Compact Manifolds Representation, a New Approach

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Abstract

We discuss a new method for representing Compact Manifolds.

Key Words: compact manifold, topology.

1 Introduction

This is not a formal mathematical paper but rather the outline of a possible method that can be used to represent compact manifolds and that requires further mathematical research. Some reasonable hypothesis are provided but, till a formal proof is given and as always happen in math, appearances may be deceiving. However, if those hypothesis turn out to be true, the method we propose may result very interesting.

One advantage of this method is that it reduceds by one the dimension of the problem. For example, 3D closed manifolds are represented by 2D closed surfaces, which in general are not manifolds.

2 Representing Manifolds

2.1 Definitions

Definition 2.1: n-dimensional Thick Hyper-surface
Let X a closed (n-1)-dimensional Δ-complex with $n \geq 2$.

1. *For each (n-2)-simplex S_i in Δ embed X locally in a n-disk D_i shaped such that S_i is all in the interior of D_i and none of the other (n-2) simplices $S_{j \neq i}$ of X are in it.*
2. *For each S_i embed the relevant n-disk D_i in $\mathbb{R}^{n+1}$. Let $\hat{\eta}$ be the axis orthogonal to the n-hyper-plane on which D_i lays. Give an orientation to $\hat{\eta}$ and choose the ordered sequence you encounter the (n-1)-surfaces (i.e. (n-1) other simplices of X) attached to S_i by going around $\hat{\eta}$ clockwise. Since*

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we locally embed X in each D_i, (n-1)-simplices cannot cross each other in the D_i disks and the ordered sequence we give to them is well defined. We will call this data set the arrangement $\mathbf{A}_X$ of the Thick surface.

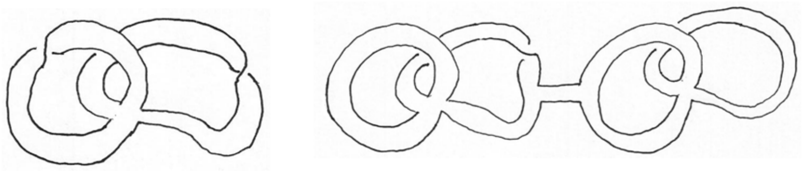
3. *Give a third dimension to X by expanding it by a small δL orthogonally to each (n-1)-surface and keeping the final space locally embedded in each D_i. This will turn X in a 3D manifold with boundaries, and each k-simplex (for $k = 0 ... n-1$) of X in a (k+1)-dimensional object.*

4. *(For $n = 2$ only) To each edge (i.e. now a strip) decide which one has to be twisted. We will call this data set $\mathbf{T}_x$. Cut midway all the edges (i.e. strips) to be twisted and glue back the two sides of the cut by identifying them in the opposite direction.*

The n-dimensional manifold with boundaries we get after applying the above procedure is what we call the n-dimensional Thick Surface $\hat{X}$.

Note that we thicken X just to freeze the information $\mathbf{O}_X$ and for $n = 2$ the information $\mathbf{T}_x$ but we do not really need to do it. We can say that for $n > 2$ a Thick Surface is the couple $(X, \mathbf{O}_X)$ and for $n = 2$ a Thick Surface is the triple $(X, \mathbf{O}_X, \mathbf{T}_x)$.

For $n = 2$ Thick Surfaces may also be called Thick Lines or strip configurations. Fig. 1 shows examples of Thick Lines (i.e. Thick Surfaces for the case $n = 2$):



(a) **(b)**

Figure 1: Thick Lines

Note that the reason why for $n > 3$ we do not need the last step, in which we change some strips in twisted strips, is because for $n > 2$, $(n-1)$-surfaces have already a property of orientability, which does not make sense for lines, and twisting a strip is equivalent to change the general orientability of the space it is attached if this is an orientable surface.

Definition 2.2: Associated Manifold
Let $\hat{X}$ be a n-dimensional Thick Surface. Let γ_i be the path connected subspaces of the boundary of $\hat{X}$. For each γ_i we define a (n-1)-dimensional Δ-complex on it. We attach a n-simplex on each (n-1)-simplex of the above defined complex of γ_i. We attach the remaining hyper-faces of each n-simplex to each other by identifying together the remaining free vertices of the n-simplices and following, for hyper faces of the n-simplices, the same pattern of which the relevant (n-1)-simpices on γ_i are attached to their neighbour simplices. By doing so

we get a compact n-dimensional manifold $\Omega(\hat{X})$ which we will call the associated compact manifold to the Thick Surface $\hat{X}$.

