Another method to solve the grasshopper problem(IMO)

Yasushi Ieno (vei04156@nifty.com)

Abstract

The 6th problem of the 50th International Mathematical Olympiad (IMO), held in Germany, 2009, is called 'the grasshopper problem'. To this problem Kos[1] developed theory from unique viewpoints by reference of Noga Alon's combinatorial Nullstellensatz.

We have tried to solve this problem by an original method inspired by a polynomial function that Kos defined in [1] , then researched for n=3 and n=4. For almost cases the problem can be solved, but there remains imperfection due to 'singularity'.

Keywords. inductive, combinatorial Nullstellensatz, Vandermonde polynomial

0.Introduction

The 6th problem of the 50th International Mathematical Olympiad (IMO), held in Germany, 2009, was the following.

Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of n$-$1 positive integers not containing s=a_1+a_2+$\cdots$+a_n. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

According to [1], Kos says that up to now, all known solutions to this problem, so called 'the grasshopper problem', are elementary and inductive, for example, by drawing a real axis on paper. In fact a solution of ours is one of its examples.

Then in [1], Kos tried to apply Noga Alon's combinatorial Nullstellensatz [2]. This theorem is effective but we couldn't rely on this entirely. So another method has proved to be necessary for further researches.

We try to present a way to solve the problem and prove it even if partially.

1. Alon's combinatorial Nullstellensatz

Now we introduce an interesting tool which may help our research.

Lemma 1 (Nonvanishing combinatorial Nullstellensatz).

Let $S_1, \ldots, S_n$ be nonempty subsets of a field F, and let $t_1, \ldots, t_n$ be nonnegative integers such that $t_i<|S_i|$ for i=1,2,. . .,n. Let P$(x_1, \ldots, x_n)$ be a polynomial over F with total degree $t_1+\cdots+t_n$, and suppose that the coefficient of ${x_1}^{t_1}{x_2}^{t_2}\cdots{x_n}^{t_n}$ in P$(x_1, \ldots, x_n)$ is nonzero. Then there exist elements $s_1{\in}S_1, \ldots, s_n{\in}S_n$ for which P$(s_1, \ldots, s_n){\neq}0$.

Also we present a polynomial function f$(x_1, x_2, \ldots, x_n)$ by reference of [1] as follows.

$$\begin{aligned}\mathrm{f}(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n) =&: (\mathrm{x}_1{-}\mathrm{m}_1)(\mathrm{x}_1{-}\mathrm{m}_2)\ldots(\mathrm{x}_1{-}\mathrm{m}_{n-1})(\mathrm{x}_1{+}\mathrm{x}_2{-}\mathrm{m}_1)\ldots\\ &(x_1{+}x_2{-}m_2)\ldots(x_1{+}x_2{-}m_{n-1})(x_1{+}\ldots{+}x_{n-1}{-}m_1)\ldots\\ &(\mathrm{x}_1{+}\ldots{+}\mathrm{x}_{n-1}{-}\mathrm{m}_2)\ldots(x_1{+}\ldots{+}x_{n-1}{-}m_{n-1})\\ &= \prod_{l=1}^{n-1}\prod_{i=1}^{n-1}((x_1{+}x_2+\ldots+x_l){-}m_i) \end{aligned} \tag{1}$$

On the grasshopper problem now if we fix the jumping order as a_1,a_2,. . ., a_n, then a grasshopper succeeds in its jumping without blocked if and only if f$(a_1, \ldots, a_n) \neq 0$, then the degree of f$(a_1, \ldots, a_n)$ is (n$-$1)^2. And $x_1^{\mathrm{n}-1}x_2^{\mathrm{n}-1}\cdots x_{n-1}^{\mathrm{n}-1}$ is a monomial the total degree of which is (n$-$1)^2, and the coefficient of which is 1.

Now we define n sets $S_1, S_2, \cdots, S_n$ such that $S_1{=}S_2{=}\cdots{=}S_n{=}\{a_1, a_2, \cdots, a_n\}$, then the number of elements of these n sets are $|S_1|{=}|S_2|{=}\cdots{=}|S_n|{=}$n $>$n$-$1, so we can adopt Lemma 1 to this polynomial function (1).

But there remains imperfection because the elements $a_1, a_2, ..., a_n$ considered in Lemma 1 are not necessarily distinct, that is to say, a pair of $(a_1, \ldots, a_n)$ may be the same number.

