Solid Strips Configurations

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Dec. 2019†

Abstract

We introduce the idea of Solid Strip Configurations which is a way of describing 3-dimensional compact manifolds alternative to Δ-complexes and CW complexes. The proposed method is just an idea which we believe deserve further formal mathematical investigation.

Key Words: compact manifolds, manifold decomposition.

1 Introduction

Compact manifolds of dimension higher then 2 are very hard to study and classify. Starting from a method for the 2D case and focusing on 3D manifolds, we propose in this paper a method, alternative to Δ-complexes and CW complexes, to describe these manifolds which, if further developed, we believe may results very convenient.

2 Strip Configurations in 2-Dimensions

2.1 Main Definitions

A **Strip** is a 2-dimensional manifold with boundaries obtained by identifying 2 opposite edges of the 4 edges of a square. It can be done without a twist (Untwisted Strip) or with a twist (Mobius strip).

A **Strip Configuration** is a finite set of strips, crossing each other or not, such that it exist a compact 2-dimensional manifold in which the set of strips can be embedded. An example of two strings that do not form a string configuration is given in Fig. 1a. Once we embed the strips on such a manifold we are allowed to move the strips on the manifold at will. If a and b are two strips then we will use the notation $a \diamond b$ for the configuration obtained by making a and b crossing 1 time.

A non path connected strip configuration can always be changed in a path connected one according to the following procedure: 1-embed the strips in a compact two dimensional manifold; 2- bring two strips from two non path connected subset of the configuration close each other without changing the configuration of the two subset(see Fig. 1b); 3- overlap the two trips so that they cross in two points (see Fig. 1b).

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†Posted at: http://vixra.org/abs/1912.0161 - Current version: v2 - Jan. 2020

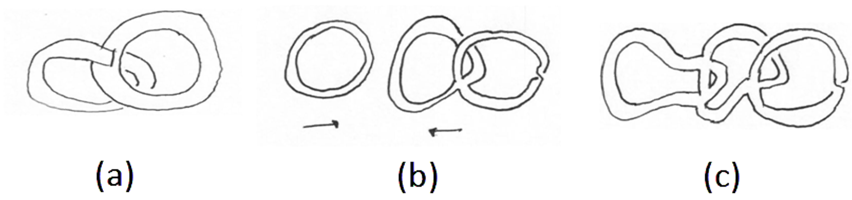

Figure 1: Definition of String Configuration

Note that the boundary of a string configuration is made of a finite number of **sub-boundaries** (i.e. non path connected parts) each of which being a circles (i.e. $\mathbf{S}^1$). The **Associated (Compact) Manifold** to a strip configuration is the compact manifold obtained by making the configuration path connected (if it is not) and identifying the boundary of a disk ($\mathbf{D}^2$) to each sub-boundaries of the strip configuration. We will use the notation $\Omega(A)$ for the associated manifold to the strip configuration A.

Two strip configurations are **Homeomorphic Associated Equivalent** if their associated manifolds are homeomorphic or, which is the same, if once embedded in the associated manifold one string configuration can be changed into the other by moving the strips on the manifold and deforming the manifold by means of continuous transformations. In the process each strip shall always keep its own identity even when it crosses other strips with continuous transformations meaning that a strip cannot be cut and glued to form other strips. Two strip configurations are **Homotopy Associated Equivalent** if their associated manifolds are homotopy equivalent.

