

# Another research for solving the grasshopper problem(IMO)

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abstract

The 6th problem of the 50th International Mathematical Olympiad (IMO), held in Germany, 2009, is called 'the grasshopper problem'. To this problem Kos[1] developed theory from unique viewpoints by reference of Noga Alon's combinatorial Nullstellensatz.

We have tried to solve this problem by an original method inspired by a polynomial that Kos defined in [1], then researched for  $n=3$  and  $n=4$ . For almost cases the problem can be solved, but there remains imperfection due to 'singularity'.

## 0.Introduction

The 6th problem of the 50th International Mathematical Olympiad (IMO), held in Germany, 2009, was the following.

Let  $a_1, a_2, \dots, a_n$  be distinct positive integers and let  $M$  be a set of  $n-1$  positive integers not containing  $s = a_1 + a_2 + \dots + a_n$ . A grasshopper is to jump along the real axis, starting at the point 0 and making  $n$  jumps to the right with lengths  $a_1, a_2, \dots, a_n$  in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in  $M$ .

According to [1], Kos says that up to now, all known solutions to this problem, so called 'the grasshopper problem', are elementary and inductive, for example, by drawing a real axis on paper. In fact a solution of ours is one of its examples.

Then in [1], Kos tried to apply Noga Alon's combinatorial Nullstellensatz [2]. This theorem is effective but we couldn't rely on this entirely.

So another method proves to be necessary for further researches. We try to present the way to solve the problem and prove it partially

## 1. Alon's combinatorial Nullstellensatz

Now we introduce an interesting tool which helps our research.

Lemma 1 (Nonvanishing combinatorial Nullstellensatz).

Let  $S_1, \dots, S_n$  be nonempty subsets of a field  $F$ , and let  $t_1, \dots, t_n$  be nonnegative integers such that  $t_i < |S_i|$  for  $i = 1, 2, \dots, n$ . Let  $P(x_1, \dots, x_n)$

be a polynomial over  $F$  with total degree  $t_1 + \dots + t_n$ , and suppose that the coefficient of  $x_1^{t_1} x_2^{t_2} \dots x_n^{t_n}$  in  $P(x_1, \dots, x_n)$  is nonzero. Then there exist elements  $s_1 \in S_1, \dots, s_n \in S_n$  for which  $P(s_1, \dots, s_n) \neq 0$ .

Also we define a polynomial function  $f(x_1, x_2, \dots, x_n)$  as follows.

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= (x_1 - m_1)(x_1 - m_2) \dots (x_1 - m_{n-1})(x_1 + x_2 - m_1) \dots \\ &\quad (x_1 + x_2 - m_2) \dots (x_1 + x_2 - m_{n-1})(x_1 + \dots + x_{n-1} - m_1) \dots \\ &\quad (x_1 + \dots + x_{n-1} - m_2) \dots (x_1 + \dots + x_{n-1} - m_{n-1}) \\ &= \prod_{l=1}^{n-1} \prod_{i=1}^l ((x_1 + x_2 + \dots + x_l) - m_i) \end{aligned} \quad (1)$$

On this grasshopper problem now if we fix the jumping order as  $a_1, a_2, \dots, a_n$ , then a grasshopper succeeds in its jumping without blocked if and only if  $f(a_1, \dots, a_n) \neq 0$ , then the degree of  $f(a_1, \dots, a_n)$  is  $(n-1)^2$ . And  $x_1^{n-1} x_2^{n-1} \dots x_{n-1}^{n-1}$  is a monomial the total degree of which is  $(n-1)^2$ , and the coefficient of which is 1.

Now we define  $n$  sets  $S_1, S_2, \dots, S_n$  such that  $S_1 = S_2 = \dots = S_n = \{a_1, a_2, \dots, a_n\}$ , then the number of elements of these  $n$  sets are  $|S_1| = |S_2| = \dots = |S_n| = n > n-1$ , so we can adopt Lemma 1 to this polynomial function (1).

But there remains imperfection because the elements  $a_1, a_2, \dots, a_n$  considered in Lemma 1 are not necessarily distinct, that is to say, a pair of  $(a_1, \dots, a_n)$  may be the same number.

