Restricted Proof of the polynomial boundedness of $\sum_{A \in EdgeCovers(G)} 2^{-|A|}$

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Claim . $\sum_{A \in \mathbf{A}} -2^{|A|}$ is in $\mathcal{O}(Poly(|V|, |E|))$ for a graph $G = (V, E)$, where $\mathbf{A}$ is the set of all edge covers of G **under the assumption that** $|E| \leq 2|V| - 2$

Proof. Let $(A_m) = \{S \mid S \in \mathbf{A} \wedge |S| = m\}$.

Thus, we define $T(m) = \sum_{A \in \mathbf{A_m}} -2^{|\mathbf{A}|}$

Lemma 1 : If $|(A_n)\mathcal{O}(Poly(|V|, |E|))| = |(A_{n-1})|$ **.** Consider an arbitrary edge cover (C) of size n. We can remove one of its n edges to create (potentially)new edge covers of size $n-1$. So, the lemma follows.

Lemma 2 : $|(A_{n-1})\mathcal{O}(Poly(|V|, |E|))| = |(A_n)|$ **.** Consider an arbitrary edge cover (C) of size $n-1$. We can add one of the $m-n+1$ edges not in it to create (potentially)new edge covers of size n. So, the lemma follows.

Now, we consider the following statement $P(k) : T(n-1-k)+T(n-1+k) \equiv \mathcal{O}(Poly(|V|, |E|))$ To prove this statement inductively, we first prove $P(0)$. We note that $T(n-1)$ is in (P) since it is the case which spanning trees(which all have size $|V|-1$.

$$\begin{aligned} P(0) &: T(|V|-1-0) + T(|V|-1+0) \\ &= 2T(|V|-1) \\ &= \mathcal{O}(Poly(|V|, |E|)) \end{aligned} \tag{1}$$

Next, we prove $P(k)$ for some arbitrary $k \in \mathbf{N}$ give that $P(m)$ holds $\forall m < k$, $m \in \mathbf{N}$:

$$
\begin{aligned}
LHS &= T(|V|-1-k) + T(|V|-1+k) \\
&= \sum_{A \in \mathbf{A_{|V|-1-k}}} -2^{|\mathbf{A}|} + \sum_{A' \in \mathbf{A'_{|V|-1+k}}} -2^{|\mathbf{A'}|} \\
&= |(A_{|V|-1-k})|(-2)^{|V|-1-k} + |(A_{|V|-1+k})|(-2)^{|V|-1+k} \\
&\le \mathcal{O}(Poly(|V|,|E|))|(A_{|V|-k})|(-2)^{|V|-1-k} \\
&\quad + \mathcal{O}(Poly(|V|,|E|))|(A_{|V|-2+k})|(-2)^{|V|-1+k} \\
&\qquad\qquad \text{(Lemma 1 and 2)} \\
&= \mathcal{O}(Poly(|V|,|E|))[|(A_{|V|-1-(k-1)})|\frac{(-2)^{|V|-k}}{2} + |(A_{|V|-1+(k-1)})|(-2)^{|V|-1+k-1} * 2] \\
&= \mathcal{O}(Poly(|V|,|E|)) * \left[\sum_{A \in \mathbf{A_{|V|-k}}} -2^{|\mathbf{A}|} + \sum_{A' \in \mathbf{A'_{|V|-2+k}}} -2^{|\mathbf{A'}|} \right] \\
&= \mathcal{O}(Poly(|V|,|E|)) * [T(|V|-1-(k-1)) + T(|V|-1+(k-1))] \\
&= \mathcal{O}(Poly(|V|,|E|)) * \mathcal{O}(Poly(|V|,|E|)) \text{(By the Induction Hypothesis)} \\
&= \mathcal{O}(Poly(|V|,|E|))
\end{aligned}
\tag{2}
$$

Hence, from (1) and (2), we have proved inductively that for $k \in \mathbf{N}$, $P(k)$ holds. (Continued on next page)

Therefore,

$$
\begin{aligned}
\sum_{A \in \mathbf{A}} -2^{|\mathbf{A}|} &= \sum_{m=0}^{|E|} \sum_{A \in \mathbf{A_m}} -2^{|A|} \\
&\leq \sum_{m=0}^{2|V|-2} \sum_{A \in \mathbf{A_m}} -2^{|A|} (\text{By our assumption that } |E| <= 2|V|-1) \\
&= \sum_{m=0}^{2|V|-2} T(m) \\
&= -T(|V|-1) + \sum_{m=0}^{|V|-1} T(|V|-1-m) + T(|V|-1+m) \\
&\equiv \frac{\mathcal{O}(Poly(|V|,|E|))}{2} + \sum_{m=0}^{|V|-1} \mathcal{O}(Poly(|V|,|E|)) (\text{Using P(0) and P(m)}) \\
&\equiv (0.5 + |V| - 1) | \mathcal{O}(Poly(|V|,|E|)) \\
&\equiv \mathcal{O}(Poly(|V|,|E|))
\end{aligned}
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□