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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 27(2017), No. 2, 83-100

PROPERTIES OF α-OPEN SETS IN IDEAL MINIMAL SPACES

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Abstract. The purpose of this paper is to introduce and characterize the concept of α-open set and several related notions in ideal minimal spaces.

Dedicated to Professor Valeriu Popa on the Occasion of His 80th Birthday

1. Introduction and preliminaries

Popa and Noiri [10] introduced the notion of minimal structures which is a generalization of a topology on a given nonempty set. They also introduced the notion of m-continuous functions as a function defined between an m-space and a topological space. They showed that the m-continuous functions have properties similar to those of continuous functions between topological spaces. Let X be a topological space and $A \subset X$.

Keywords and phrases: Ideal minimal space, m-α-$\mathcal{I}$-open set, m-α-$\mathcal{I}$-closed set.
(2010) Mathematics Subject Classification: 54D10.

The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subfamily m of the power set $P(X)$ of a nonempty set X is called a minimal structure [10] on X if $\emptyset$ and X belong to m. By (X, m), we denote a nonempty set X with a minimal structure m on X. The members of the minimal structure m are called m-open sets [10], and the pair (X, m) is called an m-space. The complement of an m-open set is said to be m-closed [10]. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [6] and Vaidyanathas[12]. An ideal $\mathcal{I}$ on a nonempty set X is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given an m-space (X, m) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*_m$: $\mathcal{P}(X) \to \mathcal{P}(X)$ called the local minimal function [11] of A with respect to m and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*_m(\mathcal{I}, m) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in m(x)\}$, where $m(x) = \{U \in m | x \in U\}$. The set operator $m\,\mathrm{Cl}^*(.)$, called a minimal $*$-closure, is defined as $m\,\mathrm{Cl}^*(A) = A \cup A^*_m$ for $A \subset X$. The minimal structure$m^*(\mathcal{I}, m)$, generated by $m^*(\mathcal{I}, m) = \{U \subset X \mid mCl^*(X \setminus U) = X \setminus U\}$, is called a $*$-minimal structure, which is finer than m. And $m\,\mathrm{Int}^*(A)$ denotes the interior of A in $m^*(\mathcal{I}, m)$ (see [11]).

Definition 1.1. [10] *Let (X, m) be an m-space. For a subset A of X, the m-interior of A and the m-closure of A are defined by* $m\,\mathrm{Int}(A) = \cup\{W/W \in m, W \subseteq A\}$ *and* $m\,\mathrm{Cl}(A) = \cap\{F/A \subseteq F, X \setminus F \in m\}$, *respectively.*

Theorem 1.2. [10] *Let (X, m) be an m-space, and A, B subsets of X. Then $x \in m\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \varnothing$ for every $U \in m$ containing x. Further, the following properties hold:*

(i) $m\,\mathrm{Cl}(m\,\mathrm{Cl}(A)) = m\,\mathrm{Cl}(A)$.
(ii) $m\,\mathrm{Int}(m\,\mathrm{Int}(A)) = m\,\mathrm{Int}(A)$.
(iii) $m\,\mathrm{Int}(X \setminus A) = X \setminus m\,\mathrm{Cl}(A)$.
(iv) $m\,\mathrm{Cl}(X \setminus A) = X \setminus m\,\mathrm{Int}(A)$.
(v) *If $A \subset B$ then $m\,\mathrm{Cl}(A) \subset m\,\mathrm{Cl}(B)$.*
(vi) $m\,\mathrm{Cl}(A \cup B) \subset m\,\mathrm{Cl}(A) \cup m\,\mathrm{Cl}(B)$.
(vii) *$A \subset m\,\mathrm{Cl}(A)$ and $m\,\mathrm{Int}(A) \subset A$.*

Observe that any collection $\varnothing \neq \mathcal{J} \subset P(X)$ is always contained in an m-structure that have the property $\mathcal{B}$ [7]: A minimal structure m_X is said to have property $\mathcal{B}$ if the union of any family of subsets belonging to m_X belongs to m_X, that is, $m(\mathcal{J})=\{\varnothing, X\}\cup \{\bigcup_{M\in J} M : \varnothing \neq J \subset \mathcal{J}\}$.

Theorem 1.3. [10] *Let (X, m) be an m-space and m satisfy the property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(i) *$A \in m$ if and only if $m\,\mathrm{Int}(A)=A$.*
(ii) *A is m-closed if and only if $m\,\mathrm{Cl}(A)=A$.*
(iii) *$m\,\mathrm{Int}(A) \in m$ and $m\,\mathrm{Cl}(A)$ is m-closed.*

Definition 1.4. *A subset A of an m-space (X, m) is said to be αm-open* [8] *if $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(m\,\mathrm{Int}(A)))$.*
The complement of an αm-open set is called an αm-closed set.

Definition 1.5. [8] *Let (X, m) be an m-space and $A \subset X$.*

(i) *The intersection of all αm-closed sets containing A is called the αm-closure of A and is denoted by $\alpha m\,\mathrm{Cl}(S)$.*
(ii) *The union of all αm-open sets contained in A is called the αm-interior of A and is denoted by $\alpha m\,\mathrm{Int}(S)$.*

Definition 1.6. *A function $f : (X, m) \to (Y, \tau)$ is said to be αm-continuous* [8] *if the inverse image of every open set of Y is αm-open in (X, m).*

An m-space (X, m) with an ideal $\mathcal{I}$ on X is called an ideal minimal space and is denoted by $(X, m, \mathcal{I})$.

Definition 1.7. *A subset A of an ideal minimal space $(X, m, \mathcal{I})$ is said to be*

(i) *m-R-$\mathcal{I}$-open* [2] *if $A = m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))$.*
(ii) *m-semi-$\mathcal{I}$-open* [3] *if $A \subset m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A))$.*
(iii) *m-pre-$\mathcal{I}$-open* [1] *if $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))$.*
(iv) *m-β-$\mathcal{I}$-open* [4] *if $A \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)))$.*
(v) *m-δ-$\mathcal{I}$-open* [2] *if $m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A) \subset m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A))$.*

The complement of an m-pre-$\mathcal{I}$-open (resp. m-β-$\mathcal{I}$-open) set is called an m-pre-$\mathcal{I}$-closed (resp. m-β-$\mathcal{I}$-closed) set.