If we multiple  $f(x_1, \dots, x_n)$  by the so-called Vandermonde polynomial (see, for example, [3, pp. 346–347]), a new polynomial is created as follows.

$$\prod_{1 \leq k < j \leq n-1} (x_k - x_j) \prod_{l=1}^{n-1} \prod_{i=1}^{n-1} ((x_1 + x_2 + \dots + x_l) - m_i) \quad (2)$$

The elements  $a_1, a_2, \dots, a_n$  are required to be distinct if the new polynomial is nonzero when  $x_i = a_i$  for any  $i$  such that  $1 \leq i \leq n$ . But any monomial of (2) the total degree of which is equal to the degree of (2),  $(n-1)^2 +_{n-1} C_2$ , has a factor the power of which is over  $n-1$ . Thus Lemma 1 can't be applied.

## 2. Attempts to use new polynomials by permutations

We could not apply Lemma 1 to  $f(x_1, x_2, \dots, x_n)$  if  $a_1, a_2, \dots, a_n$  are distinct.

We want to find out an effective polynomial, on the condition that the total degree is kept, if possible.

By a permutation  $\pi \in \text{Sym}(n)$ , we get

$$f(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$$

$$= \prod_{l=1}^{n-1} \prod_{i=1}^{n-1} ((x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(l)}) - m_i) \quad (3)$$

There are totally  $(n-1)^2$  factors in (3).

Then we multiple each (3) by the signature of each permutation, that is +1 or -1, and make their summation as follows.

$$\begin{aligned} & \sum_{\pi \in Sym(n)} sgn(\pi) f(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}) \\ &= \sum_{\pi \in Sym(n)} sgn(\pi) \prod_{l=1}^{n-1} \prod_{i=1}^{n-1} ((x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(l)}) - m_i) \quad (4) \end{aligned}$$

In (4)  $x_i$  and  $x_j$  is anti-symmetric if  $i$  is not equal to  $j$ , so it may be a multiple of the above-mentioned Vandermonde polynomial.

### 3. Real example for this case

#### 3-1. $n=3$ 's case

Unfortunately Alon's combinatorial Nullstellensatz can't be applied now, because by simple calculations we can see that nothing but unsuitable 4-degree monomials like  $x_1^3 x_2$ ,  $x_1^3 x_3$  exist. In this case  $|S_1|$  must be larger than 3, applying Lemma 1 is impossible.

We calculate (4) for  $n=3$  by summing up  $3!=6$  polynomials as follows.

$$\begin{aligned} & \sum_{\pi \in Sym(3)} sgn(\pi) f(x_{\pi(1)}, x_{\pi(2)}) \\ &= \sum_{\pi \in Sym(3)} sgn(\pi) \prod_{l=1}^2 \prod_{i=1}^2 ((x_{\pi(1)} + x_{\pi(2)}) - m_i) \quad (5) \end{aligned}$$

The calculation of (5) is the following.

$$\begin{aligned} (5) &= f(x_1, x_2, x_3) - f(x_1, x_3, x_2) - f(x_2, x_1, x_3) \\ & \quad + f(x_2, x_3, x_1) + f(x_3, x_1, x_2) - f(x_3, x_2, x_1) \\ &= (x_1 - m_1)(x_1 - m_2)(x_1 + x_2 - m_1)(x_1 + x_2 - m_2) \\ & \quad - (x_1 - m_1)(x_1 - m_2)(x_1 + x_3 - m_1)(x_1 + x_3 - m_2) \\ & \quad - (x_2 - m_1)(x_2 - m_2)(x_2 + x_1 - m_1)(x_2 + x_1 - m_2) \\ & \quad + (x_2 - m_1)(x_2 - m_2)(x_2 + x_3 - m_1)(x_2 + x_3 - m_2) \\ & \quad + (x_3 - m_1)(x_3 - m_2)(x_3 + x_1 - m_1)(x_3 + x_1 - m_2) \\ & \quad - (x_3 - m_1)(x_3 - m_2)(x_3 + x_2 - m_1)(x_3 + x_2 - m_2) \\ &= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)((x_1 + x_2 + x_3) - (m_1 + m_2)) \quad (6) \end{aligned}$$

We present other calculations. 3 pairs of the above 6 polynomials appear by turns.

$$\begin{aligned}
& f(x_1, x_2, x_3) - f(x_1, x_3, x_2) \\
&= (x_1 - m_1)(x_1 - m_2)(x_1 + x_2 - m_1)(x_1 + x_2 - m_2) \\
&\quad - (x_1 - m_1)(x_1 - m_2)(x_1 + x_3 - m_1)(x_1 + x_3 - m_2) \\
&= (x_1 - m_1)(x_1 - m_2)((2x_1 + x_2 + x_3) - (m_1 + m_2)) \tag{7}
\end{aligned}$$