We note that for 2-dimensional Thick Surfaces (i.e. thick lines), the boundaries γ_i are circles and the above procedure is equivalent to identifying the boundary of 2-disks D_i to the γ_i (i.e. fill the holes with disks).

Definition 2.3: Equivalent Thick Surfaces
Two Thick Surfaces are said to be equivalent if their associated manifolds are homeomorphic.

Definition 2.4: Trivial Intersection.
Let $\hat{X}$ be a n-dimensional Thick Surface. If we can split $\hat{X}$ in two parts A and B which overlap in a region such that A and B cannot be embedded in $\mathbb{R}^n$ while C can, then the Thick surfaces $\hat{X}$ can always be decomposed in two non path connected Thick Surfaces $\hat{X}_A$ and $\hat{X}_A$ such that $\Omega(\hat{X}_A)\#\Omega(\hat{X}_B)$ (i.e. connected sum) is homeomorphic to $\Omega(\hat{X})$ (See [1]). In this case we say that the two Thick Surfaces $\hat{X}_A$ and $\hat{X}_A$ cross in a trivial intersection.

Note that, given a Thick Surface $\hat{X}$, if we can split $\hat{X}$ in two parts A and B such that B can be embedded in $\mathbb{R}^3$, then the Thick Surfaces $\hat{X}$ can always be decomposed in two non path connected Thick Surfaces $\hat{X}_A$ and $\hat{X}_A$ such that $\hat{X}_A$ is equivalent to $\hat{X}$ and $\Omega(\hat{B})$ is an n-sphere. Fig, 2 shows an example of a trivial intersection.

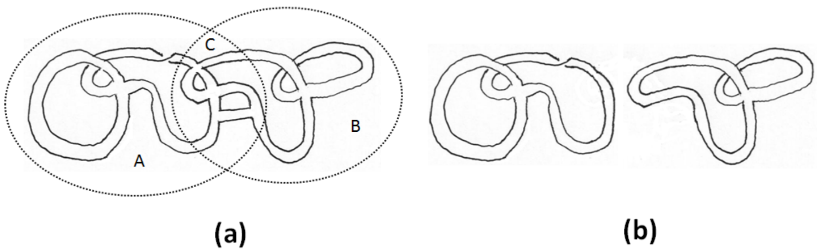


Figure 2: Thick Lines Decomposition

In the next definition we introduce the idea of a Singular Regions. In general it is not possible to embed $\hat{X}$ in $\mathbb{R}^n$. However, if we remove a finite number of n-disks from $\mathbb{R}^3$ we can embed part of $\hat{X}$ letting the non embeddable parts out from $\mathbb{R}^n$ through the boundary of the relevant n-disks. Let suppose we maximise the number of n-disks in order to maximise the part of $\hat{X}$ which is embedded in $\mathbb{R}^n$. We will call the regions of $\hat{X}$, that are not embedded in $\mathbb{R}^n$, the Singular Regions of $\hat{X}$.

Definition 2.5: Singular Regions.
Let $\hat{X}$ be a n-dimensional Thick Surface. A singular region S is a triple $(Y, f, \partial D)$ where $Y \in X$ is a minimal part of $\hat{X}$ that cannot be embedded in $\mathbb{R}^n$, ∂D is the boundary of a removed n-disk from $\mathbb{R}^n$ trough which Y is connected

to X and $f = Y \cup \partial D$ is the part of the boundary of Y which is laying on ∂D and it is called the interface of S.

The interface f is a non path connected union of thick (n-1)-spheres (e.g. couple of segments for $n = 2$ and annuli for $n = 3$).

The Order of S is the number of such (n-1)-spheres. The order may be zero.

Fig, 3 shows an example of Singular Regions. Note that in two dimensions there are only two types of Singular Regions: 1- Missed intersection (regions a and b in the figure) of order two, 2- Twisted strip (region c in the figure) of order one.

