The Collatz Conjecture - a proof

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Abstract

Originated by Lothar Collatz in 1937 [1], the conjecture states: given the recursive function, y=3x+1 if x is odd, or y=x/2 if x is even, for any positive integer x, y will equal 1 after a finite number of steps. This analysis examines number form and uses a tree type graph to prove the process.

examples

An example for a random selection of 7, using the original method:

S=(7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1)

An example for a random selection of 12, using the original method:

S= (12, 6, 3, 10, 5, 16, 8, 4, 2, 1)

functions

The recursive function is replaced with function d for odd values (2n-1), with

$$d(x) = 3x+1 = u = 2^k y \quad (2.0)$$

and function e for even values, which removes all factors of 2,

$$e(u) = y \quad (2.1)$$

The function e can be defined as a short program with a loop that repeatedly divides u by 2 until the output is an odd integer. This eliminates the redundancy and clutter of repeated division by 2.
After k divisions by 2, u = y, an odd integer. The value of y becomes the input x, and the cycle is repeated until y=1. The application of e(d(7)) results in S=(7 11 17 13 5 1), the revised format used in this analysis, with the understanding of a 2^k factor between each pair of odd integers. Notation is upper case for sets, lower case for elements of a set.

reverse sequences

If all sequences converge to the value 1, then it should be possible to form all reverse sequences R, diverging from 1. For this purpose the odd integers are classified into 3 subsets, 0 mod 3, 2 mod 3, and 1 mod 3, labeled as Y_0, Y_1, and Y_2.

Y_0 = {3 9 15 21 27 ...} or y = 6n-3
Y_1 = {5 11 17 23 29 ...} or y = 6n-1
Y_2 = {1 7 13 19 25 ...} or y = 6n-5

Rearranging (2.0), we can find x, given y, while requiring y to be a (1 mod 3) value.
If $y \equiv 1 \bmod 3$, then k is even and if $y \equiv 2 \bmod 3$, then k is odd.

If y = 6n-1 then x = (12n-3)/3 = 4n-1. (2.2)

If y = 6n-5 then x = (24n-21)/3 = 8n-7. (2.3)

In ascending mode,

input is Y_1 = {5 11 17 23 29 ...}
output is X_1 = {3 7 11 15 19 ... }

input is Y_2 = {1 7 13 19 25 ...}
output is X_2 = {1 9 17 25 33 ... }

growing the tree

Beginning with 1 in Y_2 the output is 1, resulting in zero growth. This requires a method of bypassing 1 with a new branch.
From eq. (2.0), if k increases by 2, then for successive values of U,

$$U_{r+1} = 4U_r \tag{2.4}$$

If $u = 3x_r+1$ and $4u = 3x_{r+1}+1$, then for successive values of x,

$$x_{r+1} = 4x_r+1 \tag{2.5}$$

Using the d function, $3(4x_r+1)+1 = 12\,x_r+4 = 4(3x_r+1)$

The sequence of x terms formed with eq. (2.5) form an unlimited set labeled B_y for branching terms. All members of B_y transform to y via the function e(d(x)) in descending mode and allow horizontal extension of a tree graph.
For the remaining set $X_{3.}$ = {5 13 21 29 ...}, the expression (8n-3) can be rearranged as 4(2n-1)+1, the same form as eq. (2.5), thus X_3 becomes a sequence of B terms for the set of odd integers.

The set of B terms begins,

B_1 = {1 5 **21** 85 ... }
B_5 = {**3** 13 53 **213** ... }

B_7 = {**9** 37 149 **597** ... }
B_{11} = {7 29 **117** 469 ... }
...

Bold fonts are (0 mod 3) terms.
This results in fig.1.

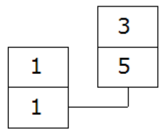

fig.1

Since the function d() cannot generate (0 mod 3) terms, there is no X that corresponds to Y_0. A (0 mod 3) term can only begin a descending sequence, thus a reverse sequence with that term would terminate.

