,2),(4,3),(4,4)} and let U = {1,2,3,4,5}$\times${1,2,3,4,5} be a $^{*}g\alpha$-open in ($Z^2$, $K^2$).

Then cl(A) = {2,3,4}$\times${2,3,4}. Hence cl(A) $\subseteq$ U. Therefore, A is a $^{**}g\alpha$-closed set in ($Z^2$, $K^2$).

But cl(A) $\neq$ A, therefore A is not a closed set in ($Z^2$, $K^2$).

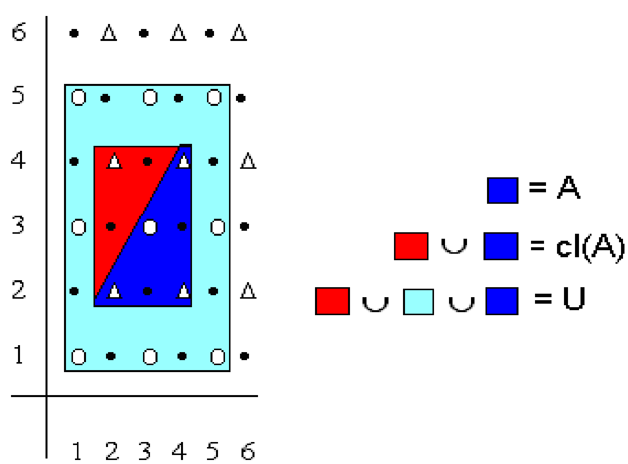
Example 3.4 can be explained by the following figure:

**Figure 1:**




**A ⊆ U**

**Figure 2:**




**cl(A) ⊆ U & cl(A) ≠ A**

**Theorem 3.5**

If a set A is open in $(Z^2, K^2)$, then it is a $^{**}g\alpha$-open set in $(Z^2, K^2)$.

**Proof**

**Case(i):** Consider the case that A contains all odd points ($E_{K^2} \subseteq A$)

Let A = (2n+1,2m+1) and U = $\phi$ for any m, n.

Assume that U $\subseteq$ A and A is $^{*}g\alpha$-closed in $(Z^2, K^2)$. Then U$\subseteq$ A = int(A) and since A is open in $(Z^2, K^2)$, we have U $\subseteq$ int(A). Therefore, A is $^{**}g\alpha$-open in $(Z^2, K^2)$.

**Case(ii):** Consider the case that A contains all even, odd and mixed points ($E_{mix} \cup E_F \cup E_{K^2} \subseteq A$)

Let A = $\{2n, 2n\pm 1, 2n\pm 2,\ldots,2n\pm q\}\times\{2m, 2m\pm 1, 2m\pm 2,\ldots,2m\pm q\}$for any n, m and q is an odd integer.

Let U = $\{2n, 2n\pm 1, 2n\pm 2,\ldots,2n\pm q, 2n\pm (q-1)\}\times\{2m, 2m\pm 1, 2m\pm 2,\ldots,2m\pm q, 2m\pm (q-1)\}$for any n, m and q is an odd integer.

The proof is analogous to case (i).

**Case(iii):** Consider the case that A contains all odd and mixed points ($E_{k2}\cup E_{mix} \subseteq A$)

Let A = $\{2n+1\}\times\{2m, 2m\pm 1, 2m\pm 2,\ldots,2m\pm q\}$for any n,m and q is an odd integer and let U = $\phi$.

The proof is analogous to case (i).

Similarly we prove for A = $\{2n, 2n\pm 1, 2n\pm 2,\ldots,2n\pm q\}\times\{2m+1\}$ for any n,m and q is an odd integer.

**Remark 3.6**

The converse of the above theorem need not be true. It can be seen by the following example.

**Example 3.7**

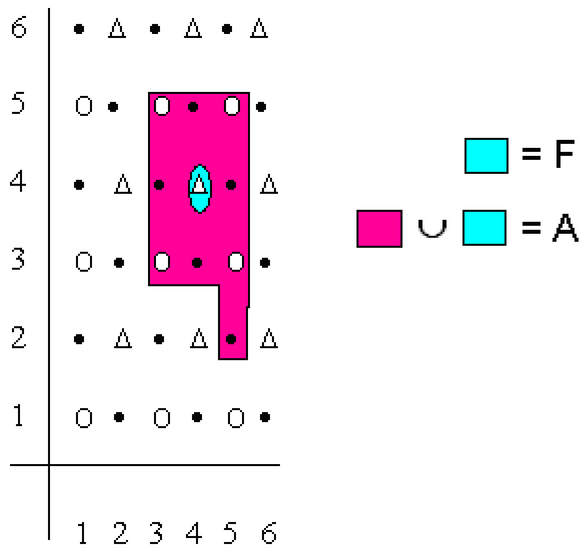
Let A = ({3,4,5}×{3,4,5},(5,2)) and let U = (4,4) be a $^{*}g\alpha$-closed in $(Z^2, K^2)$.