Lemma 1.8. *Let $(X, m, \mathcal{I})$ be an ideal minimal space and $A \subset X$. Then*

(i) *A subset A is m-pre-$\mathcal{I}$-closed if and only if $m\,\mathrm{Cl}(m\,\mathrm{Int}^*(A)) \subset A$* [1];
(ii) *A subset A is m-β-$\mathcal{I}$-closed if and only if $m\,\mathrm{Int}(m\,\mathrm{Cl}(m\,\mathrm{Int}^*(A)))$ $\subset A$* [4].

Definition 1.9. *A function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is said to be*

(i) *m-pre-$\mathcal{I}$-continuous* [1] *if the inverse image of every open set of Y is m-pre-$\mathcal{I}$-open in X.*
(ii) *m-semi-$\mathcal{I}$-continuous* [3] *if the inverse image of every open set of Y is m-semi-$\mathcal{I}$-open in X.*
(iii) *m-β-$\mathcal{I}$-continuous* [4] *if the inverse image of every open set of Y is m-β-$\mathcal{I}$-open in X.*
(iv) *m-δ-$\mathcal{I}$-continuous* [3] *if the inverse image of every open set of Y is m-δ-$\mathcal{I}$-open in X.*

2. m-α-$\mathcal{I}$-open sets

Definition 2.1. *A subset A of an ideal minimal space $(X, m, \mathcal{I})$ is said to be m-α-$\mathcal{I}$-open if and only if $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$. The family of all m-α-$\mathcal{I}$-open sets of $(X, m, \mathcal{I})$ is denoted by $\alpha\mathcal{I}O(X, m)$. Also, the family of all m-α-$\mathcal{I}$-open sets of $(X, m, \mathcal{I})$ containing x is denoted by $m\alpha\mathcal{I}O(X, x)$.*

Proposition 2.2. (i) *Every m-open set is m-α-$\mathcal{I}$-open.*
(ii) *Every m-α-$\mathcal{I}$-open set is m-semi-$\mathcal{I}$-open.*
(iii) *Every m-α-$\mathcal{I}$-open set is αm-open.*
(iv) *Every m-α-$\mathcal{I}$-open set is m-pre-$\mathcal{I}$-open.*

Proof. The proof follows from the definitions. □

The following examples show that the converses of Proposition 2.2 are not true in general.

Example 2.3. *Let $X = \{a, b, c\}$ $m = \{\emptyset, \{a\}, \{b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the set $\{a, b\}$ is m-α-$\mathcal{I}$-open but not m-open, the set $\{b, c\}$ is m-semi-$\mathcal{I}$-open but not m-α-$\mathcal{I}$-open.*

Example 2.4. *Let $X = \{a, b, c\}$ $m = \{\emptyset, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the set $\{b, c\}$ is m-pre-$\mathcal{I}$-open but not m-α-$\mathcal{I}$-open.*

Example 2.5. *Let $X = \{a, b, c\}$ $m = \{\emptyset, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the set $\{a, b\}$ is αm-$\mathcal{I}$-open but not m-α-$\mathcal{I}$-open.*

Proposition 2.6. *Let $(X, m, \{\emptyset\})$ be an ideal minimal space and $A \subset X$. Then A is m-α-$\mathcal{I}$-open if and only if it is αm-open.*

Proof. The proof follows from the fact that, if $\mathcal{I} = \{\emptyset\}$, then $A^*_m = m\,\mathrm{Cl}(A)$ and $m\,\mathrm{Cl}^*(A) = m\,\mathrm{Cl}(A)$ by Remark 2.3 of [11]. □

Proposition 2.7. *Let A be a subset of an ideal minimal space $(X, m, \mathcal{I})$. If B is an m-semi-$\mathcal{I}$-open set of X such that $B \subset A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(B))$, then A is an m-α-$\mathcal{I}$-open set of X.*

Proof. Since B is an m-semi-$\mathcal{I}$-open set of X, $B \subset m\,\mathrm{Cl}^*(m\,\mathrm{Int}(B))$. Thus, $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(B)) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(B)))) = m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(B))) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$, and so A is an m-α-$\mathcal{I}$-open set of X. □

Proposition 2.8. *Let $(X, m, \mathcal{I})$ be an ideal minimal space. Then a subset of X is m-α-$\mathcal{I}$-open if and only if it is both m-δ-$\mathcal{I}$-open and m-pre-$\mathcal{I}$-open.*

Proof. Let A be an m-α-$\mathcal{I}$-open set. By Proposition 2.2, every m-α-$\mathcal{I}$-open set is m-semi-$\mathcal{I}$-open and m-pre-$\mathcal{I}$-open. Hence, we have $m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Cl}^*(Int(A)))) \subset m\,\mathrm{Cl}^*(Int(A))$. Hence A is an m-δ-$\mathcal{I}$-open. Conversely, let A be an m-δ-$\mathcal{I}$-open and m-pre-$\mathcal{I}$-open set. Then we have $m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)) \subset m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A))$ and hence $m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$. Since A is m-pre-$\mathcal{I}$-open, $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))$. Therefore, we obtain that $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$; hence A is m-α-$\mathcal{I}$-open. □

Lemma 2.9. *A subset A is m-α-$\mathcal{I}$-open if and only if m-semi-$\mathcal{I}$-open and m-pre-$\mathcal{I}$-open.*

Proof. Let A be m-semi-$\mathcal{I}$-open and m-pre-$\mathcal{I}$-open subset of $(X, m, \mathcal{I})$. Then, $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A)) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))) = m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$. Hence A is m-α-$\mathcal{I}$-open. The converse is obvious. □

Corollary 2.10. *The following properties are equivalent for subsets of an ideal minimal space $(X, m, \mathcal{I})$:*

(i) *Every m-pre-$\mathcal{I}$-open set is m-semi-$\mathcal{I}$-open.*
(ii) *A subset A of X is m-α-$\mathcal{I}$-open if and only if it is m-pre-$\mathcal{I}$-open.*

Corollary 2.11. *The following properties are equivalent for subsets of an ideal minimal space $(X, m, \mathcal{I})$:*

(i) *Every m-semi-$\mathcal{I}$-open set is m-pre-$\mathcal{I}$-open.*
(ii) *A subset A of X is m-α-$\mathcal{I}$-open if and only if it is m-semi-$\mathcal{I}$-open.*

Proposition 2.12. *Let A be a subset of an ideal minimal space $(X, m, \mathcal{I})$ and m satisfy the property of $\mathcal{B}$. If A is m-pre-$\mathcal{I}$-closed and m-α-$\mathcal{I}$-open, then it is m-open.*