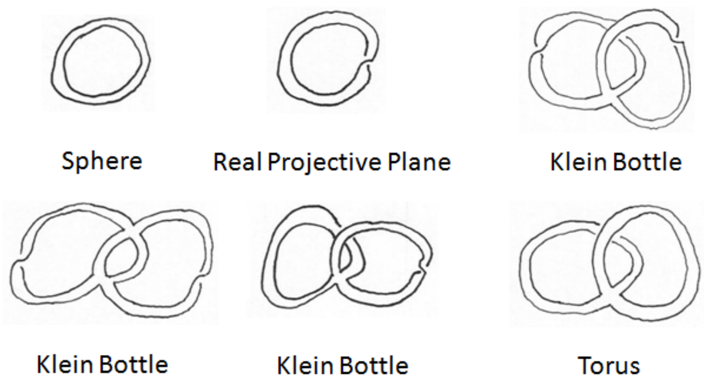

Figure 2: 1 and 2 Strip Configurations

In a strip configuration a string can be twisted n times (with $n \geq 0$) (if n is even then the string is homomorphic to an untwisted strip, if n is odd to a Mobius strip) and two strips can cross each other m times (with $m \geq 0$).

We want to give now some criteria for two strip configurations to be homeomorphic associated equivalent. Some of these criteria are not obvious and should be formally proved.

1. A non path connected strip configuration and the path connected one obtained from it using the procedure explained in the paragraph above

are equivalent.

2. An untwisted strip that does not cross any other strip can be removed from the configuration because this is equivalent to removing from the associated manifold a sphere which is sum connected to the manifold.

3. Given a strip configuration, this is equivalent to the same strip configuration where strips that are twisted an odd number of times are replaced by Mobius strips and strips that are twisted an even number of times are replaced with untwisted strips.

We note that the direct sum of 2-dimensional manifolds has a non path connected strip configuration given by the union of the two strip configurations of the two manifolds.

However, the above criteria are not enough and we want to evaluate equivalences by calculating topological invariants on the configurations. Strip configurations are very convenient from this point of view because the fundamental group of the associated manifold can be easily computed from its strip configuration using the van Kampen theorem.

To evaluate the fundamental group, the generators are given by the open maximal spanning graph obtained from the graph we get homotopyng each strips to a 1- dimensional space (i.e. we turn strips into lines) while the conditions to present the group can be evaluated on the strip configuration itself.

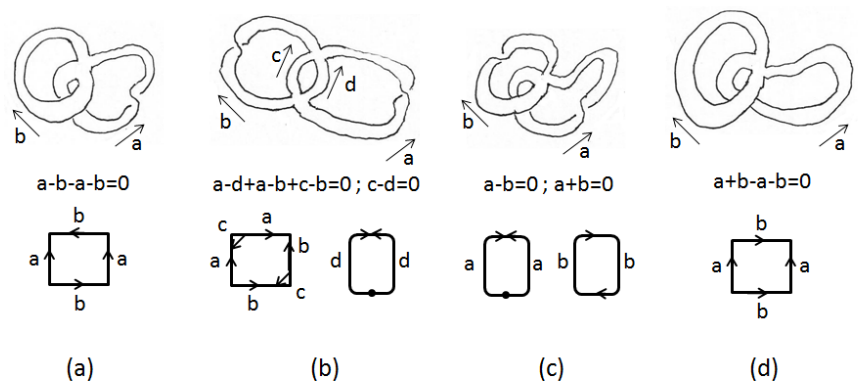

Figure 3: Strip Configurations Fundamental Groups

We will show this with some examples. In figure Fig. 3 we show some strips configurations with the generators used to have the free non commutative groups. The conditions to present the fundamental group of the associated manifold are drawn in a "polygonal picture" under each configuration. These conditions are obtained starting from a point and adding the generators (group are presented with an additive operation although unusual for non commutative groups) that we encounter on the boundary going all around till we get to the same point.

For case of Fig. 3a the condition lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. For case 3b, we have $c = d$ which, with a simple algebraic manipulations give the condition presented in the two polygon under the configuration in the figure. These lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. For case 3.c, from the two conditions we have that $a = b$ and therefore the two conditions became $a - a = 0$ and $b + b = 0$. Once again these lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. Condition of Fig. 3d leads to the commutative free group on two generators which is the group of the torus $\pi_1 = \mathbb{Z}^2$.

We note that for cases of Fig. 3b and 3c we need to manipulate the conditions algebraically to permute the names of edges for the polygonal representation end this because in each polygon we want to have pairs of edges with the same name.

