Solid Strips

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Abstract

In this paper we introduce the idea of a Solid Strip which is the generalization to higher dimensions of 2-dimensional untwisted Mobius strips.

Key Words: compact manifold, topology.

1 Introduction

For the purpose of this paper, we will call a strip is a 2-dimensional manifold with boundary obtained by identifying 2 opposite edges of the 4 edges of a square. It can be done without a twist (Untwisted Strip) or with a twist (Mobius strip).

In 2-dimensions we have only two possible configurations. An untwisted strip which has a boundary composed by two circles and it is orientable and a Mobius Strip that has a boundary composed only by one circles and it is not orientable. We want to generalise the idea of a strip to the 3-dimensional case.

2 Solid Strips

The idea of a strip described above can be easily extended to 3-dimensional manifolds. In 2-dimensions we have 2-D strips obtained by identifying one couple of opposite edges of the two couples of edges of a square. In 3-dimensions we will have "Solid Strips" which are 3-D "strips" obtained by identifying two couples of opposite faces of the three couples of faces of a cube.

Solid trips are like "rings" in $\mathbb{R}^4$ meaning that they can be linked like you link in $\mathbb{R}^3$ the links of a common steel chain you can buy in an hardware shop, and they cannot be separated without breaking them. There are 8 ways of identifying two opposite faces of a cube (group of symmetries of a square keeping the other one fixed) which, for two couples, gives a total of 64 different possible manifolds. However these manifolds form homeomorphic classes and therefore the number of actual spaces is much lower.

All the above 64 mentioned configurations are reported in the two tables in Appendix A1 where vertices numbering of the cube used to build the configurations is defined in Fig. 1. The 2^{nd} and 3^{rd} columns of the two tables contain the way opposite faces of the cube are identified for that specific configuration and

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the 4^{th} and 5^{th} columns contain the way edges and vertices of the two remaining faces (the up and bottom faces in Fig. 1) get identified (to better understand how to read these 2 columns, refer to Fig. 1).

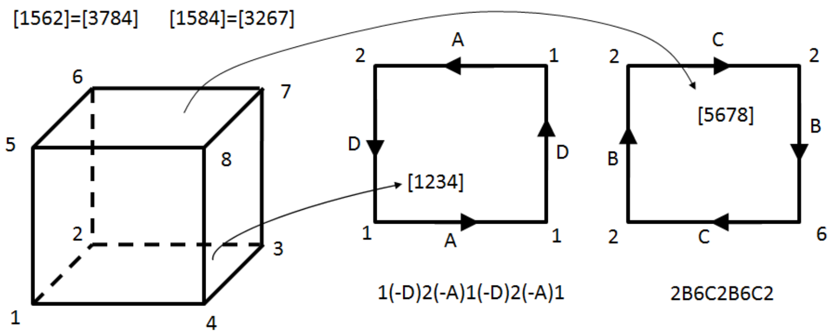

Figure 1: Example of a Solid Strip Configuration - Boundary $\mathbf{RP}^2 \vee \mathbf{RP}^2$

Given the above configurations, 2 out of 64 (configurations 63 and 64 in the tables in Appendix A1) lead to a not feasible space. To see this let us consider configuration 63. The identification requirements from columns 2 ask edge [15] to be identified with edge [73] while from column 3 we are asked to identify edge [15] to edge [73] and the two identification are obviously impossible to be done at the same time.

The 64 configurations in some cases are equivalent each other. For example, in configuration 40 in table A1.2 we are requested to rotate a face of a couple by 90deg and then identify it to the other face. In configuration 41 we are requested to do the same thing to the other couple of faces. The two configurations are obviously equivalent and we can go from one to another by rotating the cube by 90deg around the z axis. More analytically we have that configurations 40 and 41 are given by the following identification requirements:

$$\begin{array}{lll} C_{40}: & [1562]=[4873] & [1584]=[6732] \\ C_{41}: & [1562]=[3487] & [1584]=[2673] \end{array} \tag{1}$$

by applying a rotation of 90deg around the z axis, to configuration 40, given by the following permutation of cube vertices:

$$RotZ = \begin{pmatrix} 12345678 \\ 23416785 \end{pmatrix} \tag{2}$$

we get the following configuration:

$$C'_{40}: \quad [2673]=[1584] \quad [2651]=[7843] \tag{3}$$

which is, by permuting the order of the vertices, configuration 41. The two configuration are therefore equivalent and the two relevant spaces are homeomorphic.