$$\begin{aligned}
& f(x_2, x_1, x_3) - f(x_2, x_3, x_1) \\
&= (x_2 - m_1)(x_2 - m_2)(x_2 + x_1 - m_1)(x_2 + x_1 - m_2) \\
&\quad - (x_2 - m_1)(x_2 - m_2)(x_2 + x_3 - m_1)(x_2 + x_3 - m_2) \\
&= (x_2 - m_1)(x_2 - m_2)((x_1 + 2x_2 + x_3) - (m_1 + m_2)) \tag{8}
\end{aligned}$$

$$\begin{aligned}
& f(x_3, x_1, x_2) - f(x_3, x_2, x_1) \\
&= (x_3 - m_1)(x_3 - m_2)(x_3 + x_1 - m_1)(x_3 + x_1 - m_2) \\
&\quad - (x_3 - m_1)(x_3 - m_2)(x_3 + x_2 - m_1)(x_3 + x_2 - m_2) \\
&= (x_3 - m_1)(x_3 - m_2)((x_1 + x_2 + 2x_3) - (m_1 + m_2)) \tag{9}
\end{aligned}$$

Theorem 1.

Let  $a_1, a_2, a_3$  be distinct positive integers, and  $m_1, m_2$  be distinct positive integers, then there exists  $\pi \in Sym(3)$  that holds

$$\begin{aligned}
& f(a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}) = \\
& (a_{\pi(1)} - m_1)(a_{\pi(1)} - m_2)(a_{\pi(1)} + a_{\pi(2)} - m_1)(a_{\pi(1)} + a_{\pi(2)} - m_2) \\
& \neq 0. \tag{10}
\end{aligned}$$

*Proof.*

If for any  $\pi \in Sym(3)$   $f(a_{\pi(1)}, a_{\pi(2)}) = (a_{\pi(1)} - m_1)(a_{\pi(1)} - m_2)(a_{\pi(1)} + a_{\pi(2)} - m_1)(a_{\pi(1)} + a_{\pi(2)} - m_2) = 0$ , then 4 equations hold as below by (6), (7), (8) and (9).

$$(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)((a_1 + a_2 + a_3) - (m_1 + m_2)) = 0. \tag{11}$$

$$(a_1 - m_1)(a_1 - m_2)((2a_1 + a_2 + a_3) - (m_1 + m_2)) = 0. \tag{12}$$

$$(a_2 - m_1)(a_2 - m_2)((a_1 + 2a_2 + a_3) - (m_1 + m_2)) = 0. \tag{13}$$

$$(a_3 - m_1)(a_3 - m_2)((a_1 + a_2 + 3a_3) - (m_1 + m_2)) = 0. \tag{14}$$

From (11),  $(a_1 + a_2 + a_3) - (m_1 + m_2) = 0$  follows, because  $a_1, a_2, a_3$  are distinct. Then neither  $2(a_1 + a_2 + a_3) - (m_1 + m_2)$  nor  $(a_1 + 2a_2 + a_3) - (m_1 + m_2)$  nor  $(a_1 + a_2 + 3a_3) - (m_1 + m_2)$  is equal to 0, so  $(a_1 - m_1)(a_1 - m_2) = 0$  and  $(a_2 - m_1)(a_2 - m_2) = 0$  and  $(a_3 - m_1)(a_3 - m_2) = 0$  at (12), (13) and (14), which does not happen at the same time, this is because  $a_1, a_2$  and  $a_3$  are distinct and  $m_1$  and  $m_2$  are also distinct.

It follows that the above-mentioned assumption does not come true.

This completes the proof. □

If  $f(a_1, a_2, a_3) \neq 0$ , then at least one of the above-mentioned six polynomials consisting of (6) is not 0. Therefore the claim of the grasshopper problem follows for  $n=3$ , that is to say, a grasshopper succeeds in jumping without landing on  $m_1$  or  $m_2$  by choosing one order  $(a_{i1}, a_{i2}, a_{i3})$  out of six possible jumping orders, such that  $f(a_{i1}, a_{i2}, a_{i3}) = (a_{i1} - m_1)(a_{i1} - m_2)(a_{i1} + a_{i2} - m_1)(a_{i1} + a_{i2} - m_2) \neq 0$ .

For the  $n=3$ 's case of the grasshopper problem,  $\{(a_1, a_2, a_3) | (a_1 + a_2 + a_3) - (m_1 + m_2) = 0\}$  is a 'singularity' set that may vanish the possibility of a grasshopper's safe jumping. But by comparing (6) with (7), (8) and (9), this possibility has been easily denied.