2.2 Represented 2D Manifolds

A question we may ask is how many compact 2D manifolds we can represent with strip configurations. We have the following proposition:

Proposition 2.1: *If a 2D compact manifold has a Δ-complex representation, then it has also a strip configuration representation.*

Rather then proving the above proposition we will sketch an argument that shows the relationship between homology groups and strip configurations which lead to a constructive procedure for defining strip configurations (may be more then one) associated to a 2D compact manifold from its Δ-complex. This argument, if fully developed, would lead to the proof of the above proposition.

We will show our argument with an example. Given the Klein bottle, its $\mathbf{H}_1$ homology group is $\mathbf{H}_1 = \mathbb{Z} + \mathbb{Z}_2$. This group tells us that there are two separate classes of loops in the manifold, which is one class (of trivial homotopy loops) plus the rank of $\mathbf{H}_1$. The first loop (i.e. class of loops) can be ignored because it is contractable to a point. We have also the $\mathbb{Z}_2$ term that tells use that in the above two classes there is somewhere a second loop (or even one for each class) which needs to go around twice to be contractable to a point. An analysis of the manifold shows indeed that such a loop exists and therefore we end up with two separate loops. With abuse of terminology we may say that the first loop is orientable, meaning that a flat man living in the surface and walking on the loop would go back to the original point staying on the same side of the surface with respect of the loop, and the second loop is not orientable.

The above loops may be used to define strips. To have a strip we need two loops. We have 3 possibility. $1-$ we may take two copies of the orientable loop, the second one displaced by a δL from the first one, together with the surface between the two loops. This defines an untwisted strip. $2-$ we may take two copies of the non orientable loop with the area between them, once again with the second loop sightly displaced, but this time with the two loops concatenated otherwise they would cross each other. This defines a Mobius strip. $3-$ we may take a copy of the orientable loop and a copy on the non orientable loop concatenated, so that they do not cross, move them close each other and take the area between them. This would defines a Mobius strip.

We see that we have three separate strips that can be embedded in the surface. However a configuration with three strip would be redundant since half of the boundary of two strips would coincide with the two half boundaries of the third strip. The strip configurations we are looking for are therefore formed by two strips and we may take any two strips of the three above.

This is why we have two strip configurations associated to the Klein bottle. The first one is formed by an untwisted strip crossing a Mobius strip and the second one is formed by two Mobius strip crossing. The above example together with the relevant procedure to derive strip configurations can be extended to any 2D compact manifold.

3 Motivation for 3-Dimensions Strip Configurations

If we think for a moment to what we did in the previous paragraph we see that we represent 2D manifold starting from strip configurations or, another way to see it, we use 2D strips to probe a 2D space in a similar way homotopy theory does with loops. Given a strip configurations, this may not be embedded in $\mathbb{R}^2$ but it does exist a minimal (in away that may be made precise using the concept of associated space) 2D compact manifold where this strip configuration can be embedded. In other words a strip configuration defines a compact 2D manifolds in the same way a CW complex or a Δ-complex does.

This way to probe spaces has the advantage to see differences in some spaces that are homotopy equivalent. The most trivial example (although with boundaries) is the Mobius strip which is homotopy equivalent to $\mathbf{S}^1$. However, in this space obviously a smaller Mobius Strip can be embedded while the same cannot be done in the circle.

Me may think to have a look to a strip configuration and see immediately what "strip loop" are present and tell in this way if two spaces are the same. However, the examples from Fig. 3c show that this is not so straight forward. In order to solve the problem we have build groups based on the boundaries of the strip configurations, using the the van Kampen theorem, that are eventually fundamental groups.

We will show later that we may defines some sort of 3D strips and there are at least 15 of them. Once combined in configurations, this leads to an huge amount of combinations, which may somehow be used to represent 3-manifold in a convenient way.

Obviously, before we do that, we need to show what a 3D strip is and what their configurations are. This will be done in the following sections.