We have written a simple code that applies all the 48 symmetries of a cube to each configuration in Appendix A1 and compare the results with the other configurations looking for equivalences. By using this code, we have classified all 64 spaces in 22 equivalent classes (one of which composed of non feasible configurations) leading to a total of 21 equivalent homeomorphic classes of spaces reported in the table below:

For each class, we have finally evaluated the boundary of each space (see Appendix A2) and we have found 12 different boundaries which are reported in the last column of the table below.

Cla.	Ident. 1	Ident. 2	Face 1 [1234]	Face 2 [5678]	Bound.
1	[1562]=[4873]	[1584]=[2673]	1(-C)1(-E)1C1E1	5A5D5(-A)5(-D)5	$\mathbf{T}^2 ; \mathbf{T}^2$
2	[1562]=[4873]	[1584]=[3762]	1(-C)1E1C1E1	5A5(-D)5(-A)5(-D)5	$\mathbf{K} ; \mathbf{K}$
3	[1562]=[3784]	[1584]=[3762]	1(-D)2F1(-D)2F1	5B6(-E)5B6(-E)5	$\mathbf{RP}^2 ; \mathbf{RP}^2$
4	[1562]=[3784]	[1584]=[3267]	1(-D)2(-A)1(-D)2(-A)1	2B6C2B6C2	$\mathbf{RP}^2 \vee \mathbf{RP}^2$
5	[1562]=[3487]	[1584]=[3267]	1(-D)2(-A)1A2D1	2B6(-B)2(-C)6C2	$\mathbf{X}_1 \vee \mathbf{X}_1$
6	[1562]=[7348]	[1584]=[7326]	1(-D)2(-E)3B4F1	3B4F1(-D)2(-E)3	$\mathbf{S}^2$
7	[1562]=[7348]	[1584]=[2376]	1(-D)1A3B4C1	3B4C1(-D)1A3	$\mathbf{X}_1$
8	[1562]=[4873]	[1584]=[6237]	1(-C)2D2C1E1	2A1(-E)1(-A)2(-D)2	$\mathbf{T}^2$
9	[1562]=[4873]	[1584]=[7326]	1(-C)2(-D)2C1E1	2A1E1(-A)2(-D)2	$\mathbf{T}^2$
10	[1562]=[8437]	[1584]=[6237]	1(-C)2D1(-A)2E1	2A1(-E)2C1(-D)2	$\mathbf{T}^2$
11	[1562]=[8437]	[1584]=[3762]	1(-C)2E1(-A)2E1	2A1(-D)2C1(-D)2	$\mathbf{K}$
12	[1562]=[8437]	[1584]=[3267]	1(-D)2(-A)1(-B)2(-C)1	2B1A2D1C2	$\mathbf{X}_2$
13	[1562]=[8437]	[1584]=[7326]	1(-C)1(-D)3(-A)3E1	3A3E1C1(-D)3	$\mathbf{K}$
14	[1562]=[8437]	[1584]=[2376]	1(-D)1A3(-B)3C1	3B3(-A)1D1(-C)3	$\mathbf{X}_2$
15	[1562]=[3487]	[1584]=[6732]	1(-D)2B1A2(-C)1	2B1A2(-C)1(-D)2	$\mathbf{X}_2$
16	[1562]=[4378]	[1584]=[2376]	1(-D)1A3(-A)1C1	3B1(-D)1C1(-B)3	$\mathbf{X}_2$
17	[1562]=[4873]	[1584]=[3267]	1(-D)1(-A)1D1C1	1B1(-A)1(-B)1C1	$\mathbf{Y}_1$
18	[1562]=[4873]	[1584]=[2376]	1(-D)1A1D1C1	1B1A1(-B)1C1	$\mathbf{Y}_1$
19	[1562]=[3784]	[1584]=[2376]	1(-D)1A1(-D)1C1	1B1(-C)1B1(-A)1	$\mathbf{Y}_1$
20	[1562]=[3487]	[1584]=[7326]	1(-B)1(-C)1B1D1	1A1D1(-A)1(-C)1	$\mathbf{Y}_1$
21	[1562]=[3487]	[1584]=[2376]	1(-D)1A1A1C1	1B1B1(-C)1D1	$\mathbf{Z}_1$