Proof. Suppose A is m-pre-$\mathcal{I}$-closed and m-α-$\mathcal{I}$-open. Then by Lemma 1.8 $m\,\mathrm{Cl}(m\,\mathrm{Int}^*(A)) \subset A$ and $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$. Now

$m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)) \subset m\,\mathrm{Cl}(m\,\mathrm{Int}(A)) \subset m\,\mathrm{Cl}(m\,\mathrm{Int}^*(A)) \subset A$ and so $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)) \subset m\,\mathrm{Int}(A)$. Therefore, A is m-open. □

Lemma 2.13. [2] *If A is any subset of an ideal minimal space $(X, m, \mathcal{I})$, then $m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))$ is m-R-$\mathcal{I}$-open.*

Proposition 2.14. *Let A be a subset of an ideal minimal space $(X, m, \mathcal{I})$. If A is m-α-$\mathcal{I}$-open and m-β-$\mathcal{I}$-closed, then it is m-R-$\mathcal{I}$-open.*

Proof. Let A be an m-α-$\mathcal{I}$-open and m-β-$\mathcal{I}$-closed subset of $(X, m, \mathcal{I})$. By Lemma 1.8, $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$ and $m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A))) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(m\,\mathrm{Int}^*(A))) \subset A$; hence $A = m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A)))$. Thus, by Lemma 2.13, A is m-R-$\mathcal{I}$-open. □

Remark 2.15. *The intersection of two m-α-$\mathcal{I}$-open sets need not be m-α-$\mathcal{I}$-open as it can be seen from the following example.*

Example 2.16. *Let $X = \{a, b, c\}$, $m = \{\varnothing, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then the sets $\{a, b\}$ and $\{a, c\}$ are m-α-$\mathcal{I}$-open sets of $(X, m, \mathcal{I})$ but their intersection $\{a\}$ is not an m-α-$\mathcal{I}$-open set of $(X, m, \mathcal{I})$.*

Theorem 2.17. *If $\{A_\alpha\}_{\alpha\in\Omega}$ be a family of m-α-$\mathcal{I}$-open sets in $(X, m, \mathcal{I})$, then $\bigcup_{\alpha\in\Omega} A_\alpha$ is m-α-$\mathcal{I}$-open in $(X, m, \mathcal{I})$.*

Proof. Since $\{A_\alpha : \alpha \in \Omega\} \subset m\alpha\mathcal{I}O(X)$, $A_\alpha \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A_\alpha)))$ for every $\alpha \in \Omega$. Thus, $\cup_{\alpha\in\Omega} A_\alpha \subset \cup_{\alpha\in\Omega} m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(A_\alpha))) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(\cup_{\alpha\in\Omega} A_\alpha)))$ and $\cup_{\alpha\in\Omega} A_\alpha \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(\cup_{\alpha\in\Omega} A_\alpha))$. Hence any union of m-α-$\mathcal{I}$-open sets is m-α-$\mathcal{I}$-open. □

Definition 2.18. *In an ideal minimal space $(X, m, \mathcal{I})$, $A \subset X$ is said to be m-α-$\mathcal{I}$-closed if $X\backslash A$ is m-α-$\mathcal{I}$-open in X.*
The family of all m-α-$\mathcal{I}$-closed sets of $(X, m, \mathcal{I})$ is denoted by $\alpha\mathcal{I}C(X, m)$.

Theorem 2.19. *Let $(X, m, \mathcal{I})$ be an ideal minimal space. Then, A is m-α-$\mathcal{I}$-closed if and only if $m\,\mathrm{Cl}(m\,\mathrm{Int}^*(m\,\mathrm{Cl}(A))) \subset A$.*

Proof. The proof follows from the definitions. □

Theorem 2.20. *If A is an m-α-$\mathcal{I}$-closed set in an ideal minimal space $(X, m, \mathcal{I})$, then $m\,\mathrm{Cl}(m\,\mathrm{Int}(m\,\mathrm{Cl}^*(A))) \subset A$.*

Proof. It follows from Theorem 2.19 that $m\operatorname{Cl}(m\operatorname{Int}(m\operatorname{Cl}^*(A))) \subset m\operatorname{Cl}(m\operatorname{Int}^*(m\operatorname{Cl}(A))) \subset A$. □

Theorem 2.21. *Arbitrary intersection of m-α-$\mathcal{I}$-closed sets is always m-α-$\mathcal{I}$-closed.*

Proof. This follows from Theorems 2.17. □

Definition 2.22. *Let $(X, m, \mathcal{I})$ be an ideal minimal space, S a subset of X and x be a point of X. Then*

(i) *x is called an m-α-$\mathcal{I}$-interior point of S if there exists $V \in \alpha\mathcal{I}O(X, m)$ such that $x \in V \subset S$.*
(ii) *the set of all m-α-$\mathcal{I}$-interior points of S is called the m-α-$\mathcal{I}$-interior of S and is denoted by $m\alpha\mathcal{I}\operatorname{Int}(S)$.*

Theorem 2.23. *Let A and B be subsets of $(X, m, \mathcal{I})$. Then the following properties hold:*

(i) $m\alpha\mathcal{I}\operatorname{Int}(A) = \cup\{T : T \subset A$ *and* $T \in \alpha\mathcal{I}O(X, m)\}$.
(ii) $m\alpha\mathcal{I}\operatorname{Int}(A)$ *is the largest m-α-$\mathcal{I}$-open subset of X contained in A.*
(iii) *A is m-α-$\mathcal{I}$-open if and only if $A = m\alpha\mathcal{I}\operatorname{Int}(A)$.*
(iv) $m\alpha\mathcal{I}\operatorname{Int}(m\alpha\mathcal{I}\operatorname{Int}(A)) = m\alpha\mathcal{I}\operatorname{Int}(A)$.
(v) *If $A \subset B$, then $m\alpha\mathcal{I}\operatorname{Int}(A) \subset m\alpha\mathcal{I}\operatorname{Int}(B)$.*
(vi) $m\alpha\mathcal{I}\operatorname{Int}(A) \cup m\alpha\mathcal{I}\operatorname{Int}(B) \subset m\alpha\mathcal{I}\operatorname{Int}(A \cup B)$.
(vii) $m\alpha\mathcal{I}\operatorname{Int}(A \cap B) \subset m\alpha\mathcal{I}\operatorname{Int}(A) \cap m\alpha\mathcal{I}\operatorname{Int}(B)$.