Assume that U $\subseteq$ A and U is $^{*}g\alpha$-closed set. Then U$\subseteq$ int(A), since int(A) = {3,4,5}×{3,4,5}. Therefore, A is a $^{**}g\alpha$-open set in $(Z^2, K^2)$.

But int(A) $\neq$ A, therefore A is not a open in $(Z^2, K^2)$.

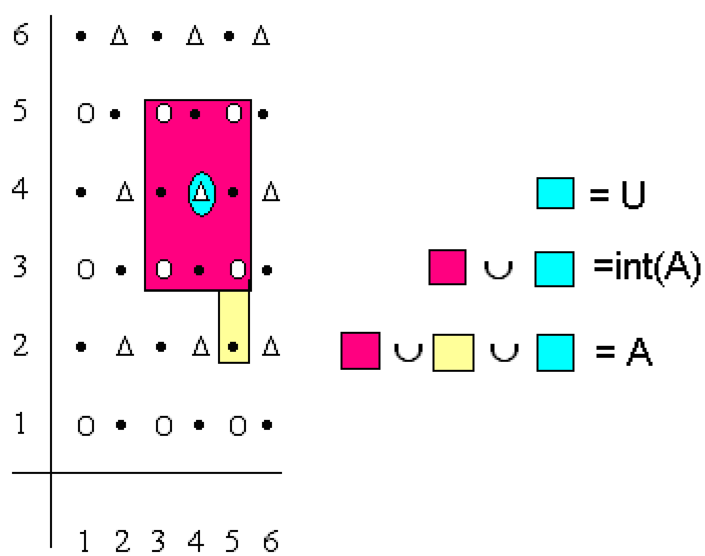
The above example can be explained by Figure 3:

**Figure 3:**




**U ⊆ A**

**Figure 4:**




**U ⊆ int(A) & int(A) ≠ A**

**Remark 3.8**

(i) If a set A contains all odd points then it is not a $^{**}g\alpha$-closed set in $(Z^2, K^2)$. The negation of the above statement is proved by Theorem 3.9.

(ii) If a sets A contains all even points then it is not $^{**}g\alpha$-open set in $(Z^2, K^2)$. The negation of the above statement is proved by Theorem 3.10.

**Theorem 3.9**

Let A be a $^{**}g\alpha$-closed set in $(Z^2, K^2)$, then it does not contain all odd points alone ($E_{k2} \not\subset A$).

**Proof**

Let A be a set in $(Z^2, K^2)$ containing all odd points. Suppose A = (2n+1,2m+1) for any n,m and U = (2n+1,2m+1) be a $^{*}g\alpha$-open set in $(Z^2, K^2)$.

Assume that $A \subseteq U$, then $cl(A) \not\subset U$, sine $cl(A) = \{2n,2n+1,2n+2\} \times \{2m,2m+1,2m+2\}$.This is a contradiction to our assumption. Therefore $^{**}g\alpha$-closed set in $(Z^2, K^2)$ does not contain all odd points alone.

**Theorem 3.10**

Let A be a $^{**}g\alpha$-open set in $(Z^2, K^2)$, then it does not contain all even points alone. ($E_F \not\subset A$).

**Proof**

Let A be a set in $(Z^2, K^2)$ which contains all even points. Let A = (2n,2m) for any n,m and let U = (2n,2m) be a $^{*}g\alpha$-closed set in $(Z^2, k^2)$.

Assume that $U \subseteq A$, then $U \not\subset int(A)$, since $int(A) = \phi$. This is a contradiction to our assumption. Therefore $^{**}g\alpha$-open set in $(Z^2, K^2)$ does not contain all even points alone.

## Theorem 3.11

Let A and E be subsets of $(Z^2, K^2)$.

(i) If E is non - empty a $^{**}g\alpha$-closed set, then $E_F \neq \phi$

(ii) If E is $^{**}g\alpha$-closed and $E \subseteq B_{mix} \cup B_{k^2}$ holds for some subset B of $(Z^2, K^2)$, then $E = \phi$.

(iii) The set $U(A_F) \cup A_{mix} \cup A_{K^2}$ is a $^{**}g\alpha$-open set containing A.