Table 1 : Classes of Equivalent Solid Strip Configurations

where:

- Space $\mathbf{X}_1$: is a 2-sphere where two separate points of the sphere are identified (see also Appendix A2). This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{X}_1 \vee \mathbf{X}_1$: is a wedge sum of two $\mathbf{X}_1$ spaces (see also Appendix A2). This space has three points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{X}_2$: is a 2-sphere where two couple of separate points of the sphere are identified (see also Appendix A2). This space has two points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Y}_1$: is a 2-torus where two separate points of the torus are identified (see also Appendix A2). This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Z}_1$: is a Klein Bottle where two separate points of the Klein Bottle are identified (see also Appendix A2). This space has a point where

the manifold is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.

Our hypothesis is that spaces with the same boundary are likely to be homeomorphic and therefore classes of solid trips may be further grouped in 12 classes corresponding to the 12 different type of boundary found. To prove that we should decompose the cube in simplexes and permute them looking for equivalent configurations. This may be done in a further issue of this paper.

Appendix

A.1 Full Set of Solid Strip Configurations

In this appendix the full set of 64 solid strip configurations, obtained by identifying in couples 4 of the 6 faces of a cube, is given. The configurations are grouped in 22 equivalent classes as specified in the right columns of the tables. Note that one class leads to non feasible configurations and therefore the number of possible different solid strips that can be obtained is only 21. However, it is not studied in this paper the possibility to further partition the above classes in classes of homeomorphic spaces.

n	Ident. 1	Ident. 2	Face 1 [1234]	Face 2 [5678]	Class
1	[1562]=[4873]	[1584]=[2673]	1(-C)1(-E)1C1E1	5A5D5(-A)5(-D)5	1
2	[1562]=[4873]	[1584]=[3762]	1(-C)1E1C1E1	5A5(-D)5(-A)5(-D)5	2
3	[1562]=[3784]	[1584]=[2673]	1(-C)1(-E)1(-C)1E1	5A5D5A5(-D)5	2
4	[1562]=[3784]	[1584]=[3762]	1(-D)2F1(-D)2F1	5B6(-E)5B6(-E)5	3
5	[1562]=[3784]	[1584]=[3267]	1(-D)2(-A)1(-D)2(-A)1	2B6C2B6C2	4
6	[1562]=[3784]	[1584]=[6732]	1(-D)2(-C)1(-D)2(-C)1	5B1A5B1A5	4
7	[1562]=[3487]	[1584]=[3762]	1(-B)2D1(-B)2D1	2(-A)6(-C)2(-A)6(-C)2	4
8	[1562]=[8734]	[1584]=[3762]	1(-B)2D1(-B)2D1	5A1(-C)5A1(-C)5	4
9	[1562]=[3487]	[1584]=[3267]	1(-D)2(-A)1A2D1	2B6(-B)2(-C)6C2	5
10	[1562]=[8734]	[1584]=[6732]	1(-D)2D1C2(-C)1	5B1A5(-A)1(-B)5	5
11	[1562]=[7348]	[1584]=[7326]	1(-D)2(-E)3B4F1	3B4F1(-D)2(-E)3	6
12	[1562]=[7348]	[1584]=[2376]	1(-D)1A3B4C1	3B4C1(-D)1A3	7
13	[1562]=[7348]	[1584]=[7623]	1(-D)2(-C)3B3(-A)1	3B3(-A)1(-D)2(-C)3	7
14	[1562]=[4378]	[1584]=[7326]	1(-B)2(-C)3A1D1	3A1D1(-B)2(-C)3	7
15	[1562]=[7843]	[1584]=[7326]	1(-B)2(-C)2(-A)4D1	2(-A)4D1(-B)2(-C)2	7
16	[1562]=[4873]	[1584]=[6237]	1(-C)2D2C1E1	2A1(-E)1(-A)2(-D)2	8
17	[1562]=[8437]	[1584]=[2673]	1(-C)1(-E)3(-A)3E1	3A3D1C1(-D)3	8
18	[1562]=[4873]	[1584]=[7326]	1(-C)2(-D)2C1E1	2A1E1(-A)2(-D)2	9
19	[1562]=[7348]	[1584]=[2673]	1(-C)1(-E)3A3E1	3A3D1(-C)1(-D)3	9
20	[1562]=[8437]	[1584]=[6237]	1(-C)2D1(-A)2E1	2A1(-E)2C1(-D)2	10