### 3-2. $n=4$ 's case

I sum up  $4! = 24$  polynomials which were made by permutation; as follows.

$$\begin{aligned} & \sum_{\pi \in Sym(4)} sgn(\pi) f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) \\ &= \sum_{\pi \in Sym(4)} sgn(\pi) \prod_{l=1}^3 \prod_{i=1}^3 ((x_{\pi(1)} + x_{\pi(2)} + x_{\pi(3)}) - m_i) \quad (15) \end{aligned}$$

The degree is  $3^2 = 9$  and the permutation number is  $4! = 24$ , so the calculation of (15) is more complicated. We present the calculating results for the  $n=4$ 's case, similarly as the  $n=3$ 's case, as below.

$$\begin{aligned} (15) &= (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) \\ &\times (3(x_1 + x_2 + x_3 + x_4) - 2(m_1 + m_2 + m_3)) \\ &\times (6(x_1^2 + x_2^2 + x_3^2 + x_4^2) + 8(m_1 m_2 + m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4) \\ &\quad - 7(m_1 + m_2 + m_3)(x_1 + x_2 + x_3 + x_4) \\ &\quad + (m_1^2 + m_2^2 + m_3^2 + 6m_1 m_2 + 6m_2 m_3 + 6m_3 m_1)) \quad (16) \end{aligned}$$

$$\begin{aligned} & f(x_1, x_2, x_3, x_4) - f(x_1, x_2, x_4, x_3) \\ &= (x_3 - m_4) \times \\ & ((3x_1^2 + 3x_2^2 + x_3^2 + x_4^2) + (6x_1 x_2 + 3x_1 x_3 + 3x_1 x_4 + 3x_2 x_3 + 3x_2 x_4 + x_3 x_4) \\ & \quad - (m_1 + m_2)(2x_1 + 2x_2 + x_3 + x_4) + m_1 m_2 m_3) \quad (17) \end{aligned}$$

Now generalizing (17), for  $(x_{j1}, x_{j2}, x_{j3}, x_{j4})$ , any permutation of  $(x_1, x_2, x_3, x_4)$ , we obtain

$$\begin{aligned} & f(x_{j1}, x_{j2}, x_{j3}, x_{j4}) - f(x_{j1}, x_{j2}, x_{j4}, x_{j3}) \\ &= (m_{j3} - m_{j4}) \times \\ & ((3x_{j1}^2 + 3x_{j2}^2 + x_{j3}^2 + x_{j4}^2) + (6x_{j1} x_{j2} + 3x_{j1} x_{j3} + 3x_{j1} x_{j4} + 3x_{j2} x_{j3} + 3x_{j2} x_{j4} + x_{j3} x_{j4}) \\ & \quad - (m_{j1} + m_{j2})(2x_{j1} + 2x_{j2} + x_{j3} + x_{j4}) + m_{j1} m_{j2} m_{j3}) \quad (18) \end{aligned}$$

From (16), for the n=4's case of the grasshopper problem, we can obtain that

$$\begin{aligned} & \{(a_1, a_2, a_3, a_4) | \\ & (3(a_1+a_2+a_3+a_4)-2(m_1+m_2+m_3)) \\ & \times (6(a_1^2+a_2^2+a_3^2+a_4^2) \\ & +8(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4) \\ & -7(m_1+m_2+m_3)(a_1+a_2+a_3+a_4) \\ & +(m_1^2+m_2^2+m_3^2+6m_1m_2+6m_2m_3+6m_3m_1)) = 0\} \quad (19) \end{aligned}$$

is a 'singularity' set that may vanish the possibility of a grasshopper's safe jumping.

Unlike the n=3's case, the comparison of (18) and (19) does not lead to the solution of the grasshopper problem for n=4 for now.

#### 4. Discussion and conclusion

As I explained in the introduction, this grasshopper problem can be proved by elementary and inductive methods(see [1]) by drawing a real axis on paper.

But if they intend to solve with the current method we have shown, there is not perfection yet.

We can easily assume anti-symmetry of the polynomial (4). But there is a big drawback, that is 'singularity'. It is not easy to analyze when n is more than 3. In short, we are still destined to solve elementarily and deductively, though in most cases, except for 'singularity', a grasshopper succeeds in jumping, judging from (4).

In the future we want to solve the grasshopper problem by analyzing equations for greater n's.

#### references

- [1] Geza Kos,6, arxiv, available at <https://arxiv.org/abs/1008.2936>
- [2] N. Alon, Combinatorial Nullstellensatz, Combin. Probab. Comput. 8 (1999) 7–29.
- [3] M. Aigner, A Course in Enumeration, Springer-Verlag, Berlin, 2007.