Counting Partitions

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Abstract

In this lecture we count the number of integer partitions $P(n)$ using an elementary algorithm based on the combinatorics of trees, coded using free apps on the iPad.

Integer partitions appear throughout mathematics. We will count them using the combinatorics of trees and cubes, fundamental to categorical methods in physics.

Here an ordinal n takes a value in $\mathbb{N}^+ = \{1, 2, 3, 4, \cdots\}$ and $P(n)$ denotes the number of partitions of n into any number of parts r [1]. We write

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_r,$$

where it is assumed that $\lambda_i \geq \lambda_{i+1}$ for any i, and each part is at least 1, so that an expression like $6 = 3+2+1$ is unique. Now $P(3) = 3$ counts the list 3, $2+1$, $1+1+1$.

These ordinary partitions are commutative, because we do not differentiate between $3+2+1$ and $3+1+2$. But noncommutative partitions are far easier to count than commutative ones. Figure 1 shows how any noncommutative partition of n is represented by either a rooted tree with two levels, or a set of $n-1$ signs corresponding to the areas between the leaves on the tree. There are clearly 2^{n-1} such partitions. Beginning with this set of signs, for any given n, we construct an algorithm for computing $P(n)$.

First, observe that the integer partitions only see how the $+$ signs are partitioned within a string of signs. For example, in figure 1 for $P(4) = 5$, the partition $4 = 2+2$ corresponds to the string $+-+$, which we now think of as the two part string of strings $(+,+)$. A single $+$ sign in such a string denotes the number 2, a string $++$ denotes 3 and so on, so that $+-++-$ stands for the partition $6 = 3+2+1$, as does any other string of the form $(++,+)$.

We count $P(n)$ by counting these sign strings. Let k be the number of entries in a string of strings, so that for $(++,+)$ we have $k = 2$. For each value of k up to the maximum, we first generate a list of *perversities* of length k, so that $(++,+)$ will become the numerical word 12. The maximum k is simply the floor of $n/2$. A perversity (for homology [2]) is, by definition, a finite list $(x_1, x_2, \cdots, x_k)$ of ordinals such that $x_i \leq x_{i+1}$ for all $1 \leq i \leq k-1$. In other words, a perversity is a partition. For example, each row in the list of matrices

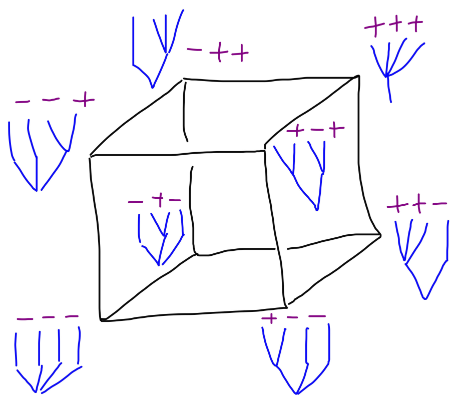


Figure 1: Noncommutative partitions as two level trees

in figure 4 corresponds to a perversity for the calculation of $n = 6$. At $k = 1$, we have up to 5 plus signs in the string. At $k = 2$, the maximum digit value is 3, because the space between the two parts is a minus sign. In general, the maximum digit is

$$rr = n - 1 - 2(k - 1).$$

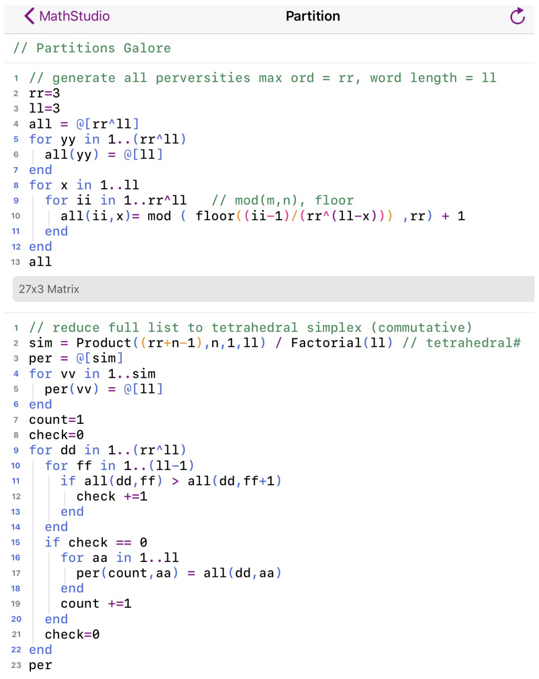
This set of perversities $\mathcal{P}(n)$ is then truncated as follows. Let s equal the sum of digits in a perversity. In the calculation of $P(n)$, we require that the maximum value of s be $n - k$, because we only have $n - 1$ signs and we need $k - 1$ spaces. Let $N(n, k)$ be the number of elements of $\mathcal{P}(n)$ satisfying this condition. Then

$$P(n) = 1 + \sum_{k=1}^{\text{floor}(n/2)} N(n, k).$$

Although $N(n, k)$ is itself made up of partitions, the computer may evaluate it with a blind search.