Proof. (i). Let $x \in \cup\{T : T \subset A$ and $T \in \alpha\mathcal{I}O(X, m)\}$. Then, there exists $T \in \alpha\mathcal{I}O(X, x)$ such that $x \in T \subset A$ and hence $x \in m\alpha\mathcal{I}\operatorname{Int}(A)$. This shows that $\cup\{T : T \subset A$ and $A \in \alpha\mathcal{I}O(X, m)\} \subset m\alpha\mathcal{I}\operatorname{Int}(A)$. For the reverse inclusion, let $x \in m\alpha\mathcal{I}\operatorname{Int}(A)$. Then there exists $T \in m\alpha\mathcal{I}O(X, x)$ such that $x \in T \subset A$. we obtain $x \in \cup\{T : T \subset A$ and $T \in \alpha\mathcal{I}O(X, m)\}$. This shows that $m\alpha\mathcal{I}\operatorname{Int}(A) \subset \cup\{T : T \subset A$ and $T \in \alpha\mathcal{I}O(X, m)\}$. Therefore, we obtain $m\alpha\mathcal{I}\operatorname{Int}(A) = \cup\{T : T \subset A$ and $T \in \alpha\mathcal{I}O(X, m)\}$.
The proofs of $(ii) - (vii)$ are obvious.

□

Corollary 2.24 ([8], Theorem 3.8)**.** *Let A and B be subsets of (X, m). Then the following properties hold:*

(i) $\alpha m\operatorname{Int}(A) \subset A$.
(ii) *A is αm-open if and only if $A = \alpha m\operatorname{Int}(A)$.*
(iii) $\alpha m\operatorname{Int}(\alpha m\operatorname{Int}(A)) = \alpha m\operatorname{Int}(A)$.
(iv) *If $A \subset B$, then $\alpha m\operatorname{Int}(A) \subset \alpha m\operatorname{Int}(B)$.*

Proof. The proof follows from Theorem 2.23, if $\mathcal{I} = \{\emptyset\}$. □

Definition 2.25. *Let $(X, m, \mathcal{I})$ be an ideal minimal space, S a subset of X and x be a point of X. Then*

(i) *x is called an m-α-$\mathcal{I}$-cluster point of S if $V \cap S \neq \emptyset$ for every $V \in m\alpha\mathcal{I}O(X, x)$.*
(ii) *the set of all m-α-$\mathcal{I}$-cluster points of S is called the m-α-$\mathcal{I}$-closure of S and is denoted by $m\alpha\mathcal{I}\,\mathrm{Cl}(S)$.*

Theorem 2.26. *Let A and B be subsets of $(X, m, \mathcal{I})$. Then the following properties hold:*

(i) $m\alpha\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F \text{ and } F \in \alpha\mathcal{I}C(X, m)\}$.
(ii) *$m\alpha\mathcal{I}\,\mathrm{Cl}(A)$ is the smallest m-α-$\mathcal{I}$-closed subset of X containing A.*
(iii) *A is m-α-$\mathcal{I}$-closed if and only if $A = m\alpha\mathcal{I}\,\mathrm{Cl}(A)$.*
(iv) $m\alpha\mathcal{I}\,\mathrm{Cl}(m\alpha\mathcal{I}\,\mathrm{Cl}(A) = m\alpha\mathcal{I}\,\mathrm{Cl}(A)$.
(v) *If $A \subset B$, then $m\alpha\mathcal{I}\,\mathrm{Cl}(A) \subset m\alpha\mathcal{I}\,\mathrm{Cl}(B)$.*
(vi) $m\alpha\mathcal{I}\,\mathrm{Cl}(A \cup B) = m\alpha\mathcal{I}\,\mathrm{Cl}(A) \cup m\alpha\mathcal{I}\,\mathrm{Cl}(B)$.
(vii) $m\alpha\mathcal{I}\,\mathrm{Cl}(A \cap B) \subset m\alpha\mathcal{I}\,\mathrm{Cl}(A) \cap m\alpha\mathcal{I}\,\mathrm{Cl}(B)$.

Proof. (i). Suppose that $x \notin m\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Then there exists $V \in m\alpha\mathcal{I}O(X, x)$ such that $V \cap A = \emptyset$. Since $X \backslash V$ is an m-α-$\mathcal{I}$-closed set containing A and $x \notin X \backslash V$, we obtain $x \notin \cap\{F : A \subset F$ and $F \in \alpha\mathcal{I}C(X, m)\}$. Conversely, suppose that $x \notin \cap\{F \mid A \subset F \text{ and } F \in \alpha\mathcal{I}C(X, m)\}$. Then there exists $F \in \alpha\mathcal{I}C(X, m)$ such that $A \subset F$ and $x \notin F$. Since $X \backslash F$ is an m-α-$\mathcal{I}$-open set containing x, we obtain $(X \backslash F) \cap A = \emptyset$. This shows that $x \notin m\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain $m\alpha\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F$ and $F \in \alpha\mathcal{I}C(X, m)\}$.
The other proofs are obvious. □

Corollary 2.27 ([8], Theorem 3.9)**.** *Let A and B be subsets of (X, m). Then the following properties hold:*

(i) $A \subset \alpha m\,\mathrm{Cl}(A)$.
(ii) *A is αm-closed if and only if $A = \alpha m\,\mathrm{Cl}(A)$.*
(iii) $\alpha m\,\mathrm{Cl}(\alpha m\,\mathrm{Cl}(A) = \alpha m\,\mathrm{Cl}(A)$.
(iv) *If $A \subset B$, then $\alpha m\,\mathrm{Cl}(A) \subset \alpha m\,\mathrm{Cl}(B)$.*

Proof. The proof follows from Theorem 2.26, if $\mathcal{I} = \{\emptyset\}$. □

Theorem 2.28. *Let $(X, m, \mathcal{I})$ be an ideal minimal space and $A \subset X$. Then a point $x \in m\alpha\mathcal{I}\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m\alpha\mathcal{I}O(X, x)$.*

Proof. This follows immediately from Definition 2.25. □

Corollary 2.29 ([8], Theorem 3.10). *Let (X, m) be an ideal minimal space and $A \subset X$. Then*

(i) *$x \in \alpha m\,\mathrm{Cl}(A)$ if and only if $A \cap V \neq \emptyset$ for every αm-open set V containing x.*
(ii) *$x \in \alpha m\,\mathrm{Int}(A)$ if and only if there exists an αm-open set U such that $x \in U \subset A$.*