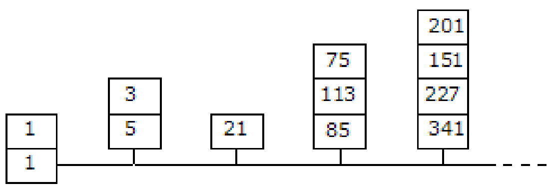

fig. 2

The B terms solve that problem bypassing the (0 mod 3) term with the next B term.
In descending mode, a complete branch is one that begins with a (0 mod 3) term and ends with a B term.

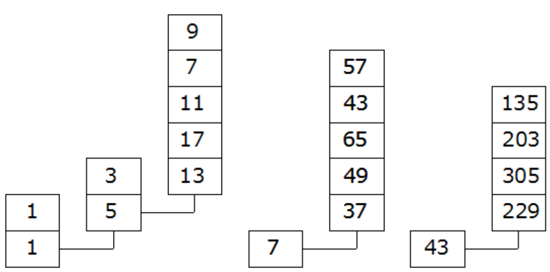

fig.3

Extending the tree from the trunk 1, the next term 13, allows a new branch and extension of R, as shown in fig.3. A new branch can be formed from any term in the current branch except (0 mod 3). In the example, the next to last term is arbitrarily selected. Using the B

terms for each successive x, extends the branch vertically to the next termination value 9. This process is repeated with 7, 43, 203, etc., and can be extended without limit.
All terms in the branch ending in 3 are 1 branch distant from the trunk .
All terms in the branch ending in 9 are 2 branches distant from the trunk.
Branches are the horizontal measurement for distance from the trunk.
The ascent to 135 is actually 4 branches R3 + R9 + R57 + R135.

expansion of 2k range

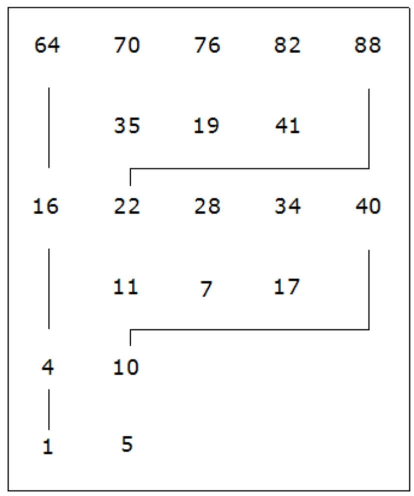

fig.4

As the value of u moves into larger ranges of 2^k, each pair of adjacent terms is expanded by a factor of 4 with 3 additional terms between them. This allows longer sequences in a branch, and larger divisors, as shown in fig.4.

5. even integer selection

All reverse sequences for even integer selection, can be formed by appending a 2^k progression times an odd integer, presented here as a list, using 1, 3, 5, 7, 9, ...

(2 4 8 16 32 ...)
(6 12 24 48 ...)
(10 20 40 80 ...)
(14 28 56 122 ...)
(18 36 72 144 ...)
...

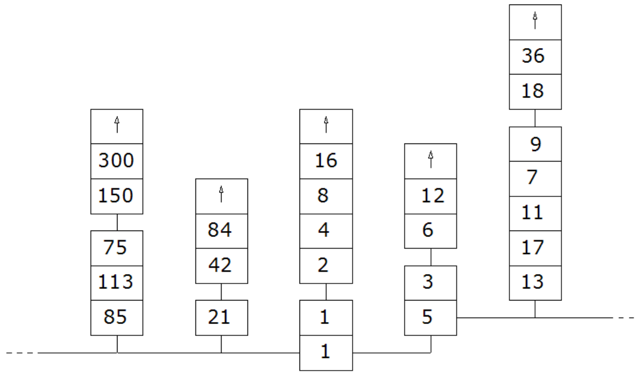

fig.5

This provides a means of extending the Y_0 termination values to sequences of unlimited length as shown in fig.5.