4 Strip Configurations in 3-Dimensions

4.1 Main Definitions

In 2-dimensions we use 2-D strips obtained by identifying one couple of opposite edges of the two couples of edges of a square. In 3-dimensions we will use **Solids Strips** which are 3-D "strips" obtained by identifying two couples of opposite faces of the three couples of faces of a cube.

This manifolds have been studied in the paper [1] where they are named "Solid Strips".

The boundary of a solid trip is build by identifying the edges of two squares and what we get may form one or two sub-boundaries. The total homeomorphic configurations of Solid Strips are 21 (reported in Appendix A.1) but they may be further reduced to 15 Homology equivalent classes of solid strips with the same boundary and same homology groups (see [1]).

Solid strips are 3-manifolds or, another way we see it, they are **Thick Compact 2D Surfaces** where by that we mean that they are like surfaces expanded by a δL in the third dimension which, by sake of visualization for the reasoning that will follow, we may think to be small with respect to the surface itself when needed.

Broadly speaking, and taking into account the approximation that the following sentence has, solid strips are thick surfaces that look like tori or Klein bottles because they are like pipes that are joint at their far ends in various way. In $\mathbb{R}^3$, tori intersect in two ways. In one circle (type 1 intersection) as shown in Fig 5a or in two circles (type 2 intersection) as shown in fig 5b. Solid strips cross (i.e. intersect) in the same way tori do but being tick surfaces they intersect in solid tori rather then circles. When locally embedded (i.e. just a little piece of them) in $\mathbb{R}^3$, they cross in one solid torus (type 1 intersection) or 2 path disconnected solid tori (type 2 intersection). We are always interested in the last kind of crossing (type 2 intersection) and from now on when we say that two strips intersect we mean that we have a type two intersection.

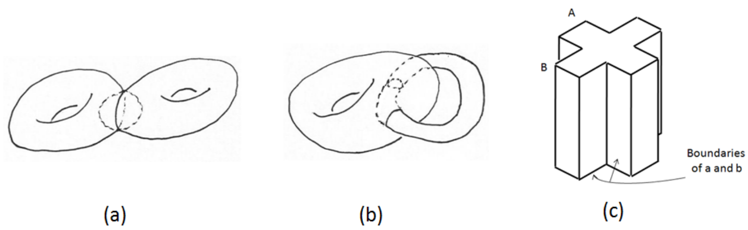

Figure 4: Crossing of two Tiles

We can get solid strips by identifying opposite faces of cubes. Being tick surfaces, for sake of representation, we can imagine the above cubes to have one dimension smaller then the others so that they look like tiles. Fig 4c, shows a type 2 intersection between strips represented by the above mentioned tiles. This is a type two intersection because in $\mathbb{R}^3$, when we identify the up and down faces and two opposite faces of the tiles to get the relevant strips, we cannot avoid to have the tiles to cross a second time. However, in higher dimensions we can identify the faces to get the strips without having a second intersection. In higher dimensions two solid strips can cross (intersect) with a type 2 intersection any zero, odd or even number of times in perfect analogy with the 2D case. Another way to see that in higher dimensions two solid strips can cross only once, is from Fig 4b. If we take one of the two intersections and we move one of the two tori in the intersection along the 4^{th} dimensions, this will not intersect the other torus any more.

We will call a **Solid Strip Configuration** a bunch of solid strips that cross each other (type 2 intersection) a finite number of times in the same way 2-strips cross forming the 2-strip configurations described in the paragraphs above. If a and b are two solid strip then we will use the notation $a \diamond b$ for the configuration obtained by making a and b crossing 1 time. The boundary of a solid strip configurations is formed by **Sub-Boundaries** exactly as in the 2D case.

In analogy with the 2D case, we will call the **Associated (Compact) Manifold** to a solid strip configurations the 3D compact manifolds that we get by filing the holes defined by its sub-boundaries (i.e. we attached manifolds to its boundaries till we get a compact space) in the "most simple" topological way where the meaning of the "most simple" will be clarified further on. In analogy with the 2D case we will use the notation $\Omega(A)$ for the associated manifold to the solid strip configuration A. We note explicitly that a non path connected strip configuration can be made connected using the same procedure we had for the 2D case.