**PROOF**

***(i):*** Let y be a point in E. Then, $y \in cl(E)=E= E_F \cup E_{mix} \cup E_{K^2}$ . We consider the following three cases for the point y.

***Case1:*** $y \in E_{K^2}$ : Let y =(2n+1,2m+1) for some $n,m \in Z$.

Then $cl(\{y\})=\{2n,2n+1,2n+2\}\times\{2m,2m+1,2m+2\} \subseteq E$. Thus, there exists a point x = $(2n,2m) \in E_F$.

***Case2:*** $y \in E_{mix}$: Let y =(2n+1,2m) for some $n,m \in z$. Then $cl(\{y\}) =\{2n,2n+1,2n+2\}\times\{2m\} \subseteq E$.

Thus, there exists a point x = $(2n,2m) \in E_F$. The proof for y =(2n,2m+1) is by the same token as above, where $n, m \in Z$.

***Case3:*** $y \in E_F$: Then $E_F \neq \phi$.

Now we prove that $E_F \neq \phi$ for all the three cases.

***(ii):*** Suppose that $E \neq \phi$. By (i), $E_F \neq \phi$. Since $E_F \subseteq (B_{mix} \cup B_{k2})_F = \phi$, we have a contradiction.

***(iii):*** We claim that $A_{mix} \cup A_{k2}$ is a $^{**}g\alpha$-open set.

Let F be any non-empty $g\alpha$-closed set such that $F \subseteq A_{mix} \cup A_{K2}$ . Then by (ii), $F = \phi$. Thus, we have that $F \subseteq Int (A_{mix} \cup A_{K^2})$ , then $A_{mix} \cup A_{K^2}$ is $^{**}g\alpha$ - open . But we know that $U(A_F)$ is an open set.

Then $U(A_F) \cup A_{mix} \cup A_{K^2}$ is $^{**}g\alpha$-open(cf., Theorem 3.29 [17]). But $A = A_F \cup A_{mix} \cup A_{K^2}$.

then $A \subseteq U(A_F) \cup A_{mix} \cup A_{K^2}$. This implies that $^{**}g\alpha$-open set contains A.

**Theorem 3.12**

Let A be a subset of $(Z^2 , K^2)$. The $^{**}G\alpha o$– kernel of A and the $^{*}G\alpha o$-kernel of A are obtained precisely as follows

(i) $^{**}G\alpha o$–ker (A) = $U (A_F) \cup A_{mix} \cup A_{K^2}$ , where $U (A_F) = \cup \{ U(x) \mid x \in A_F\}$.

(ii) $^{*}G\alpha o$ -ker (A) = U(A) , where $U (A) = \cup \{ U (x) \mid x \in A\}$.

***PROOF***

**(i):** Let $U_A = U (A_F) \cup A_{mix} \cup A_{K^2}$. By Theorem 3.11(iii), $^{**}G\alpha o$ - ker (A) $\subseteq U_A$.

To prove $U_A \subseteq {}^{**}G\alpha o$-ker (A), it is claimed that (1) if there exists a $^{**}g\alpha$-open set V such that $A \subseteq V \subset U_A$ then $V = U_A$. Let x be any point of $U_A$. There are three cases for the point x.

**Case (1):**

$x \in (U_A)_F$. we note that $(U_A)_F = (U(A_F))_F \cup (A_{mix} \cup A_k^2)_F = A_F$.

Then we have that $x \in A_F \subseteq A \subseteq V$.

**Case (2):**

$x \in (U_A)_K^2$. We note that

$(U_A)_K^2 = (U(A_F)_k^2) \cup (A_{mix})_K^2 \cup (A_K^2)_K^2 = (U(A_F))_K^2 \cup A_K^2$.

Firstly, suppose that $x \in U(A_F)$. Then $x \in U(y)$ for some $y \in A_F$. Since $y \in A_F \subseteq A \subseteq V$ and V is $^{**}g\alpha$-open, we have $\{y\} \subseteq Int(V)$. Then $U(y) \subseteq Int(V)$. Since Int (V) is open.Thus, we have that $x \in V$.

Secondly, suppose $x \in A_k^2$. Then we have $x \in V$, since $x \in A_k^2 \subseteq A \subseteq V$.