If we multiple f$(x_1, \ldots, x_n)$ by the so-called Vandermonde polynomial (see, for example, [3, pp. 346–347]), a new polynomial is created as follows.

$$\prod_{1\leq k<j\leq \mathrm{n}-1}(x_k - x_j)\prod_{l=1}^{\mathrm{n}-1}\prod_{i=1}^{\mathrm{n}-1}((x_1 + x_2 + \cdots + x_l) - m_i) \tag{2}$$

The elements $a_1, a_2, ..., a_n$ are required to be distinct if the new polynomial is nonzero when $x_i{=}a_i$ for any i such that 1≤i≤n. But any monomial of (2) the total degree of which is equal to the degree of (2), (n$-$1)^2+$_{\mathrm{n}-1}$C$_2$, has a factor the power of which is over n−1. Thus Lemma 1 can't be applied.

2. Attempts to use new polynomials by permutations

We could not apply Lemma 1 to f(x_1,x_2,...,x_n) if a_1,a_2,...,a_n are distinct.

We want to find out an effective polynomial function, on the condition that the total degree is kept, if possible.

By a permutation $\pi \in Sym$(n), we get

$$f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}) = \prod_{l=1}^{\mathrm{n}-1} \prod_{i=1}^{\mathrm{n}-1} ((x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(l)}) - m_i) \tag{3}$$

There are totally (n−1)^2 factors in (3).

Then we multiple each (3) by the signature of each permutation, that is +1 or −1, and make their summation as follows.

$$\sum_{\pi \in Sym(n)} sgn(\pi) f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}) = \sum_{\pi \in Sym(n)} sgn(\pi) \prod_{l=1}^{n-1} \prod_{i=1}^{n-1} ((x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(l)}) - m_i) \tag{4}$$

In (4) x_i and x_j is anti-symmetric if i is not equal to j, so it may be a multiple of the above-mentioned Vandermonde polynomial.

3. Real example for this case

3-1. n=3's case

Unfortunately Alon's combinatorial Nullstellensatz can't be applied now, because by simple calculations we can see that nothing but unsuitable 4-degree monomials like $x_1^3 x_2$, $x_1^3 x_3$ exist. In this case $|S_1|$ must be larger than 3, applying Lemma 1 is impossible.

We calculate (4) for n=3 by summing up 3!=6 polynomials as follows.

$$\sum_{\pi \in Sym(3)} sgn(\pi) f(x_{\pi(1)}, x_{\pi(2)}) = \sum_{\pi \in Sym(3)} sgn(\pi) \prod_{l=1}^{2} \prod_{i=1}^{2} ((x_{\pi(1)} + x_{\pi(2)}) - m_i) \tag{5}$$

The calculation of (5) is the following.

$$
\begin{aligned}
(5) = {} & f(x_1,x_2,x_3) - f(x_1,x_3,x_2) - f(x_2,x_1,x_3) \\
& + f(x_2,x_3,x_1) + f(x_3,x_1,x_2) - f(x_3,x_2,x_1) \\
& = (x_1 - m_1)(x_1 - m_2)(x_1 + x_2 - m_1)(x_1 + x_2 - m_2) \\
& - (x_1 - m_1)(x_1 - m_2)(x_1 + x_3 - m_1)(x_1 + x_3 - m_2) \\
& - (x_2 - m_1)(x_2 - m_2)(x_2 + x_1 - m_1)(x_2 + x_1 - m_2) \\
& + (x_2 - m_1)(x_2 - m_2)(x_2 + x_3 - m_1)(x_2 + x_3 - m_2) \\
& + (x_3 - m_1)(x_3 - m_2)(x_3 + x_1 - m_1)(x_3 + x_1 - m_2) \\
& - (x_3 - m_1)(x_3 - m_2)(x_3 + x_2 - m_1)(x_3 + x_2 - m_2) \\
& = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)((x_1 + x_2 + x_3) - (m_1 + m_2)) \qquad (6)
\end{aligned}
$$

We present other calculations. 3 pairs of the above 6 polynomials appear by turns.