Proof. The proof follows from Theorem 2.28, if $\mathcal{I} = \{\emptyset\}$. □

Theorem 2.30. *Let $(X, m, \mathcal{I})$ be an ideal minimal space and $A \subset X$. Then the following propeties hold:*

(i) *$m\alpha\mathcal{I}\,\mathrm{Int}(X \backslash A) = X \backslash m\alpha\mathcal{I}\,\mathrm{Cl}(A)$;*
(ii) *$m\alpha\mathcal{I}\,\mathrm{Cl}(X \backslash A) = X \backslash m\alpha\mathcal{I}\,\mathrm{Int}(A)$.*

Proof. (i). Let $x \in X \setminus m\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Since $x \notin m\alpha\mathcal{I}\,\mathrm{Cl}(A)$, there exists $V \in m\alpha\mathcal{I}O(X, x)$ such that $V \cap A = \emptyset$; hence we obtain $x \in m\alpha\mathcal{I}\,\mathrm{Int}(X \backslash A)$. This shows that $X \backslash m\alpha\mathcal{I}\,\mathrm{Cl}(A) \subset m\alpha\mathcal{I}\,\mathrm{Int}(X \backslash A)$. Let $x \in m\alpha\mathcal{I}\,\mathrm{Int}(X \backslash A)$. Since $m\alpha\mathcal{I}\,\mathrm{Int}(X \backslash A) \cap A = \emptyset$, we obtain $x \notin m\alpha\mathcal{I}\,\mathrm{Cl}(A)$; hence $x \in X \backslash m\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain $m\alpha\mathcal{I}\,\mathrm{Int}(X \backslash A) = X \backslash m\alpha\mathcal{I}\,\mathrm{Cl}(A)$.
(ii). This follows from (i). □

Corollary 2.31 ([8], Theorem 3.8(v)). *Let (X, m) be an ideal minimal space and $A \subset X$. Then the following propeties hold:*

(i) *$\alpha m\,\mathrm{Int}(X \backslash A) = X \backslash \alpha m\,\mathrm{Cl}(A)$;*
(ii) *$\alpha m\,\mathrm{Cl}(X \backslash A) = X \backslash \alpha m\,\mathrm{Int}(A)$.*

Proof. The proof follows from Theorem 2.30, if $\mathcal{I} = \{\emptyset\}$. □

Definition 2.32. *A subset B_x of an ideal minimal space $(X, m, \mathcal{I})$ is called an m-α-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists an m-α-$\mathcal{I}$-open set U such that $x \in U \subset B_x$.*

Theorem 2.33. *A subset of an ideal minimal space $(X, m, \mathcal{I})$ is m-α-$\mathcal{I}$-open if and only if it is an m-α-$\mathcal{I}$-neighbourhood of each of its points.*

Proof. Let G be an m-α-$\mathcal{I}$-open set of X. Then by definition, it is clear that G is an m-α-$\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is m-α-$\mathcal{I}$-open. Conversely, suppose G is an m-α-$\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in \alpha\mathcal{I}O(X, m)$ such that $S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each S_x is m-α-$\mathcal{I}$-open, G is m-α-$\mathcal{I}$-open in $(X, m, \mathcal{I})$. □

3. m-α-$\mathcal{I}$-continuous functions

Definition 3.1. *A function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is said to be m-α-$\mathcal{I}$-continuous if the inverse image of every open set of Y is m-α-$\mathcal{I}$-open in X.*

Proposition 3.2. *For a function $f : (X, m, \mathcal{I}) \to (Y, \tau)$, the following properties hold:*

(i) *Every m-α-$\mathcal{I}$-continuous function is m-semi-$\mathcal{I}$-continuous but not conversely.*
(ii) *Every m-α-$\mathcal{I}$-continuous function is αm-continuous but not conversely.*
(iii) *Every m-α-$\mathcal{I}$-continuous function is m-pre-$\mathcal{I}$-continuous but not conversely.*

Proof. The proof follows from Proposition 2.2, Examples 2.3 and 2.4. □

Theorem 3.3. *A function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is m-α-$\mathcal{I}$-continuous if and only if it is m-semi-$\mathcal{I}$-continuous and m-pre-$\mathcal{I}$-continuous.*

Proof. This is an immediate consequence of Lemma 2.9. □

Theorem 3.4. *For a function $f : (X, m, \mathcal{I}) \to (Y, \tau)$, the following statements are equivalent:*

(i) *f is m-α-$\mathcal{I}$-continuous;*
(ii) *For each point x in X and each open set F in Y such that $f(x) \in F$, there is an m-α-$\mathcal{I}$-open set A in X such that $x \in A$, $f(A) \subset F$;*
(iii) *The inverse image of each closed set in Y is m-α-$\mathcal{I}$-closed in X;*
(iv) *For each subset A of X, $f(m\alpha\mathcal{I}\operatorname{Cl}(A)) \subset \operatorname{Cl}(f(A))$;*
(v) *For each subset B of Y, $m\alpha\mathcal{I}\operatorname{Cl}(f^{-1}(B)) \subset f^{-1}(\operatorname{Cl}(B))$;*
(vi) *For each subset C of Y, $f^{-1}(\operatorname{Int}(C)) \subset m\alpha\mathcal{I}\operatorname{Int}(f^{-1}(C))$.*
(vii) *$m\operatorname{Cl}(m\operatorname{Int}^*(m\operatorname{Cl}(f^{-1}(B)))) \subset f^{-1}(\operatorname{Cl}(B))$ for each subset B of Y.*
(viii) *$f(m\operatorname{Cl}(m\operatorname{Int}^*(m\operatorname{Cl}(A)))) \subset \operatorname{Cl}(f(A))$ for each subset A of X.*

Proof. *(i)* ⇔ *(ii)*: Let $x \in X$ and F be an open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is m-α-$\mathcal{I}$-open in X. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$. Conversely, let F be open in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (ii), there is an m-α-$\mathcal{I}$-open set U_x in X such that $x \in U_x$ and $f(U_x) \subset F$. Then $x \in U_x \subset f^{-1}(F)$ and $f^{-1}(F) = \cup\{U_x \mid x \in f^{-1}(F)\}$. Hence $f^{-1}(F)$ is m-α-$\mathcal{I}$-open in X.