Table A1.1 : Solid Strip Configurations with Relevant Class (1-20)

n	Ident. 1	Ident. 2	Face 1 [1234]	Face 2 [5678]	Class
21	[1562]=[8437]	[1584]=[3762]	1(-C)2E1(-A)2E1	2A1(-D)2C1(-D)2	11
22	[1562]=[3784]	[1584]=[6237]	1(-C)2D1(-C)2E1	2A1(-E)2A1(-D)2	11
23	[1562]=[8437]	[1584]=[3267]	1(-D)2(-A)1(-B)2(-C)1	2B1A2D1C2	12
24	[1562]=[8437]	[1584]=[6732]	1(-D)2(-A)1(-B)2(-C)1	2B1A2D1C2	12
25	[1562]=[3487]	[1584]=[6237]	1(-B)2C1A2D1	2B1(-D)2(-A)1(-C)2	12
26	[1562]=[8734]	[1584]=[6237]	1(-B)2C1A2D1	2B1(-D)2(-A)1(-C)2	12
27	[1562]=[8437]	[1584]=[7326]	1(-C)1(-D)3(-A)3E1	3A3E1C1(-D)3	13
28	[1562]=[7348]	[1584]=[6237]	1(-C)2D2A1E1	2A1(-E)1(-C)2(-D)2	13
29	[1562]=[8437]	[1584]=[2376]	1(-D)1A3(-B)3C1	3B3(-A)1D1(-C)3	14
30	[1562]=[8437]	[1584]=[7623]	1(-D)1A3(-B)3C1	3B3(-A)1D1(-C)3	14
31	[1562]=[4378]	[1584]=[6237]	1(-B)2C2(-A)1D1	2B1(-D)1A2(-C)2	14
32	[1562]=[7843]	[1584]=[6237]	1(-B)2C2(-A)1D1	2B1(-D)1A2(-C)2	14
33	[1562]=[3487]	[1584]=[6732]	1(-D)2B1A2(-C)1	2B1A2(-C)1(-D)2	15
34	[1562]=[8734]	[1584]=[3267]	1(-D)2(-A)1C2B1	2B1(-D)2(-A)1C2	15
35	[1562]=[4378]	[1584]=[2376]	1(-D)1A3(-A)1C1	3B1(-D)1C1(-B)3	16
36	[1562]=[4378]	[1584]=[7623]	1(-D)2D1(-A)1B1	1B1(-A)1C2(-C)1	16
37	[1562]=[7843]	[1584]=[2376]	1(-D)1A1(-C)4C1	1B4(-B)1A1(-D)1	16
38	[1562]=[7843]	[1584]=[7623]	1(-D)2B2(-C)2D1	2B2(-A)1A2(-C)2	16
39	[1562]=[4873]	[1584]=[3267]	1(-D)1(-A)1D1C1	1B1(-A)1(-B)1C1	17
40	[1562]=[4873]	[1584]=[6732]	1(-D)1A1D1(-C)1	1B1A1(-B)1(-C)1	17
41	[1562]=[3487]	[1584]=[2673]	1(-B)1(-D)1(-A)1D1	1(-B)1C1(-A)1(-C)1	17
42	[1562]=[8734]	[1584]=[2673]	1(-B)1(-D)1A1D1	1(-B)1C1A1(-C)1	17
43	[1562]=[4873]	[1584]=[2376]	1(-D)1A1D1C1	1B1A1(-B)1C1	18
44	[1562]=[4873]	[1584]=[7623]	1(-D)1(-A)1D1(-C)1	1B1(-A)1(-B)1(-C)1	18
45	[1562]=[4378]	[1584]=[2673]	1(-B)1(-D)1A1D1	1(-B)1C1A1(-C)1	18
46	[1562]=[7843]	[1584]=[2673]	1(-B)1(-D)1(-A)1D1	1(-B)1C1(-A)1(-C)1	18
47	[1562]=[3784]	[1584]=[2376]	1(-D)1A1(-D)1C1	1B1(-C)1B1(-A)1	19
48	[1562]=[3784]	[1584]=[7623]	1(-D)1C1(-D)1A1	1B1(-A)1B1(-C)1	19
49	[1562]=[4378]	[1584]=[3762]	1(-B)1D1(-A)1D1	1A1(-C)1B1(-C)1	19
50	[1562]=[7843]	[1584]=[3762]	1(-B)1D1(-A)1D1	1A1(-C)1B1(-C)1	19
51	[1562]=[3487]	[1584]=[7326]	1(-B)1(-C)1B1D1	1A1D1(-A)1(-C)1	20
52	[1562]=[8734]	[1584]=[7326]	1(-B)1(-C)1B1D1	1A1D1(-A)1(-C)1	20
53	[1562]=[7348]	[1584]=[3267]	1(-D)1(-A)1B1A1	1B1(-C)1(-D)1C1	20
54	[1562]=[7348]	[1584]=[6732]	1(-D)1C1B1(-C)1	1B1A1(-D)1(-A)1	20
55	[1562]=[3487]	[1584]=[2376]	1(-D)1A1A1C1	1B1B1(-C)1D1	21
56	[1562]=[3487]	[1584]=[7623]	1(-D)1(-B)1A1(-D)1	1B1(-A)1(-C)1(-C)1	21
57	[1562]=[8734]	[1584]=[2376]	1(-D)1A1C1C1	1B1D1(-A)1B1	21
58	[1562]=[8734]	[1584]=[7623]	1(-D)1(-D)1C1(-B)1	1B1(-A)1(-A)1(-C)1	21
59	[1562]=[4378]	[1584]=[3267]	1(-D)1(-A)1(-A)1(-B)1	1B1D1C1C1	21
60	[1562]=[4378]	[1584]=[6732]	1(-D)1(-D)1(-A)1(-C)1	1B1A1C1B1	21
61	[1562]=[7843]	[1584]=[3267]	1(-D)1(-A)1(-C)1(-D)1	1B1B1A1C1	21
62	[1562]=[7843]	[1584]=[6732]	1(-D)1(-B)1(-C)1(-C)1	1B1A1A1D1	21
63	[1562]=[3784]	[1584]=[7326]	Not Feasible	Not Feasible	22
64	[1562]=[7348]	[1584]=[3762]	Not Feasible	Not Feasible	22