$$
\begin{aligned}
& f(x_1,x_2,x_3) - f(x_1,x_3,x_2) \\
& \quad = (x_1 - m_1)(x_1 - m_2)(x_1 + x_2 - m_1)(x_1 + x_2 - m_2) \\
& \quad - (x_1 - m_1)(x_1 - m_2)(x_1 + x_3 - m_1)(x_1 + x_3 - m_2) \\
& \quad = (x_1 - m_1)(x_1 - m_2)((2x_1 + x_2 + x_3) - (m_1 + m_2)) \qquad (7) \\
& f(x_2,x_1,x_3) - f(x_2,x_3,x_1) \\
& \quad = (x_2 - m_1)(x_2 - m_2)(x_2 + x_1 - m_1)(x_2 + x_1 - m_2) \\
& \quad - (x_2 - m_1)(x_2 - m_2)(x_2 + x_3 - m_1)(x_2 + x_3 - m_2) \\
& \quad = (x_2 - m_1)(x_2 - m_2)((x_1 + 2x_2 + x_3) - (m_1 + m_2)) \qquad (8) \\
& f(x_3,x_1,x_2) - f(x_3,x_2,x_1) \\
& \quad = (x_3 - m_1)(x_3 - m_2)(x_3 + x_1 - m_1)(x_3 + x_1 - m_2) \\
& \quad - (x_3 - m_1)(x_3 - m_2)(x_3 + x_2 - m_1)(x_3 + x_2 - m_2) \\
& \quad = (x_3 - m_1)(x_3 - m_2)((x_1 + x_2 + 2x_3) - (m_1 + m_2)) \qquad (9)
\end{aligned}
$$

Theorem 1.

Let a_1, a_2, a_3 be distinct positive integers, and m_1, m_2 be distinct positive integers, then there exists $\pi \in Sym(3)$ that holds

$$
\begin{aligned}
& f(a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}) = \\
& (a_{\pi(1)} - m_1)(a_{\pi(1)} - m_2)(a_{\pi(1)} + a_{\pi(2)} - m_1)(a_{\pi(1)} + a_{\pi(2)} - m_2) \\
& \quad \neq 0. \qquad (10)
\end{aligned}
$$

Proof.

If $f(a_{\pi(1)}, a_{\pi(2)})=(a_{\pi(1)}-m_1)(a_{\pi(1)}-m_2)(a_{\pi(1)}+a_{\pi(2)}-m_1)$
$\times(a_{\pi(1)}+a_{\pi(2)}-m_2)=0$ for any $\pi\in Sym(3)$,
then four equations hold as below by (6),(7),(8) and(9).

$$(a_1-a_2)(a_1-a_3)(a_2-a_3)((a_1+a_2+a_3)-(m_1+m_2)) = 0. \tag{11}$$
$$(a_1-m_1)(a_1-m_2)((2a_1+a_2+a_3)-(m_1+m_2)) = 0. \tag{12}$$
$$(a_2-m_1)(a_2-m_2)((a_1+2a_2+a_3)-(m_1+m_2)) = 0. \tag{13}$$
$$(a_3-m_1)(a_3-m_2)((a_1+a_2+3a_3)-(m_1+m_2)) = 0. \tag{14}$$

From (11), $(a_1+a_2+a_3)-(m_1+m_2)=0$ follows, because a_1, a_2, a_3 are distinct. Then neither $2(a_1+a_2+a_3)-(m_1+m_2)$ nor $(a_1+2a_2+a_3)-(m_1+m_2)$ nor $(a_1+a_2+3a_3)-(m_1+m_2)$ is equal to 0, so $(a_1-m_1)(a_1-m_2)=0$
and $(a_2-m_1)(a_2-m_2)=0$ and $(a_3-m_1)(a_3-m_2)=0$ at (12), (13) and (14), which does not happen at the same time, this is because a_1, a_2 and a_3 are distinct and m_1 and m_2 are also distinct.

It follows that the above-mentioned assumption does not come true.

This completes the proof.

□

If $f(a_1, a_2, a_3) \neq 0$, then at least one of the above-mentioned six polynomials consisting of (6) is not 0. Therefore the claim of the grasshopper problem follows for n=3, that is to say, a grasshopper succeeds in jumping without landing on m_1 or m_2 by choosing one order (a_{i1}, a_{i2}, a_{i3}) out of six possible jumping orders, such that $f(a_{i1}, a_{i2}, a_{i3})=(a_{i1}-m_1)(a_{i1}-m_2)(a_{i1}+a_{i2}-m_1)$
$(a_{i1}+a_{i2}-m_2)\neq 0$.

For the n=3's case of the grasshopper problem, $\{(a_1, a_2, a_3)|(a_1+a_2+a_3)$
$-(m_1+m_2) = 0\}$ is a 'singularity' set that may vanish the possibility of a grasshopper's safe jumping. But by comparing (6) with (7),(8) and (9), this possibility bas been easily denied.