**Case (3):**

$x \in (U_A)_{mix}$. Observe that

$(U_A)_{mix} = (U(A_F))_{mix} \cup (A_K^2)_{mix} \cup (A_{mix})_{mix}$

$= (U(A_F))_{mix} \cup A_{mix}$

Now, suppose that $x \in U(A_F)$. Then $x \in U(y)$ for some $y \in A_F$. Then y is a $^*g\alpha$-closed point since every closed point is $^*g\alpha$-closed point. Since $y \in A_F \subseteq A \subseteq V$, $\{y\}$ is $^*g\alpha$-closed and V is $^{**}g\alpha$-open set, we have $\{y\} \subseteq Int(V)$. Then $U(y) \subseteq Int(V)$ and so $x \in V$.

Let $x \in A_{mix}$. Then $x \in A_{mix} \subseteq A \subseteq V$.

For all cases we assume that $x \in U_A$. Then we show that $x \in V$, and therefore $U_A \subseteq V$. But we know that $V \subseteq U_A$. From the above cases we conclude that $V = U_A$. Thus, we shown (1).

Let $^{**}G\alpha o(A)$ be the family of all $^{**}g\alpha$-open sets containing A. Then, we have that $U_A \subseteq W$ for each $W \in {}^{**}G\alpha o(A)$, using (1) above and properties that $A \subseteq W \cap U_A \subseteq U_A$ and $W \cap U_A$ is $^{**}g\alpha$-open set(Remark 3.3 [16]). Hence, we show that $U_A \subseteq \cap\{W / W \in {}^{**}G\alpha o(A)\} = {}^{**}G\alpha o\text{–ker}(A)$ and thus $U_A \subseteq {}^{**}G\alpha o\text{–ker}(A)$. Therefore $^{**}G\alpha o\text{–ker}(A) = U_A$.

**(ii):** Since U(A) is open, $^*G\alpha O\text{-ker}(A) \subseteq U(A)$ holds.To prove $U(A) \subseteq {}^*G\alpha O\text{-ker}(A)$, it is prove that (2) if there exists an open set V such that $A \subseteq V \subseteq U(A)$. Let W be any $^*g\alpha$-open sets containing A. Then, we have that $U(A) \subseteq W$, using (**) above and properties that $A \subseteq W \cap U(A) \subseteq U(A)$ and $W \cap U(A)$ is $^*g\alpha$-open. Hence, we show that $U(A) \subseteq \cap\{W \mid W \in {}^*G\alpha O(A) \text{ and } A \subseteq W\} = {}^*G\alpha O\text{-ker}(A)$. Therefore $^*G\alpha O\text{-ker}(A) = U(A)$.

**Theorem 3.13**

The digital plane $(Z^2, K^2)$ is not an ${}_{\alpha}T_{1/2}^{***}$ space.

## Proof

Let x be a point of $(Z^2, K^2)$.

***Case (1):*** x = (2m+1,2n+1), where n, m∈Z: The singleton {x} is open in $(Z^2, K^2)$.

***Case (2):*** x = (2m,2n), where n,m∈Z: The singleton {x} is closed in $(Z^2, K^2)$ and also it is $^*$gα-closed set[4].

***Case (3):*** x = (2m,2n+1) where n, m∈Z : Let be U any α-open set containing {x}, that is {x}⊆ U. Then cl({x})=cl({2m,2n+1})= {2m}×{2n,2n+1,2n+2} ⊄U. Therefore {x} is not $^*$gα-closed set in $(Z^2, K^2)$.

***Case (4):*** x = (2m+1,2n), where n, m∈Z : The proof is similar to that of case(3).

Thus we prove that every singleton {x} of $(Z^2, K^2)$ is pure odd or pure even which is open and $^*$gα-closed, respectively. But the singleton {x} of $(Z^2, K^2)$ is mixed point and we prove that it is not open or $^*$gα-closed.

Therefore, the statement "If (X, τ) is ${}_{\alpha}T_{1/2}^{***}$ space if every singleton of {x} is either $^*$gα-closed set or open" (Theorem 4.8 [17]) is not true. Therefore the digital plane $(Z^2, K^2)$ is not an ${}_{\alpha}T_{1/2}^{***}$ space.

**Theorem 3.14**

Let E be a subset of $(Z^2, K^2)$. If E is $^{**}$gα-closed and dense set in $(Z^2, K^2)$, then $(Z^2, K^2)$ is the only $^*$gα-open set containing E.

**Proof**

Let U be a $^*$gα-open set containing E. Then, cl(E) ⊆ U, since E is $^{**}$gα-closed set in $(Z^2, k^2)$. Observe $(Z^2, K^2)$ ⊆ U, since E is dense in $(Z^2, K^2)$. Therefore $(Z^2, K^2)$ is the only $^*$gα-open set containing E.