Let us look at the code. Using the MathStudio app on the iPad, the brute force code of figure 2 generates any given set of perversities, for rr letters in the alphabet and words of length ll. It begins by generating all possible words in the letters $\{1, 2, \cdots, rr\}$, and then selecting only those that are perversities. A sample output is given in figure 3 for the ten points on the tetractys triangle. The total number of perversities for a given pair of parameters is always a tetrahedral number. Considering the numerals $\{1, 2, 3\}$ as letters X, Y and Z in an alphabet, the usual integer simplex coordinates are given by the number of occurences of each letter in a word, so that for instance $111 \mapsto XXX \mapsto 300$, and all coordinates have entries summing to 3.

Once a library of perversities is obtained, as shown in figure 4, we count $P(n)$ by properly truncating the set. Although the computational complexity of this algorithm grows as expected with n, it only involves very elementary logic operations on lists of ordinals.



```
// Partitions Galore
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```
// generate all perversities max ord = rr, word length = ll
rr=3
ll=3
all = @[rr^ll]
for yy in 1..(rr^ll)
  all(yy) = @[ll]
end
for x in 1..ll
  for ii in 1..rr^ll   // mod(m,n), floor
    all(ii,x)= mod ( floor((ii-1)/(rr^(ll-x))) ,rr) + 1
  end
end
all
```

```
27x3 Matrix
```

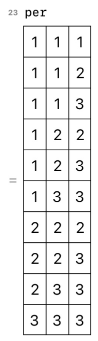
```
// reduce full list to tetrahedral simplex (commutative)
sim = Product((rr+n-1),n,1,ll) / Factorial(ll) // tetrahedral#
per = @[sim]
for vv in 1..sim
  per(vv) = @[ll]
end
count=1
check=0
for dd in 1..(rr^ll)
  for ff in 1..(ll-1)
    if all(dd,ff) > all(dd,ff+1)
      check +=1
    end
  end
  if check == 0
    for aa in 1..ll
      per(count,aa) = all(dd,aa)
    end
    count +=1
  end
  check=0
end
per
```

Figure 2: Building perversities and simplices

References

[1] H. S. Wilf, *Lectures on integer partitions*, University of Pennsylvania (2000)

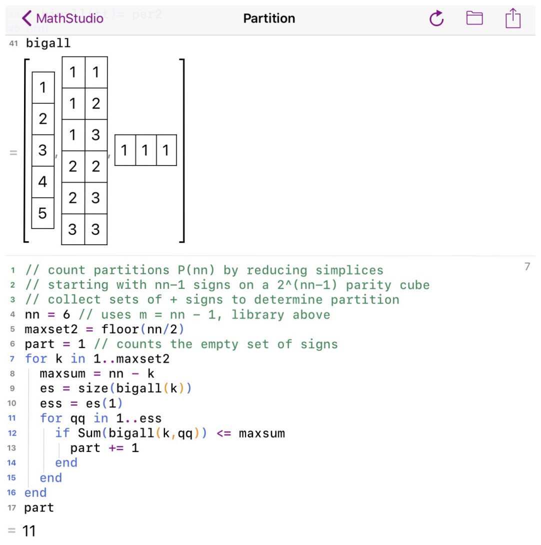
[2] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology 19, 2 (1980) 135-162



```
per
```

1	1	1
1	1	2
1	1	3
1	2	2
1	2	3
1	3	3
2	2	2
2	2	3
2	3	3
3	3	3

Figure 3: Tetractys simplex on 10 points for $rr = 3$ and $ll = 3$



```
bigall
```

$$= \left[\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \begin{bmatrix}1&1\\1&2\\1&3\\2&2\\2&3\\3&3\end{bmatrix}, \begin{bmatrix}1&1&1\end{bmatrix} \right]$$

```
// count partitions P(nn) by reducing simplices
// starting with nn-1 signs on a 2^(nn-1) parity cube
// collect sets of + signs to determine partition
nn = 6 // uses m = nn - 1, library above
maxset2 = floor(nn/2)
part = 1 // counts the empty set of signs
for k in 1..maxset2
  maxsum = nn - k
  es = size(bigall(k))
  ess = es(1)
  for qq in 1..ess
    if Sum(bigall(k,qq)) <= maxsum
      part += 1
    end
  end
end
part
```

= 11

Figure 4: Counting $P(6)$ by summing allowed perversities