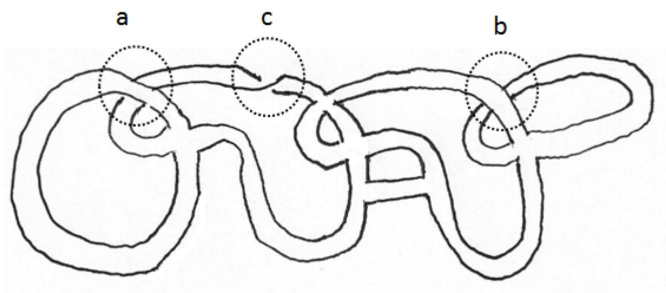


Figure 3: Singular Regions

Fig. 4 shows one more example of Singular Region and in particular a singular region composed of non path connected manifolds. Note that in three dimensions there is only one type of "manifold like" Singular Regions, which is the missed intersection between a torus and a plane showed in the figure. All other Singular Regions have at least a part of them which is not a manifold and there are an infinite number of them.

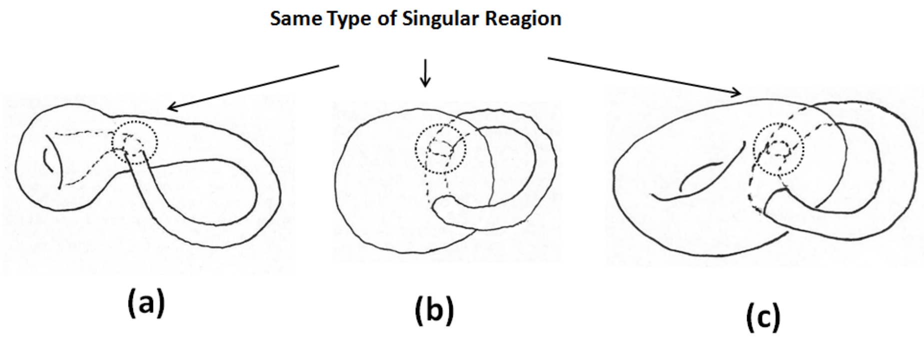


Figure 4: Singular Region for Thick Surfaces

Definition 2.6: Type of a Singular Region and Completing Thick Surface.

We say that two Singular Regions $S_A = (A, f_A, \partial D_A)$ and $S_B = (B, f_B, \partial D_B)$ are of the same type is there exists an homeomorphism between S_A and S_B and the restriction of this homeomorphism to f_A and f_B, as well as the restriction to

∂D_A and ∂D_B, are homeomorphisms. We say that a Thick Surface completes a Singular Region S if it has only a Singular Region and this region is of type S.

We note explicitly that given the Singular Region of Fig. 4, if we complete it by attaching 3 disks to the interface, we get two disconnected spheres which are not a Singular Region any more. Given the definition, this is not a possible way to complete that Singular Region.

We also note that a Singular Region can be completed to Thick Surfaces which are not homoemorphic to each other (See Fig. 4).

Definition 2.7: Natural Order of a Singular Region.

(For $n > 2$). Given two Singular Regions $S_M = (M, f_M, \partial D_M)$ and $S_m = (m, f_m, \partial D_m)$, and let γ_i be one of the path disconnected components of f_M. If by identifying the boundary of an n-disk to γ_i and pushing it inside ∂D_M we turn S_M into S_m, we say that the two Singular Regions are of the same type but different order (their order differs by one) and that γ_i is an hidden interface of S_m.

The opposite operation is also possible and we can pull out an hidden interface from a Singular Region and increase its order.

The natural order of a Singular Region $S = (A, f_A, \partial D_A)$ is the order we get by pulling out on ∂D_A all its hidden interfaces.

(For $n = 2$). In this case there are only two types of Singular Region. The possible natural orders of them is defined in Fig. 5.

Note that, the idea of natural order is well defined because there is an upper limit to the number of hidden interfaces present in a Singular Region and that are possible to be pulled out without adding structures to S.

We also note explicitly that given the Singular Region of Fig 4. if we lower the order by 1 by attaching a disk to one end of the cylinder, what we get is now embeddable in $\mathbb{R}^2$ and therefore it is not a Singular Region any more.

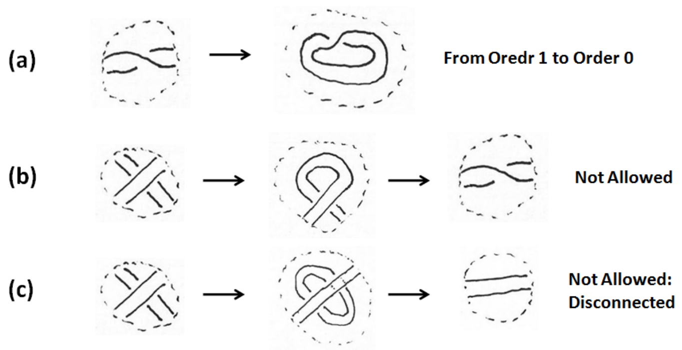


Figure 5: Order of 2-Dimensional Singular Regions

Definition 2.8: Array of Singular Regions.