$(i) \Rightarrow (iii)$: This follows due to the fact that for any subset B of Y, $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$.
$(iii) \Rightarrow (iv)$: Let A be a subset of X. Since $\mathrm{Cl}(f(A))$ is closed in Y and by (iii) $f^{-1}(\mathrm{Cl}(f(A)))$ is m-α-$\mathcal{I}$-closed in X and $A \subset f^{-1}(\mathrm{Cl}(f(A))$. Then $m\alpha\mathcal{I}\,\mathrm{Cl}(A) \subset f^{-1}(\mathrm{Cl}(A))$; hence $f(m\alpha\mathcal{I}\,\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A))$.
$(iv) \Rightarrow (v)$: Let B be any subset of Y. Now, $f(m\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)))$ $\subset \mathrm{Cl}(f(f^{-1}(B))) \subset \mathrm{Cl}(B)$. Consequently, $m\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$.
$(i) \Leftrightarrow (vi)$: Suppose that f is m-α-$\mathcal{I}$-continuous. Let B be any subset of Y. Clearly, $f^{-1}(\mathrm{Int}(B))$ is m-α-$\mathcal{I}$-open in X and we have $f^{-1}(\mathrm{Int}(B)) \subset m\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}\,\mathrm{Int}(B)) \subset m\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}B)$. Conversely, let B be an open set in Y. Then $\mathrm{Int}(B) = B$ and $f^{-1}(B) \subset f^{-1}(\mathrm{Int}(B)) \subset m\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$. Hence we have $f^{-1}(B) = m\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is m-α-$\mathcal{I}$-open in X.
$(v) \Rightarrow (vii)$: Let B any subset of Y. Since $m\alpha\,\mathrm{Cl}(f^{-1}(B))$ is m-α-$\mathcal{I}$-closed, by Theorem 2.19 and (v), $m\,\mathrm{Cl}(m\,\mathrm{Int}^*(m\,\mathrm{Cl}(f^{-1}(B)))) \subset m\,\mathrm{Cl}(m\,\mathrm{Int}^*(m\,\mathrm{Cl}(m\alpha\,\mathrm{Cl}(f^{-1}(B))))) \subset m\alpha\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$.
$(vii) \Rightarrow (viii)$: Let A be any subset of X. By (vii), $m\,\mathrm{Cl}(m\,\mathrm{Int}^*(m\,\mathrm{Cl}(A)) \subset m\,\mathrm{Cl}(m\,\mathrm{Int}^*(m\,\mathrm{Cl}(f^{-1}(f(A))))) \subset f^{-1}(\mathrm{Cl}(f(A)))$ and hence
$f(m\,\mathrm{Cl}(m\,\mathrm{Int}^*(m\,\mathrm{Cl}(A)))) \subset \mathrm{Cl}(f(A))$.
$(viii) \Rightarrow (i)$: Let $V \in \tau$. Then by (v), $f(m\,\mathrm{Cl}(m\,\mathrm{Int}^*(m\,\mathrm{Cl}(f^{-1}(Y\backslash V))))) \subset \mathrm{Cl}(f(f^{-1}(Y\backslash V))) \subset \mathrm{Cl}(Y\backslash V) = Y\backslash V$. It follows that,
$m\,\mathrm{Cl}(m\,\mathrm{Int}^*(m\,\mathrm{Cl}(f^{-1}(Y\backslash V)))) \subset f^{-1}(Y\backslash V) \subset X\backslash f^{-1}(V)$. Consequently, we obtain $f^{-1}(V) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(f^{-1}(V))))$. This shows that $f^{-1}(V)$ is m-α-$\mathcal{I}$-open. Thus, f is m-α-$\mathcal{I}$-continuous. □

Theorem 3.5. *Let $f : (X, m, \mathcal{I}) \to (Y, \tau)$ be an m-α-$\mathcal{I}$-continuous function. Then for each subset V of Y, $f^{-1}(\mathrm{Int}(V)) \subset m\,\mathrm{Cl}^*(f^{-1}(V))$.*

Proof. Let V be any subset of Y. Then $f^{-1}(\mathrm{Int}(V))$ is m-α-$\mathcal{I}$-open in X. Hence $f^{-1}(\mathrm{Int}(V)) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}^*(m\,\mathrm{Int}(f^{-1}(\mathrm{Int}(V))))) \subset$
$m\,\mathrm{Cl}^*(f^{-1}(V))$. □

Theorem 3.6. *Let $f : (X, m, \mathcal{I}) \to (Y, \tau)$ be a bijection. Then f is m-α-$\mathcal{I}$-continuous if and only if $\mathrm{Int}(f(U)) \subset f(m\alpha\mathcal{I}\,\mathrm{Int}(U))$ for each subset U of X.*

Proof. Let $U \subset X$. By Theorem 3.4, $f^{-1}(\mathrm{Int}(f(U))) \subset m\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(f(U)))$. Since f is a bijection, $\mathrm{Int}(f(U)) = f(f^{-1}(\mathrm{Int}(f(U))) \subset f(m\alpha\mathcal{I}\,\mathrm{Int}(U))$. Conversely, let $V \subset Y$.

Then $\mathrm{Int}(f(f^{-1}(V))) \subset f(m\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V)))$. Since f is a bijection, $\mathrm{Int}(V) = \mathrm{Int}(f(f^{-1}(V))) \subset f(m\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V)))$; hence $f^{-1}(\mathrm{Int}(V)) \subset m\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$. Therefore, by Theorem 3.4, f is m-α-$\mathcal{I}$-continuous. □

Proposition 3.7. *A function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is m-α-$\mathcal{I}$-continuous if and only if it is both m-δ-$\mathcal{I}$-continuous and m-pre-$\mathcal{I}$-continuous.*

Proof. The proof follows from Proposition 2.8. □

Definition 3.8. *The graph $G(f)$ of a function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is said to be m-α-$\mathcal{I}$-closed in $X \times Y$ if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in m\alpha\mathcal{I}O(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 3.9. *The graph of a function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is m-α-$\mathcal{I}$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in m\alpha\mathcal{I}O(X, x)$ and an open set V of Y containing y such that $f(U) \cap V = \emptyset$.*

Proof. The proof is an immediate consequence of Definition 3.8. □

Theorem 3.10. *If $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is an m-α-$\mathcal{I}$-continuous function and (Y, τ) is T_2, then $G(f)$ is m-α-$\mathcal{I}$-closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since Y is T_2, there exist disjoint open sets V and W of Y such that $f(x) \in W$ and $y \in V$. Since f is m-α-$\mathcal{I}$-continuous, there exists $U \in m\alpha\mathcal{I}O(X, x)$ such that $f(U) \subset W$. Therefore, $f(U) \cap V = \emptyset$. Therefore, by Lemma 3.9, $G(f)$ is m-α-$\mathcal{I}$-closed. □