We need now to make more mathematically precise the two ideas of "solid strip crossing" and "filling the holes" of a configuration. This will be done in the following two sections.

4.2 Crossing of Solid Strips

We have seen that we have a type 2 intersection when the two tiles A and B, from which we form the strips, intersect also on two of their faces see Fig. 4c. In the above figure we see that we can identify the vertical faces of the tiles as we want in order to get our strips but for the up and down faces, having them a square in common, we have some imitation in the way we can identify them.

Fig. 5a shows what the up and down face of the two strip (tiles) look like. In this case we cannot apply all the 8 symmetry of a square for identifying the up and down faces but only a subgroup of them which preserve the fact that the up faces of the two tiles are identified with the down faces of the same tile.

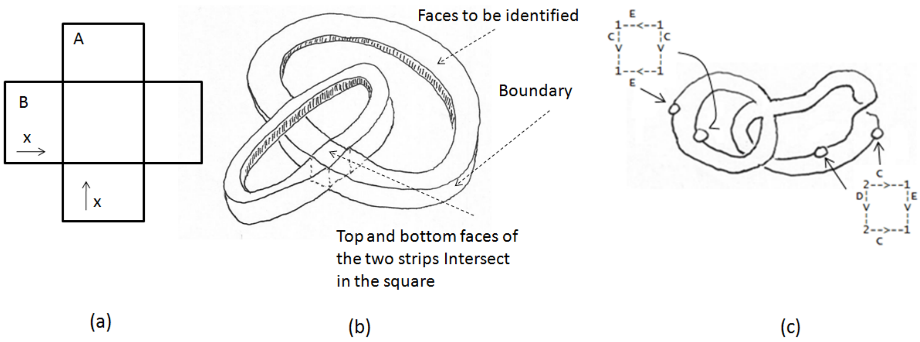

Figure 5: Crossing of two Solid Strips

The above construction shows, what we mean for a type 2 intersection between 2 solid strips, although valid only for a limited number of combinations of solid strips. This does not mean that we cannot have a type 2 intersection for any two given types of strip.

We can compress the up direction of each tile and draw solid strip configurations as in Fig 5b which give a good idea of what a type 2 intersection with only one crossing is. Although misleading, for sake of representation, if we ignore the up direction, we can draw solid strip configurations in a similar way as we do for the 2D case see Fig. 5c. In this representation the inside of the strip is represented as a surface but it is a 3D space and the boundaries of the configuration are represented as lines but they are surfaces. As for the 2D case, with the above representation, by following the boundaries of the strip configuration till we get back to the starting point, we identify sub-boundaries.

4.3 Associated Manifold

Given a strip configuration, we need to give a precise procedure to make its associated manifold. Given a strip configuration, its sub-boundaries are closed surfaces. For each sub-boundary in the configuration, we take a 2-Δ-Complex decomposition of the sub-boundary and we attach a 3 simplex to each 2-simpex of it. We identify the three 2-simplices of each of those simplices each other following the same way the edges simplices of the sub-boundary are. This will completely "fill the holes" of the solid strip configuration and will give us the compact manifold we where looking for.

4.4 Represented 3D Manifolds

In the previous section we have defined the associated manifold to a Solid Strip Configuration. We note explicitly that, in the 3D case, strip configurations may represent a large class of spaces. As for the 2D case we have:

Proposition 4.3: *If a 3D compact manifold has a Δ-complex representation, then it has also a solid strip representation.*

The above proposition may be proven following an argument similar to the one used for the 2D case although in 3D all becomes more tricky and, what was a sketch for a prof in 2D, in 3D it becomes an interesting reasoning but very far from proving anything.