Let $S_1 ... S_k$ be n-Singular Regions non necessary all of different type. The configuration we get by plugging all the various $S_i = (X_i, f_i, D_i)$ in $\mathbb{R}^n$, by *removing an n-disk form it and identifying the boundary of $\mathbb{R}^n - S_i$ with D_i, is called and array of Singular Regions* $\mathbf{Ar}(S_1, \ldots, S_k)$.

We say that a Thick surface can be embedded non trivially in an Array of Singular Regions **Ar** *if, it can be embedded in it, and it could not be embedded if only one of the Singular Regions of* $\mathbf{A}r$ *was removed.*

Note that the boundaries of the Singular Regions S_i, which do not lay on D_i, will be still boundaries of the array. The boundaries which lays on D_i (i.e. f_i) will disappear and will be used to connect $\mathbb{R}^n$ to S_i.

2.2 Propositions

Proposition 2.1: *Let A be an n-dimensional compact manifold. If A has a Δ-Complex representation then A has a Thick Surface representation.*

Proof: For $n > 2$, let X be a Δ-Complex representation of A and let $Y \in X$ be its (n-1)-skeleton (which is an (n-1) Δ-Complex). Let also A_Y be the order in which (n-1)-simplices of Y are encountered going around each (n-2) simplex of Y in X. This order is unique before removing the n-simplices from X to get Y. Then the thick surface we are looking for is the couple (Y, A_Y).

For $(n = 2)$, it is possible to find an argument to assign a twisting status to each edges of Y and this is a possible way to prove this statement. However, is is easier to note that for n=2, we are in the situation where compact manifolds are fully classified and we know how to build each of them as the Associated Manifold of a Thick Line. □

Proposition 2.2: $\Omega(\hat{X})$ *is a prime compact manifold if and only if* $\hat{X}$ *is a Thick Surface with only a Singular Region present only once.*

Proof: Once we get the associated manifold $\Omega(\hat{X})$ from $\hat{X}$, from each Singular Region we get a manifold sum connected to an n-sphere and not homeomorphic to an n-sphere and therefore from a Thick Surface with two Singular Regions we cannot get a prime manifold. On the other hand, if $\Omega(\hat{X})$ is not prime, we can deform it since we get a sphere with a manifold sum connected to it for each prime manifolds presents in $\Omega(\hat{X})$. The Thick surfaces embedded in the manifold will therefore deform into a shape with a Singular Region for each prime manifolds in $\Omega(\hat{X})$. □

We have given two proposition that we know are true although they should be proved in a more formal way. We would like now to add some additional Hypothesis that are not proven and that are based only on tenth of real examples analysed by a computer that give an hint of the fact that there is a tiny possibility they are true. The following Hypothesis and their proof, according to us, may be a good roadmap for further mathematical research on this topic.

Hypothesis 2.1: *Let S be an n-Singular Region of natural order k, clearly S can be completed to a finite number of Thick Surfaces only. Let Λ_S be the class of homomorphic Thick Surfaces obtained by completing S and let* $\Omega(\Lambda_S)$

be the relevant set of associated manifolds to the above Thick Surfaces then, for any $i, j \leq m$ with $i \neq j$ we have that $\Omega(\hat{X}_i)$ is not homeomorphic to $\Omega(\hat{X}_j)$.

Note that if S is not of natural order and S' is the same Singular Region of higher order, then S can be completed to a set of Thick Surfaces which is a subset of the Thick Surfaces to which S' can be completed. This is because to go from S' to S it is enough to attach some n-Disks to some of its interfaces which is part the process of completing a Thick surfaces from S'.

Hypothesis 2.2: *Let $\hat{X}$ and $\hat{Y}$ be two n-dimensional Thick Surfaces with only a singular region present only once. If $\Omega(\hat{X})$ is homeomorphic to $\Omega(\hat{Y})$ then the Singular Regions of $\hat{X}$ and $\hat{Y}$ are of the same type.*

For example the 3-manifold associated to the Thick 2-sphere is $\mathbf{S}^3$. We state that this is the only way to generate $\mathbf{S}^3$ by means of a thick surface. At first sight also the manifold associated to the 2-torus seams to be $\mathbf{S}^3$ however, further analysis shows that $\Omega(\hat{\mathbf{S}^2})$ has a different homology with respect to $\Omega(\hat{\mathbf{T}^2})$.