**Remark 3.15**

(i) $^{**}G\alpha O\text{-}cl(\{x\}) = \{x\}\cup(cl(\{x\}))_F$

(ii) $^{**}G\alpha O\text{-}int(\{x\}) = \{x\}\cap(int(\{x\}))_{K^2}$

**Theorem 3.16**

Let B be a non-empty subset of $(Z^2, K^2)$. If $B_F = \phi$, then B is a $^{**}g\alpha$-open set of $(Z^2, K^2)$.

**Proof**

Let F be a $^{*}g\alpha$-closed set such that $F \subset B$. Since $B_F = \phi$, we have $B = B_{mix} \cup B_{K^2}$. Then by Theorem 3.11(ii), it is obtained that $F = \phi$, because F is $^{*}g\alpha$-closed set in $(Z^2, K^2)$. Thus, we conclude that whenever F is $^{*}g\alpha$-closed and $F \subset B$, $F = \phi \subset int(B)$. Hence B is $^{**}g\alpha$-open set of $(Z^2, K^2)$.

**Theorem 3.17**

Let B be a non-empty subset of $(Z^2, K^2)$. For a subset $B_F \neq \phi$, if a subset B is a $^{**}g\alpha$-open set of $(Z^2, K^2)$, then $(U(\{x\})_{K^2} \subset B$ holds for each point $x \in B_F$.

**Proof**

Let $x \in B_F$. Since $\{x\}$ is closed, then $\{x\}$ is $^{*}g\alpha$-closed set and $\{x\}\subset B$. Since B is $^{**}g\alpha$-open set, $\{x\}\subset int(B)$. Therefore $\{x\}$ is an interior point of the set B. Thus, we have that, for the smallest open set U(x) containing x, $U(x) \subset B$.

We can set $x=(2s,2u)$ for some integers $s, u \in Z$, because $x\in(Z^2)_F$. Since $U((2s, 2u)) = \{2s-1, 2s, 2s+1\}\times\{2u-1, 2u, 2u+1\}$, we have $(U(x))_{K^2} = \{(x_1, x_2) \in U(x) \mid x_1 \text{ and } x_2 \text{ are odd}\} = \{p_1, p_2, p_3, p_4\}$, where $p_1= (2s-1, 2u-1)$, $p_2 = (2s-1, 2u+1)$, $p_3 = (2s+1, 2u-1)$, and $p_4 = (2s+1, 2u+1)$. For each point $p_i(1 \leq i \leq 4)$, $p_i\in B$ and so $\{pi\}\cap B\neq\phi$. Therefore $(U(\{x\})_{K^2}\subset B$.

**Proposition 3.18**

Let x be a point of $(Z^2, K^2)$. The following properties on the singleton {x} hold:

(i) If $x \in (Z^2)_{K^2}$, then {x} is $^{**}g\alpha$-open; it is not $^{**}g\alpha$-closed in $(Z^2, K^2)$.

(ii) If $x \in (Z^2)_F$, then {x} is $^{**}g\alpha$-closed; it is not $^{**}g\alpha$-open in $(Z^2, K^2)$.

(iii) If $x \in (Z^2)_{mix}$, then {x} is not $^{**}g\alpha$-closed; it is $^{**}g\alpha$-open in $(Z^2, K^2)$.

**Proof**

**(i):** It follows from the assumption that {x} is open in $(Z^2, K^2)$ and so it is $^{**}g\alpha$-open in $(Z^2, K^2)$ (Theorem 3.2 [16]). We prove that {x} is not $^{**}g\alpha$-closed. Indeed, let x = (2s+1, 2u+1) $\in (Z^2)_{K^2}$, where $s, u \in Z$.

Let x = (2s+1, 2u+1) $\subseteq$ U = (2s+1, 2u+1) where U is a $^{*}g\alpha$-open in $(Z^2, k^2)$. Then cl({x}) = {2s, 2s+1, 2s+2}$\times${2u, 2u+1, 2u+2}$\not\subset$ {2s+1, su+1}. Therefore {x} is not $^{**}g\alpha$-closed in $(Z^2, K^2)$.

**(ii):** For the case where $x \in (Z^2)_F$, {x} is closed in $(Z^2, K^2)$ and so it is $^{**}g\alpha$-closed in $(Z^2, K^2)$. Assume that F= {2s, 2u}$\subseteq$ {x}= {2s, 2u}, where F is $^{*}g\alpha$-closed set. Then F $\not\subset$ int({x}), since int({2s, 2u}) = $\phi$. Therefore, {x} is not $^{**}g\alpha$-open in $(Z^2, K^2)$.