In 3D we will relate solid strip configurations with the group $\mathbf{H}_2$. This group will give use a set of equivalence classes of compact surfaces in the manifold under study (the equivalent of the loops in 2D). If k is the rank of $\mathbf{H}_2$, in analogy with the 2D case, then we will have $k+1$ classes of surfaces, one of homology trivial surfaces and k classes of surfaces that cross only once to surfaces in the other classed.

We note explicitly that in the above classes there will be orientable surfaces and also non orientable surfaces, if the group $\mathbf{H}_2$ has a torsion term $\mathbb{Z}_2$, in perfect analogy with the 2D case. This non orientable surfaces correspond to surfaces that have to go twice around to be homology trivial. There is no torsion higher then $\mathbb{Z}_2$ in the group $\mathbf{H}_2$ of compact 3D manifolds and therefore we do not need to look for surfaces that go around more then twice (although this should be proven). So far so good and we have a set of $k+1$ classes of surfaces which existence can be known just from the rank of the homology group of order 2. This is exactly equivalent to what we did in the 2D case for loops.

The difference with the 2D case is that for each of the above classes of equivalence, while in 2D loops can be only of two kinds (orientable and non orientable), in 3D surfaces may be of any kind (Spheres, Klein bottles, Tori, etc...) and therefore in each class we have to find out what type of surface we have (and we can have more then one type) and how they cross. At the end of this process we will have a full set of surfaces and we will know the way they cross. Some surfaces may be removed for one of the two following reasons: 1- if they can be moved and confined to a flat region of the manifold locally homeomorphic to $\mathbb{R}^3$ and do not cross any other strip. 2- if they can be moved and superimposed to another surface homeomorphic to them in the manifold. This process will eventually give us a minimal set of surfaces that cannot be further reduced.

As said before, if k is the rank of the group $\mathbf{H}_2$, we get $k+1$ classes of surfaces each of them containing a set of different surfaces. We believe that, although this should be proven, if two surfaces belong to the same class and are homeomorphic, then they cross other surfaces in the same way and can be superimposed to each other. If that is true, searching for equivalent surfaces in each class would be much easier. This is something that deserve further investigation.

In analogy with the 2D case, we turn our set of surfaces to tick surfaces (i.e. solid strips). As in the 2D case it take two separate copies of an orientable surface and two copies of a non orientable surface that goes around twice to make a tick surface. This give us the final solid strip configuration associated to a given Δ-complex and its related compact manifold.

In the 2D case we where able to combine separate loops, two at a time, to get additional redundant strips. The process may be done also in the 3D case but it is much more difficult to be analysed and it will not be done here. This point deserve further investigation.

We note explicitly that in the process we may find that for a generic Δ-complex the possible compact surfaces lead to tick surfaces that are a subset of the solid strips defined in [1]. In this case we are fine. If we find out that we need additional tick surfaces to define any Δ-complex, then we need to add these to the set of possible solid strip defined in [1] in order for our theory to be complete. We will call Γ the set of all thick surfaces required to define any Δ-complex. Note that Γ may turn out to be an infinite set. Once again, this point is something that deserve further investigation.

Appendix

A.1 Solid Strip Configurations

This appendix contains the full set of solid strips equivalent class configurations. For more details and for the meaning of the $\xi(a_i, b_j)$ notation see [1].