The two above hypothesis, if true, are telling us that prime manifolds come in classes where the class is defined by the singular region that generates them. The same is valid for Thick Surfaces that have more then one singular region. They form classes of non prime manifolds.

Hypothesis 2.3: *Let A be a set of Singular Regions where each Singular Region can be present more then once. Let Γ_A be the sets of Thick Surfaces that can be embedded non trivially in the array $\mathbf{Ar}(A)$ generated by A and let Ω_A be the set of manifold associated to the elements of Γ_A. Then A is a class of closed manifolds. Let B, Γ_B and Ω_B another class, then if each element of Γ_A can be embedded non trivially in $\mathbf{Ar}(B)$ and viceversa, then Ω_A and Ω_B are the same class of closed manifolds (i.e. they contains the same elements up to homeomorphism).*

In Fig. 6 we show two classes for the the 2-dimensional case. In this case these classes both contain only one element. Note that the $\mathbf{RP}^2$ Thick surfaces can be embedded in the Klein bottle class but trivially (see Fig. 6a and Fig. 6b). The Klein bottle class can be generated by two different array of Singular regions (see Fig. 6c and Fig. 6d).

Hypothesis 2.4: *Let A be a set of Singular Regions where each Singular Region can be present more then once. Let γ_A be the sets of Thick Surfaces that cannot be decomposed in simpler components and that can be embedded non trivially in the array $\mathbf{Ar}(A)$ generated by A and let ω_A be the set of manifold associated to the elements of γ_A. Moreover let S be a Singular Region not present in A. Then there exist an integer $k \geq 1$ such that by adding S to A, k times, we get each time a different class A_k, γ_{A_k} and ω_{A_k} and from the time $k+1$ onward γ_{A_k} is empty.*

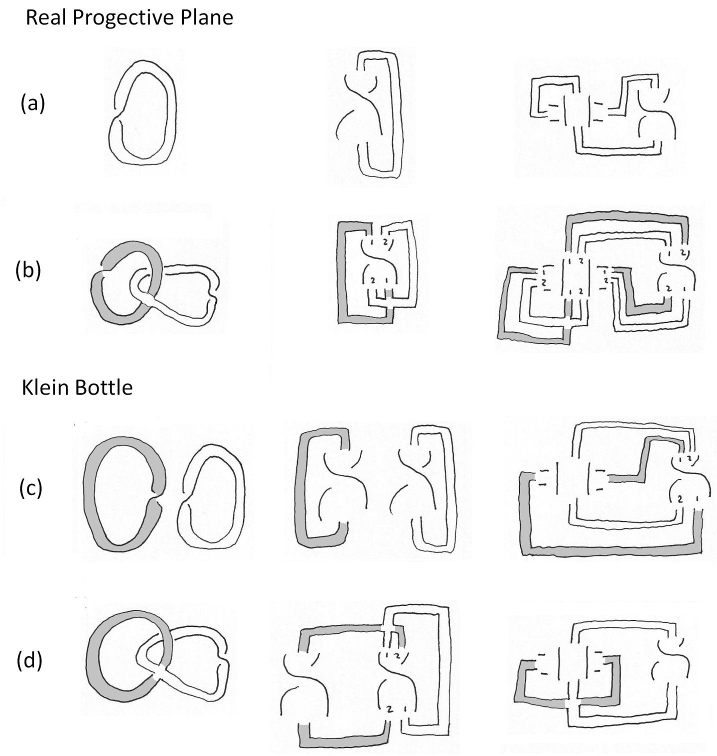


Figure 6: Examples of Manifold Classes

Appendix - Examples

In this appendix we present some examples of space build using Thick Surfaces.

The first examples in appendix A.1 are the most simple spaces that can build using the approach presented in this paper. Further examples in appendix A.2 and A.3 are the Poincaré homology sphere and some lens spaces that are classic example of 3-manifolds defined in the very early years of topology.

A.1 Some Simple Examples

The following tables presents the homology groups of 5 most simple examples of 3-manifolds that can be build using thick surfaces.