**(iii):** Let $x \in (Z^2)_{mix}$, i.e, x = (2s+1, 2u) such that cl({x}) $\not\subset$ {x} = U, where U is $^{*}g\alpha$-open set, because cl({x}) = {2s, 2s+1, 2s+2}$\times${2u, 2u+1, 2u+2}. Therefore {x} is not $^{**}g\alpha$-closed in $(Z^2, K^2)$.

Let x = (2s+1, 2u) such that F = $\phi \subseteq$ (2s+1, 2u) where F is $^{*}g\alpha$-closed set such that $\phi \subseteq$ int ({2s+1, 2u}) = $\phi$. Hence{x} is $^{**}g\alpha$-open in $(Z^2, K^2)$.

Similarly, we can prove this statement for x = (2s, 2u+1).

**Remark 3.19**

(i) The union of any collection of $^{**}g\alpha$-open set in $(Z^2, K^2)$ is $^{**}g\alpha$-open in $(Z^2, K^2)$.

(ii) The intersection of any collection of $^{**}g\alpha$-open set in $(Z^2, K^2)$ is $^{**}g\alpha$-open in $(Z^2, K^2)$.

As Corollary of Remark 3.19, we have a new topology say $^{**}G\alpha O$ $(Z^2, K^2)$ of $Z^2$. We change the topology $K^2$ of $(Z^2, K^2)$ by a new topology $^{**}G\alpha O$ $(Z^2, K^2)$ (cf. Section 3, Change the topologies [14]).

**Corollary 3.20**

Let $^{**}G\alpha O(Z^2, K^2)$ be the family of all $^{**}g\alpha$-open sets in $(Z^2, K^2)$. Then, the following properties hold:

(i) The family $^{**}G\alpha O(Z^2, K^2)$ is a topology of $Z^2$.

(ii) Let $(Z^2, {}^{**}G\alpha O(Z^2, K^2))$ be a topological space obtained by changing the topology $k^2$ of the digital plane $(Z^2, K^2)$ by $^{**}G\alpha O(Z^2, K^2)$, then $(Z^2, {}^{**}G\alpha O(Z^2, K^2))$ is a $T_{1/2}$ space.

**Proof**

**(i):** It is obvious from Remark 3.19 and definitions that the family $^{**}G\alpha O(Z^2, K^2)$ is a topology of $Z^2$.

**(ii):** Let $(Z^2, {}^{**}G\alpha O(Z^2, K^2))$ be a topological space with a new topology $^{**}G\alpha O(Z^2, K^2)$. Then, it is claimed that the topological space $(Z^2, {}^{**}G\alpha O (Z^2, K^2))$ is $T_{1/2}$ by Theorem 3.1(ii) [9]. By Proposition 3.18(i) (resp. (ii), (iii)), a singleton {x} is open (resp. closed, open) in $(Z^2, {}^{**}G\alpha O(Z^2, K^2))$, where $x \in (Z^2)_{K^2}$ (resp. $x \in (Z^2)_F$, $x \in (Z^2)_{mix}$). Therefore, every singleton {x} of $Z^2$ is open or closed in $(Z^2, {}^{**}G\alpha O (Z^2, K^2))$ and so by Theorem 3.1(ii) [6], the space $(Z^2, {}^{**}G\alpha O(Z^2, K^2))$ is $T_{1/2}$.

**Remark 3.21**

If B is a $^{**}g\alpha$-open set containing a point $x \in (Z^2)_F$, then $\{x\} \cup U(\{x\})_{K^2} \subset B$. Indeed, by Theorem 3.16, $U(\{x\})_{K^2} \subset B$ and $x \in B_F \subset B$.

**Proposition 3.22**

For a topological space $(Z^2, {}^{**}G\alpha O (Z^2, K^2))$, we have the following properties on the singletons as follows. Let x be a point of $Z^2$ and $s, u \in Z$.

(i) If $\{x\} \in (Z^2)_{K^2}$, then $^{**}G\alpha O\text{-ker}(\{x\}) = \{x\}$ and $^{**}G\alpha O\text{-ker}(\{x\}) \in {}^{**}G\alpha O(Z^2, K^2)$.

(ii) If $\{x\} \in (Z^2)_F$, then $^{**}G\alpha O\text{-ker}(\{x\}) = \{x\} \cup U(\{x\})_{K^2} = (\{2s, 2u\}) \cup \{(2s+1, 2u+1), (2s+1, 2u-1), (2s-1, 2u+1), (2s-1, 2u-1)\}$, where $x = (2s, 2u)$ and $^{**}G\alpha O\text{-ker}(\{x\}) \in {}^{**}G\alpha O(Z^2, K^2)$.