$[\xi]$	Homology Class	ξ	$\partial\xi$	$\chi(\xi)$
1	1	$\xi(g_0, a_0)$	$\mathbf{T}^2 \sqcup \mathbf{T}^2$	0
2	2	$\xi(g_4, a_0)$, $\xi(g_0, a_4)$	$\mathbf{K} \sqcup \mathbf{K}$	0
3	3	$\xi(g_4, a_4)$	$\mathbf{RP}^2 \sqcup \mathbf{RP}^2$	1
4	4	$\xi(g_3, a_4)$, $\xi(g_2, a_4)$, $\xi(g_4, a_3)$, $\xi(g_4, a_2)$	$\mathbf{RP}^2 \vee \mathbf{RP}^2$	2
5	5	$\xi(g_3, a_3)$, $\xi(g_2, a_2)$	$\mathbf{X}_1 \vee \mathbf{X}_1$	2
6	6	$\xi(g_5, a_5)$	$\mathbf{S}^2$	1
7	7	$\xi(g_1, a_1)$	$\mathbf{T}^2$	0
8	7	$\xi(g_1, a_0)$, $\xi(g_0, a_1)$	$\mathbf{T}^2$	0
9	8	$\xi(g_5, a_0)$, $\xi(g_0, a_5)$	$\mathbf{T}^2$	0
10	9	$\xi(g_4, a_1)$, $\xi(g_1, a_4)$	$\mathbf{K}$	0
11	9	$\xi(g_5, a_1)$, $\xi(g_1, a_5)$	$\mathbf{K}$	0
12	10	$\xi(g_6, a_5)$, $\xi(g_7, a_5)$, $\xi(g_5, a_6)$, $\xi(g_5, a_7)$	$\mathbf{X}_1$	2
13	11	$\xi(g_6, a_6)$, $\xi(g_7, a_6)$, $\xi(g_6, a_7)$, $\xi(g_7, a_7)$	$\mathbf{X}_2$	1
14	12	$\xi(g_2, a_3)$, $\xi(g_3, a_2)$	$\mathbf{X}_2$	1
15	12	$\xi(g_3, a_1)$, $\xi(g_2, a_1)$, $\xi(g_1, a_3)$, $\xi(g_1, a_2)$	$\mathbf{X}_2$	1
16	12	$\xi(g_6, a_1)$, $\xi(g_7, a_1)$, $\xi(g_1, a_6)$, $\xi(g_1, a_7)$	$\mathbf{X}_2$	1
17	13	$\xi(g_3, a_0)$, $\xi(g_2, a_0)$, $\xi(g_0, a_3)$, $\xi(g_0, a_2)$	$\mathbf{Y}_1$	0
18	13	$\xi(g_6, a_0)$, $\xi(g_7, a_0)$, $\xi(g_0, a_6)$, $\xi(g_0, a_7)$	$\mathbf{Y}_1$	0
19	14	$\xi(g_6, a_4)$, $\xi(g_7, a_4)$, $\xi(g_4, a_6)$, $\xi(g_4, a_7)$	$\mathbf{Y}_1$	0
20	14	$\xi(g_5, a_3)$, $\xi(g_5, a_2)$, $\xi(g_3, a_5)$, $\xi(g_2, a_5)$	$\mathbf{Y}_1$	0
21	15	$\xi(g_6, a_3)$, $\xi(g_7, a_3)$, $\xi(g_6, a_2)$, $\xi(g_7, a_2)$, $\xi(g_3, a_6)$, $\xi(g_2, a_6)$, $\xi(g_3, a_7)$, $\xi(g_2, a_7)$	$\mathbf{Z}_1$	0
22	N/A	$\xi(g_5, a_4)$, $\xi(g_4, a_5)$	Not Feasible	N/A

Table A.1.1 : Solid Strips ξ with Strip Classes $[\xi]$, Boundaries $\partial\xi$ and the Euler Characteristics $\chi(\xi)$.

where:

- With the symbol $\sqcup$ (disjoint union) we mean two separate instances of a space which are not path connected.
- Space $\mathbf{X}_1$: is a 2-sphere where two separate points of the sphere are identified. This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{X}_1 \vee \mathbf{X}_1$: is a wedge sum of two $\mathbf{X}_1$ spaces. This space has three points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.

- Space $\mathbf{X}_2$: is a 2-sphere where two couple of separate points of the sphere are identified. This space has two points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Y}_1$: is a 2-torus where two separate points of the torus are identified. This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Z}_1$: is a Klein Bottle where two separate points of the Klein Bottle are identified. This space has a point where the manifold is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.

References

[1] V. Nardozza. *Solid Strips.* http://vixra.org/abs/1910.0039 (2019)