Num	$\hat{X}$	$\Omega(\hat{X})$	$\mathbf{H}_3$	$\mathbf{H}_2$	$\mathbf{H}_1$	$\mathbf{H}_0$
1	Thick 2-Sphere	3-Sphere	$\mathbb{Z}$	0	0	$\mathbb{Z}$
2	Thick 2-Torus	-	$\mathbb{Z}$	$\mathbb{Z}^2$	0	$\mathbb{Z}$
3	Thick $\mathbf{RP}^2$	-	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}$
4	Thick Klein Bottle (see. Fig. 4a)	-	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$
5	Thick Surface of Fig. 4b	-	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$
6	Thick Surface of Fig. 4c	-	$\mathbb{Z}$	$\mathbb{Z}^3$	0	$\mathbb{Z}$

Table 1 : Homology Groups of $\Omega(\hat{X})$

Note that the Thick Klein Bottle is also known in topology as Solid Klein Bottle.

A.2 The Poincaré Sphere

The Poincaré homology sphere, first introduced by Henri Poincaré, is an example of a closed 3-manifold with homology groups homologous to a 3-sphere but which is not homeomorphic to it. As a matter of fact it has a finite fundamental group of order 120 known as the binary icosahedral group. There are many ways to construct the Poincaré homology sphere. Among all, the simplest construction is by identifying opposite faces of a dodecahedron using the minimal clockwise twist to line up the faces.

A.2.1 Cell Complex Definition

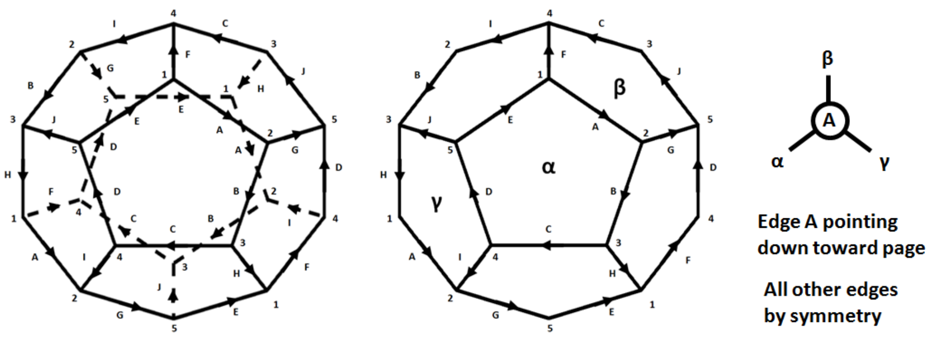
We want to find the solid strip configuration of the Poincaré homology sphere. In order to do it we have to define the Poincaré homology sphere Cell complex first.

As mentioned in the introduction a possible Cell complex of the Poincaré homology sphere is composed by a single 3-cell which is a dodecahedron where opposite faces of the dodecahedron are identified using the minimal clockwise twist to line up them. By doing so some edges an vertices of the original dodecahedron get identified themselves. We have worked out identified edges (labelled by letters from A to J) and vertices (labelled by numbers from 1 to 5) and the result is shown in Fig. 7a where an orientation for each edge is also given.

By doing so we get a final Cell complex with:

$$\begin{cases} 5 & Vertices \\ 10 & Edges \\ 6 & Faces \\ 1 & 3-Cell \end{cases} \tag{1}$$

Which gives an Euler characteristics equal to $\chi = 0$, as expected since all the homology groups of the Poincaré sphere are trivial.



(a) (b)

Figure 7: Poincaré Sphere Cell complex and Thick Surface

A.2.2 Thick Surface

We want to find the Thick surface of the Poincaré Sphere. In this case, it is obvious that the relevant surface is the one we get identifying opposite pentagons of the original tetrahedron. This is because to each pentagon there is a pentagonal base pyramid attached to it and all these pyramids make the internal of the manifold under study. However, this data itself it is not enough. We also need to define the ordered sequence surfaces are met when going around each oriented edge clockwise. This can be done by going from one surface to another from the inside of the original dodecahedron. The result is also shown in Fig. 7. The two pieces of data, the base surface and the ordered sequence of surfaces are met going around edges (i.e. the arrangement), define the thick surface $\hat{X}_p$ (p for Poincaré). This is a Thick Surface corresponding to an order 0 Singular Region. The natural order of the Singular Region associated to the Poincaré Sphere is 6, and it can be gotten by cutting a disc from each of the 6 2-cells of its complex and making the boundary of the cut an interface.