(iii) If $\{x\} \in (Z^2)_{mix}$, then $^{**}G\alpha O\text{-ker}(\{x\}) = \{x\}$ and $^{**}G\alpha O\text{-ker}(\{x\}) \in {}^{**}G\alpha O(Z^2, K^2)$.

(iv) If $\{x\} \in (Z^2)_{K^2}$, then $^{**}G\alpha O\text{-cl}(\{x\}) = \{(2s+1, 2u+1), (2s, 2u+2), (2s, 2u), (2s+2, 2u+2), (2s+2, 2u)\}$, where $x = (2s+1, 2u+1)$; and hence $\{x\}$ is not closed in $(Z^2, {}^{**}G\alpha O(Z^2, K^2))$.

(v) If $\{x\} \in (Z^2)_F$, then $^{**}G\alpha O\text{-cl}(\{x\}) = \{x\}$.

(vi) If $\{x\} \in (Z^2)_{mix}$, $^{**}G\alpha O\text{-cl}(\{x\}) = \{(2s,2u),(2s+1,2u),(2s+2,2u)\}$, where $x = (2s+1, 2u)$.

(vii) If $\{x\} \in (Z^2)_{K^2}$, then $^{**}G\alpha O\text{-int}(\{x\}) = \{x\}$.

(viii) If $\{x\} \in (Z^2)_F$, then $^{**}G\alpha O\text{-int}(\{x\}) = \phi$.

(ix) If $\{x\} \in (Z^2)_{mix}$, then $^{**}G\alpha O\text{-int}(\{x\}) = \{x\}$.

**Proof**

**(i):** By Proposition 3.18(i), $\{x\}$ is $^{**}g\alpha$-open in $(Z^2, K^2)$ for a point $x \in (Z^2)_{K^2}$. Thus, we have that $^{**}G\alpha O\text{-ker}(\{x\}) = \{x\}$ and $^{**}G\alpha O\text{-ker}(\{x\}) \in {}^{**}G\alpha O\,(Z^2, K^2)$.

**(ii):** Let B be any $^{**}g\alpha$-open set of $(Z^2, K^2)$, containing the point $x = (2s, 2u) \in (Z^2)_F$. Then by Remark 3.21, $\{x\} \cup U(\{x\})_{K^2} \subset B$ holds and $\{x\} \cup U(\{x\})_{K^2} \in {}^{**}G\alpha O(Z^2, k^2)$. Thus, we have that $^{**}G\alpha O\text{-ker}(\{x\}) = \cap\{V | \{x\} \subset V, V \in {}^{**}G\alpha O(Z^2, K^2)\} = \{x\} \cup U(\{x\})_{K^2} = \{(2s, 2u), (2s+1, 2u+1), (2s+1, 2u-1), (2s-1, 2u+1), (2s-1, 2u-1)\}$. By Theorem 3.16, the kernel $^{**}G\alpha O\text{-ker}(\{x\})$ is $^{**}g\alpha$-open in $(Z^2, K^2)$, i.e., $^{**}G\alpha O\text{-ker}(\{x\}) \in {}^{**}G\alpha O\,(Z^2, K^2)$.

**(iii):** Let $x \in (Z^2)_{mix}$. The singleton $\{x\}$ is $^{**}g\alpha$-open by Proposition 3.18(iii). Hence, we have that $^{**}G\alpha O\text{-ker}(\{x\}) = \{x\}$ and $^{**}G\alpha O\text{-ker}(\{x\}) \in {}^{**}G\alpha O\,(Z^2, K^2)$.