Definition 3.11. *An ideal minimal space $(X, m, \mathcal{I})$ is called an m-α-$\mathcal{I}$-T_2 space if for each pair of distinct points $x, y \in X$, there exist $U, V \in \alpha\mathcal{I}O(X, m)$ containing x and y, respectively, such that $U \cap V = \emptyset$.*

Definition 3.12. *An m-space (X, m) is said to be m-T_2 [10] if for any distinct points x, y of X, there exist U, $V \in m$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Theorem 3.13. *Let $(X, m, \mathcal{I})$ be an ideal minimal space and m have property $\mathcal{B}$. Then $(X, m, \mathcal{I})$ is m-T_2 if and only if it m-α-$\mathcal{I}$-T_2.*

Proof. It is obvious that every m-T_2 space is m-α-$\mathcal{I}$-T_2 since $m \subset \alpha\mathcal{I}O(X, m)$. Suppose that $(X, m, \mathcal{I})$ is m-α-$\mathcal{I}$-T_2. For any distinct points x, $y \in X$, there exist U, $V \in \alpha\mathcal{I}O(X, m)$ such that $x \in U$,

$y \in V$ and $U \cap V = \emptyset$. Since $U \cap V = \emptyset$, $mInt(V) \cap mInt(V) = \emptyset$. Since m has property $\mathcal{B}$, by Theorem 1.3 $mInt(U) \in m$ and $m \subset m^*(\mathcal{I}, m)$. Therefore, we obtain $mInt(U) \cap m\,\mathrm{Cl}^*(mInt(V)) = \emptyset$ and hence $mInt(U) \cap mInt(m\,\mathrm{Cl}^*(mInt(V))) = \emptyset$. By repeating the same argument, we obtain $mInt(m\,\mathrm{Cl}^*(mInt(U))) \cap mInt(m\,\mathrm{Cl}^*(mInt(V))) = \emptyset$. Now, $U, V \in \alpha\mathcal{I}O(X, m)$ and hence we have $x \in U \subset mInt(m\,\mathrm{Cl}^*(mInt(U)))$
$\in m$ and $y \in V \subset mInt(m\,\mathrm{Cl}^*(mInt(V))) \in m$. This shows that (X, m, I) is m-T_2. □

Theorem 3.14. *If $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is an m-α-$\mathcal{I}$-continuous injective function and Y is a T_2 space, then $(X, m, \mathcal{I})$ is an m-α-$\mathcal{I}$-T_2 space.*

Proof. The proof follows from the definitions 3.11 and 3.1. □

Theorem 3.15. *If $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is an injective m-α-$\mathcal{I}$-continuous function with an m-α-$\mathcal{I}$-closed graph, then X is an m-α-$\mathcal{I}$-T_2 space.*

Proof. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y)\backslash G(f)$. Since the graph $G(f)$ is m-α-$\mathcal{I}$-closed, there exist an m-α-$\mathcal{I}$-open set U containing x_1 and $V \in \tau$ containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since f is m-α-$\mathcal{I}$-continuous, $f^{-1}(V)$ is an m-α-$\mathcal{I}$-open set containing x_2 such that $U \cap f^{-1}(V) = \emptyset$. Hence X is m-α-$\mathcal{I}$-T_2. □

Definition 3.16. *An ideal minimal space $(X, m, \mathcal{I})$ is said to be m-α-$\mathcal{I}$-connected if X cannot be expressed as the union of two nonempty disjoint m-α-$\mathcal{I}$-open sets.*

Theorem 3.17. *A m-α-$\mathcal{I}$-continuous image of an m-α-$\mathcal{I}$-connected space is connected.*

Proof. Obvious. □

Lemma 3.18. [9] *For any function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, $f(\mathcal{I})$ is an ideal on Y.*

Definition 3.19. *A subset K of an ideal minimal space $(X, m, \mathcal{I})$ is said to be m-α-$\mathcal{I}$-compact relative to X, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of K by m-α-$\mathcal{I}$-open sets of X, there exists a finite subset Λ_0 of Λ such that $K \backslash \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X, m, \mathcal{I})$ is said to be m-α-$\mathcal{I}$-compact if X is m-α-$\mathcal{I}$-compact relative to X.*

Definition 3.20. *A subset K of an ideal minimal space $(X, m, \mathcal{I})$ is said to be countably m-α-$\mathcal{I}$-compact relative to X, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of K by countable m-α-$\mathcal{I}$-open sets of X, there exists a finite subset Λ_0 of Λ such that $K \setminus \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X, m, \mathcal{I})$ is said to be countably m-α-$\mathcal{I}$-compact if X is countable m-α-$\mathcal{I}$-compact relative to X.*

Definition 3.21. *A subset K of an ideal minimal space $(X, m, \mathcal{I})$ is said to be m-α-$\mathcal{I}$-Lindelöf relative to X, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of K by m-α-$\mathcal{I}$-open sets of X, there exists a countable subset Λ_0 of Λ such that $K \setminus \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X, m, \mathcal{I})$ is said to be m-α-$\mathcal{I}$-Lindelöf if X is m-α-$\mathcal{I}$-Lindelöf subset of X.*

Theorem 3.22. *If $f : (X, m, \mathcal{I}) \to (Y, \sigma)$ is an m-α-$\mathcal{I}$-continuous surjection and $(X, m, \mathcal{I})$ is m-α-$\mathcal{I}$-compact, then $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$-compact.*

Proof. Let $\{V_\lambda : \lambda \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is an m-α-$\mathcal{I}$-open cover of X and hence, there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup\{f^{-1}(V_\lambda) : \lambda \in \Lambda_0\} \in \mathcal{I}$. Since f is surjective, $Y \setminus \bigcup\{V_\lambda : \lambda \in \Lambda_0\} = f(X \setminus \bigcup\{f^{-1}(V_\lambda) : \lambda \in \Lambda_0\}) \in f(\mathcal{I})$. Therefore, $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$-compact. □

The proofs of the next two theorems are straight forward, we therefore omit them.