The Poincaré homology sphere is the manifold associated to $\hat{X}_p$ (i.e. $\Omega(\hat{X}_p)$ see [1]) and it is the simplest manifold in which $\hat{X}_p$ can be embedded.

A.3 Lens Spaces

Lens spaces are manifold first introduced by Heinrich Tietze for the 3-dimensional case. Alexander showed that the lens spaces $L(5;1)$ and $L(5;2)$ are not homeomorphic even though they have the same fundamental groups and the same homology.

There are many ways to construct lens spaces $L(p,q)$. Among all, the simplest construction is by using a bi-pyramid with a polygonal base having p edges and identifying faces of the up pyramid with faces of the down pyramid in a specific way.

A.3.1 Cell Complex Definition

We need to construct a Cell complex of lens spaces. A way to do it is to start from a solid bi-pyramid having for base a p-polygon. Let N and S be the two vertices of the bi-pyramid on the two opposite sides of the base. Moreover, let $V_0 ... V_{p-1}$ br the vertices of the bi-pyramid located on the vertices of the polygonal base. If we now identify the faces of the bi-pyramid, which are triangles, by identifying N to S, V_i with V_{i+q} and V_{i+1} with V_{i+q+1}, the resulting space is homeomorphic to the lens space $L(p;q)$.

The Fig. 8 shows the construction for $L(5;1)$ and $L(5;2)$ which are the two lens spaces we are focusing on.

A.3.2 Thick surface

We want to find the Thick Surface of the two above defined lens spaces. We can proceed in this case exactly as we did for the Poincaré Sphere since also in this case it is obvious that the relevant surface is the one we get identifying opposite faces of the bi-pyramid. Also in this case, we need to define the ordered sequence surfaces are met when going around each oriented edge clockwise (i.e. the arrangement, this is necessary for one edge only of the proposed complex

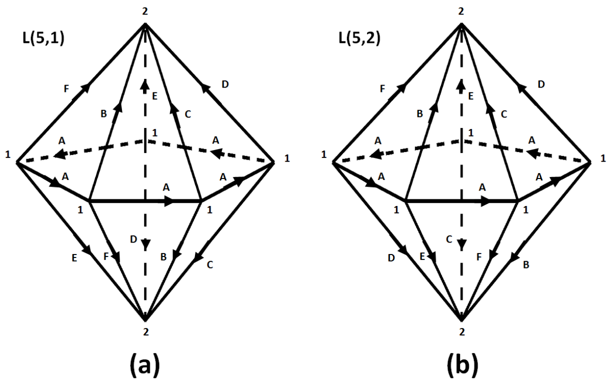


Figure 8: Lens Spaces Cell complexes

since other edges connect only two surfaces). The result is shown in the figure below.

Spaces $L(5;1)$ and $L(5;2)$ are the manifolds associated to the above defined Tick Surface and they are the simplest manifolds in which the above Thick Surfaces can be embedded. These are a Thick Surface corresponding to order 0 Singular Region. The natural order of the Singular Region associated to these Lens spaces is 5, and it can be obtained by cutting a disc from each of the 5 2-cells of its complex and making the boundary of the cut as an interface.

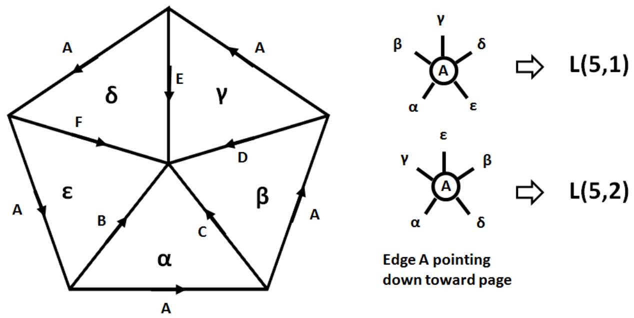


Figure 9: Lens Spaces Thick Surfaces

We note that the two lens spaces are generated by the same basic surface (Fig. 9a) but they differ by the ordered sequence faces are encountered when going around the edge A (Fig 6b). This arrangement get frozen when we thicken the surface given as a result two different Thick Surfaces which are not homeomorphic each other.

References

[1] V. Nardozza. *Solid Strips Configurations.* http://vixra.org/abs/1912.0161 (2019)