**(iv):** For a point $x \in (Z^2)_{k^2}$, we can assume that $x = (2s+1, 2u+1)$, where $s, u \in Z$. For a point $y \in (Z^2)_{K^2}$, $y \in {}^{**}G\alpha O\text{-ker}(\{x\})$ holds (i.e., $y \in ({}^{**}G\alpha O\text{-ker}(\{x\}))_{K^2}$) if and only if $x \in {}^{**}G\alpha O\text{-ker}(\{y\})$ holds (i.e., $y = x$)(c.f (i)). Thus, we have that $({}^{**}G\alpha O\text{-cl}(\{x\}))_{K^2} = \{x\}$. For a point $y \in (Z^2)_F$, $y \in {}^{**}G\alpha O\text{-cl}(\{x\})$ holds (i.e., $y \in ({}^{**}G\alpha O\text{-cl}(\{x\}))_F$) if and only if $x \in {}^{**}G\alpha O\text{-ker}(\{y\})$ holds (i.e., $x \neq y$) (c.f (ii)). Thus, we have that $({}^{**}G\alpha O\text{-cl}(\{x\}))_F = \{y \in (Z^2)_F \mid x \in \{y\} \cup U(\{y\})_{K^2}\} = W_x$; where $W_x = \{(2s, 2u), (2s, 2u+2), (2s+2, 2u), (2s+2, 2u+2)\}$ and $x = (2s+1, 2u+1)$. For a point $y \in (Z^2)_{mix}$, $^{**}G\alpha O\text{-ker}(\{y\}) = \{y\}$ holds (c.f (iii)). Since $y \neq x$, we have that $({}^{**}G\alpha O\text{-ker}(\{x\}))_{mix} = \phi$.

Thus, we obtain that $^{**}G\alpha O\text{-ker}(\{x\}) = \{x\} \cup W_x$, because $E = E_{K^2} \cup E_F \cup E_{mix}$ holds for any subset E.

**(iv):** For a point $x \in (Z^2)_F$, by Proposition 3.18(ii), it is obtained that $^{**}G\alpha O\text{-cl}(\{x\}) = \{x\}$.

**(v):** For a point $x \in (Z^2)_{mix}$, by Remark 3.18(ii), we have that

$$^{**}G\alpha O\text{-}cl(\{x\})=\{x\}\cup(cl(\{x\})_F = (2s+1, 2u) \cup \{(2s, 2u), (2s+2, 2u)\} = \{(2s+1, 2u), (2s, 2u), (2s+2, 2u)\}, \text{ where } x = (2s+1, 2u).$$

**(vii):** For a point $x\in(Z^2)_{K^2}$ (resp. $x \in(Z^2)_F$, $x \in(Z^2)_{mix}$), by Proposition 3.18(i) (resp. (ii), (iii)), it is shown that $^{**}G\alpha O\text{-}int(\{x\}) = \{x\}$. (resp. $^{**}G\alpha O\text{-}int(\{x\}) = \phi$, $^{**}G\alpha O\text{-}int(\{x\}) = \{x\}$) holds.

**Theorem 3.23**

If $x \in(Z^2)_{mix}$, i.e., x = (2s, 2u+1) or (2s+1, 2u), then {x} is not regular closed; it is semi open. But {x} is not regular open in ($Z^2$, $^{**}G\alpha O(Z^2, K^2)$).

**Proof**

For a point $x \in(Z^2)_{mix}$, by Proposition 3.22(ix) and (vi), $^{**}G\alpha O\text{-}cl[^{**}G\alpha O\text{-}int(\{x\})] = \{(2s, 2u), (2s+1, 2u), (2s+2, 2u)\} \supset \{x\}$, where x = (2s+1, 2u). Therefore {x} is not regular closed and hence it is semi-open. Let $x \in(Z^2)_{mix}$, by the Remark 3.15(i),(ii), $^{**}G\alpha O\text{-}int[^{**}G\alpha O\text{-}cl(\{x\})] = \phi$. Therefore {x} is not regular open.

**Theorem 3.24**

If $x \in(Z^2)_{k^2}$, i.e., x = (2s+1, 2u+1), then {x} is not regular closed in ($Z^2$, $^{**}G\alpha O(Z^2, K^2)$); it is semi open in ($Z^2$, $^{**}G\alpha O(Z^2, k^2)$); observe that{x} is regular open in ($Z^2$, $^{**}G\alpha O(Z^2, K^2)$).

**Proof**

Let $x = (2s+1, 2u+1) \in(Z^2)_{K^2}$, where s, u ∈. By Proposition 3.22(vii) and (iv), it is obtained that $^{**}G\alpha O\text{-}cl[^{**}G\alpha O\text{-}int(\{2s+1, 2u+1\})] = {}^{**}G\alpha O\text{-}cl[(\{2s+1, 2u+1\})] \supset \{x\}$, and hence the singleton (2s+1, 2u+1) is not regular closed and hence it is semi-open. Let $x \in(Z^2)_{mix}$, by the Proposition 3.22(iv) and (vii), $^{**}G\alpha O\text{-}int[^{**}G\alpha O\text{-}cl(\{(2s+1, 2u+1)\})] = \{(2s+1, 2u+1)\}$and hence the singleton (2s+1, 2u+1) is regular open in ($Z^2$, $^{*}G\alpha O(Z^2, K^2)$).

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