Theorem 3.23. *If $f : (X, m, \mathcal{I}) \to (Y, \sigma)$ is an m-α-$\mathcal{I}$-continuous surjection and $(X, m, \mathcal{I})$ is m-α-$\mathcal{I}$-Lindelöf, then $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$-Lindelöf.*

Theorem 3.24. *If $f : (X, m, \mathcal{I}) \to (Y, \sigma)$ is an m-α-$\mathcal{I}$-continuous surjection and $(X, m, \mathcal{I})$ is countably m-α-$\mathcal{I}$-compact, then $(Y, \sigma, f(\mathcal{I}))$ is countably $f(\mathcal{I})$-compact.*

4. m-α-$\mathcal{I}$-IRRESOLUTE FUNCTIONS

Definition 4.1. *A function $f : (X, m_1, \mathcal{I}) \to (Y, m_2, \mathcal{J})$ is said to be (m_1, m_2)-α-$\mathcal{I}$-irresolute if the inverse image of every m_2-α-$\mathcal{J}$-open set of Y is m_1-α-$\mathcal{I}$-open in X.*

Theorem 4.2. *Let $f : (X, m_1, \mathcal{I}) \to (Y, m_2, \mathcal{J})$ be a function, then the following properties are eequivalent:*

(i) *f is (m_1, m_2)-α-$\mathcal{I}$-irresolute;*

(ii) *the inverse image of each m_2-α-$\mathcal{J}$-closed subset of Y is m_1-α-$\mathcal{I}$-closed in X;*
(iii) *for each $x \in X$ and each $V \in \alpha\mathcal{J}O(Y, m_2)$ containing $f(x)$, there exists $U \in \alpha\mathcal{I}O(X, m_1)$ containing x such that $f(U) \subset V$.*

Proof. The proof is obvious from that fact that the arbitrary union of m-α-$\mathcal{I}$-open subsets is m-α-$\mathcal{I}$-open. □

Theorem 4.3. *Let $f : (X, m_1, \mathcal{I}) \to (Y, m_2, \mathcal{J})$ be a function. Then the following properties are equivalent:*
(i) *f is (m_1, m_2)-α-$\mathcal{I}$-irresolute;*
(ii) *$m_1\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Cl}(V))$ for each subset V of Y;*
(iii) *$f(m_1\alpha\mathcal{I}\,\mathrm{Cl}(U)) \subset m_2\alpha\mathcal{J}\,\mathrm{Cl}(f(U))$ for each subset U of X.*

Proof. *(i)* ⇒ *(ii)*: Let V be any subset of Y. By (i), $f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Cl}(V))$ is an m_1-α-$\mathcal{I}$-closed subset of X. Hence we have $m_1\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset m_1\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Cl}(V))) = f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Cl}(V))$.
(ii) ⇒ *(iii)*: Let U be any subset of X. Then $f(U) \subset m_2\alpha\mathcal{J}\,\mathrm{Cl}(f(U))$ and $m_1\alpha\mathcal{I}\,\mathrm{Cl}(U) \subset m_1\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(f(U))) \subset f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Cl}(f(U)))$. Then $f(m_1\alpha\mathcal{I}\,\mathrm{Cl}(U)) \subset f(f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Cl}(f(U)))) \subset m_2\alpha\mathcal{J}\,\mathrm{Cl}(f(U))$.
(iii) ⇒ *(i)*: Let V be an m_2-α-$\mathcal{J}$-closed subset of Y. Then we have $f(m_1\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset m_2\alpha\mathcal{I}\,\mathrm{Cl}(f(f^{-1}(V))) \subset m_2\alpha\mathcal{I}\,\mathrm{Cl}(V) = V$. This implies that $m_1\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(f(m_1\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)))) \subset f^{-1}(V)$. Therefore, $f^{-1}(V)$ is an m_1-α-$\mathcal{I}$-closed subset of X and consequently f is an (m_1, m_2)-α-$\mathcal{I}$-irresolute function. □

Theorem 4.4. *A function $f : (X, m_1, \mathcal{I}) \to (Y, m_2, \mathcal{J})$ is (m_1, m_2)-α-$\mathcal{I}$-irresolute if and only if $f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Int}(V)) \subset m_1\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$ for each subset V of Y.*

Proof. Suppose that f is (m_1, m_2)-α-$\mathcal{I}$-irresolute. Let V be any subset of Y. Then $m_2\alpha\mathcal{J}\,\mathrm{Int}(V) \subset V$. Since f is (m_1, m_2)-α-$\mathcal{I}$-irresolute, $f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Int}(V))$ is an m_1-α-$\mathcal{I}$-open subset of X. Hence
$f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Int}(V)) = m_1\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Int}(V))) \subset m_1\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$. Conversely, let V be an m_2-α-$\mathcal{J}$-open subset of Y. Then $f^{-1}(V) = f^{-1}(m_2\alpha\mathcal{J}\,\mathrm{Int}(V)) \subset m_1\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an m_1-α-$\mathcal{I}$-open subset of X and consequently f is an (m_1, m_2)-α-$\mathcal{I}$-irresolute function. □

The proof of the following theorems are follows from the definitions and hence omitted.

Theorem 4.5. *The (m_1, m_2)-α-$\mathcal{I}$-irresolute image of an m_1-α-$\mathcal{I}$-connected space is m_2-α-$f(\mathcal{I})$-connected.*

Theorem 4.6. *If* $f : (X, m_1, \mathcal{I}) \to (Y, m_2, \mathcal{J})$ *is an* (m_1, m_2)-α-$\mathcal{I}$-*irresolute surjection and* $(X, m_1, \mathcal{I})$ *is* m_1-α-$\mathcal{I}$-*compact, then* $(Y, m_2, f(\mathcal{I}))$ *is* m_2-α-$f(\mathcal{I})$-*compact.*

Theorem 4.7. *If* $f : (X, m_1, \mathcal{I}) \to (Y, m_2, \mathcal{J})$ *is an* (m_1, m_2)-α-$\mathcal{I}$-*irresolute surjection and* $(X, m_1, \mathcal{I})$ *is* m_1-α-$\mathcal{I}$-*Lindelöf, then* $(Y, m_2, f(\mathcal{I}))$ *is* m_2-α-$f(\mathcal{I})$-*Lindelöf.*

Theorem 4.8. *If* $f : (X, m_1, \mathcal{I}) \to (Y, m_2, \mathcal{J})$ *is an* (m_1, m_2)-α-$\mathcal{I}$-*irresolute surjection and* $(X, m_1, \mathcal{I})$ *is countably* m_1-α-$\mathcal{I}$-*compact, then* $(Y, m_2, f(\mathcal{I}))$ *is countably* m_2-α-$f(\mathcal{I})$-*compact.*

We close with the following: Find nontrivial examples for m-α-$\mathcal{I}$-compactness, countable m-α-$\mathcal{I}$-compactness and m-α-$\mathcal{I}$-Lindelöfness.

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