3-2. n=4's case

We sum up 4!=24 polynomials which were made by permutation; as follows.

$$\sum_{\pi\in Sym(4)} sgn(\pi) f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$$
$$= \sum_{\pi\in Sym(4)} sgn(\pi) \prod_{l=1}^{3}\prod_{i=1}^{3} ((x_{\pi(1)} + x_{\pi(2)} + x_{\pi(3)}) - m_i) \tag{15}$$

The degree is 3^2=9 and the permutation number is 4!=24, so the calculation of (15) is more complicated. We present the calculating results for the n=4's case, similarly as the n=3's case, as below.

$$\begin{aligned}
(15) = {} & (x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4) \\
& \times(3(x_1+x_2+x_3+x_4)-2(m_1+m_2+m_3)) \\
& \times(6(x_1^2+x_2^2+x_3^2+x_4^2)+8(m_1m_2+m_1m_3+m_1m_4+m_2m_3+m_2m_4+m_3m_4) \\
& \quad -7(m_1+m_2+m_3)(x_1+x_2+x_3+x_4) \\
& \quad +(m_1^2+m_2^2+m_3^2+6m_1m_2+6m_2m_3+6m_3m_1))
\end{aligned} \tag{16}$$

$$\begin{aligned}
& f(x_1,x_2,x_3,x_4)-f(x_1,x_2,x_4,x_3) \\
& =(x_3-m_4) \\
& \times(({3x_1}^2+{3x_2}^2+{x_3}^2+{x_4}^2)+(6x_1x_2+3x_1x_3+3x_1x_4+3x_2x_3+3x_2x_4+x_3x_4) \\
& \quad -(m_1+m_2)(2x_1+2x_2+x_3+x_4)+m_1m_2m_3)
\end{aligned} \tag{17}$$

Now generalizing (17), for $(x_{j1},x_{j2},x_{j3},x_{j4})$, any permutation of (x_1,x_2,x_3,x_4), we obtain

$$\begin{aligned}
& f(x_{j1},x_{j2},x_{j3},x_{j4})-f(x_{j1},x_{j2},x_{j4},x_{j3}) \\
& =(m_{j3}-m_{j4}) \\
& \times((3x_{j1}^2+3x_{j2}^2+x_{j3}^2+x_{j4}^2)+(6x_{j1}x_{j2}+3x_{j1}x_{j3}+3x_{j1}x_{j4}+3x_{j2}x_{j3}+3x_{j2}x_{j4}+x_{j3}x_{j4}) \\
& \quad -(m_{j1}+m_{j2})(2x_{j1}+2x_{j2}+x_{j3}+x_{j4})+m_{j1}m_{j2}m_{j3})
\end{aligned} \tag{18}$$

From (16), for the n=4's case of the grasshopper problem, we can obtain that

$$\begin{aligned}
& \{(a_1,a_2,a_3,a_4)| \\
& \quad (3(a_1+a_2+a_3+a_4)-2(m_1+m_2+m_3)) \\
& \quad \times(6(a_1^2+a_2^2+a_3^2+a_4^2) \\
& \quad\quad +8(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4) \\
& \quad\quad -7(m_1+m_2+m_3)(a_1+a_2+a_3+a_4) \\
& \quad\quad +(m_1^2+m_2^2+m_3^2+6m_1m_2+6m_2m_3+6m_3m_1)) \; = \; 0\}
\end{aligned} \tag{19}$$

is a 'singularity' set that may vanish the possibility of a grasshopper's safe jumping.

Unlike the n=3's case, the comparison of (18) and (19) does not lead to the solution of the grasshopper problem yet for n=4.

4. Discussion and conclusion

As we explained in the introduction, this grasshopper problem can be proved only by elementary and inductive methods(see [1]), for example, by drawing a real axis on paper.

And if they intend to solve by the current method we have shown, there is not perfection yet.

We can easily assume anti-symmetry of the polynomial function (4). But there is a big drawback, that is 'singularity'. It is not easy to analyze when n is more than 3. In short, we are still destined to solve elementarily and deductively, though in most cases, except for 'singularity', a grasshopper succeeds in jumping, judging from (4).

In the future we want to solve the grasshopper problem by analyzing equations for larger n's.

references

[1] Geza Kos,6, arvix, available at https://arxiv.org/abs/1008.2936
[2] N. Alon, Combinatorial Nullstellensatz, Combin. Probab. Comput. 8 (1999) 7–29.
[3] M. Aigner, A Course in Enumeration, Springer-Verlag, Berlin, 2007.