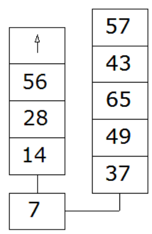

fig.6

Each term from Y_1 and Y_2 now have an extended sequence of even integers as in fig.6.

the tree

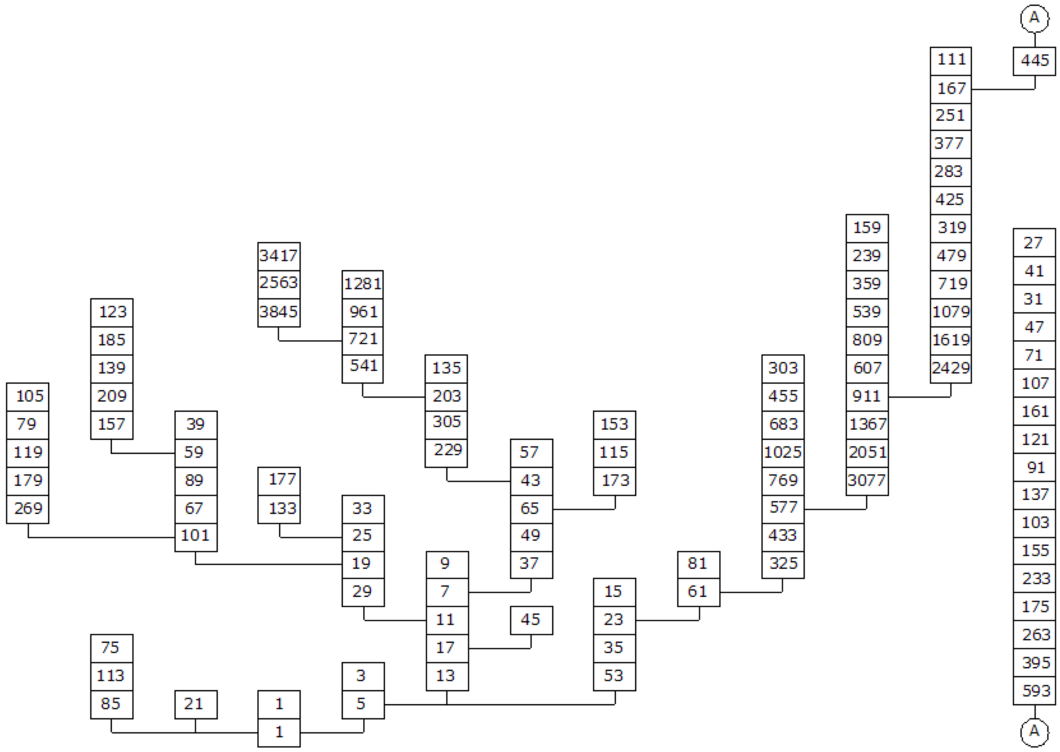

fig.7

Fig.7 shows the initial growth of the tree for odd integers only, from a 'trunk' of 1, vertically with each branch terminating in a (0 mod 3) value, and horizontally via the B terms. The descending sequence for x=27 is revealed on the right as a composite of 7 branches, $S_{27} + S_{111} + S_{159} + S_{303} + S_{81} + S_{15} + S_3$. To visualize a partial tree with all branches would require 3 dimensions.

conclusion

The reverse sequences are intended to answer the question,
Is a network possible that produces the specified results, using the specified rules?
Forming reverse sequences shows it is possible.
Descending in any branch, the values reflect x movement within the 2^k ranges, whereas the horizontal movement using B terms, moves a sequence of x closer to x=1, with decreasing values. As shown in fig.7, the number of elements in a sequence varies for the same number of branches. Therefore there is no (simple) distance function for any sequence of values relative to the trunk. The distance is determined by number of branches.
In ascending mode, choices were made in forming the 'one to many' network of paths diverging from the trunk, based on the Collatz rules. If a path can be formed from x=1 to any integer using a reverse engineering method, then a randomly selected x must return to x=1 via a 'one to one' predetermined path.
An analogy would be a multi-story building with stairways between all floors. A person placed on any floor, has a path to the ground floor, by design.

Therefore all sequences merge at x=1 in descending mode. The Collatz conjecture applies only to finite length sequences, in the descending mode

reference

1. Wikipedia.org/Collatz Conjecture, Mar 2018