Table A1.2 : Solid Strip Configurations with Relevant Class (21-64)

A.2 Boundaries of Strip Configurations by Class

This appendix contains the boundaries of the 21 solid strip configurations in a pictorial form.

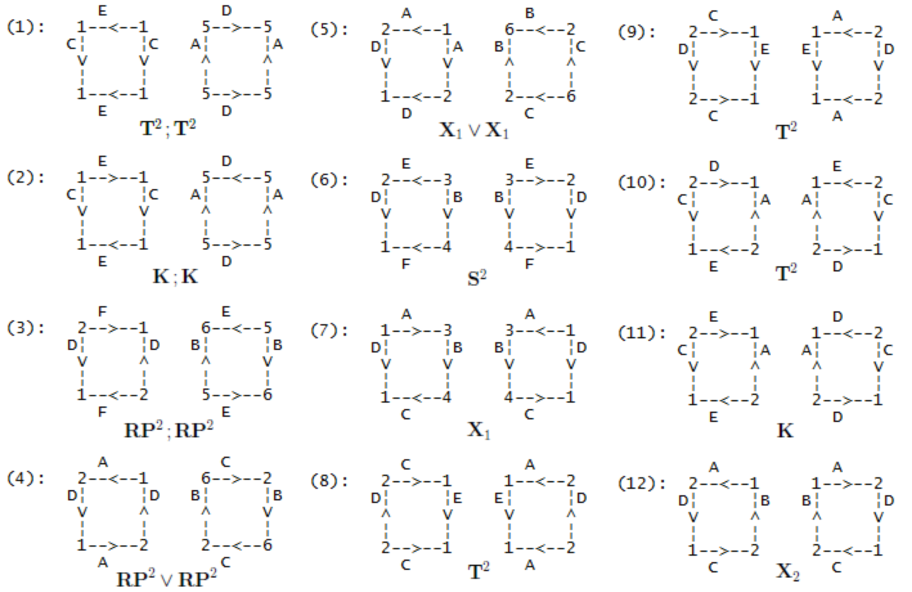

Figure 2: Solid Strip Configuration Boundaries for Classes 1-12

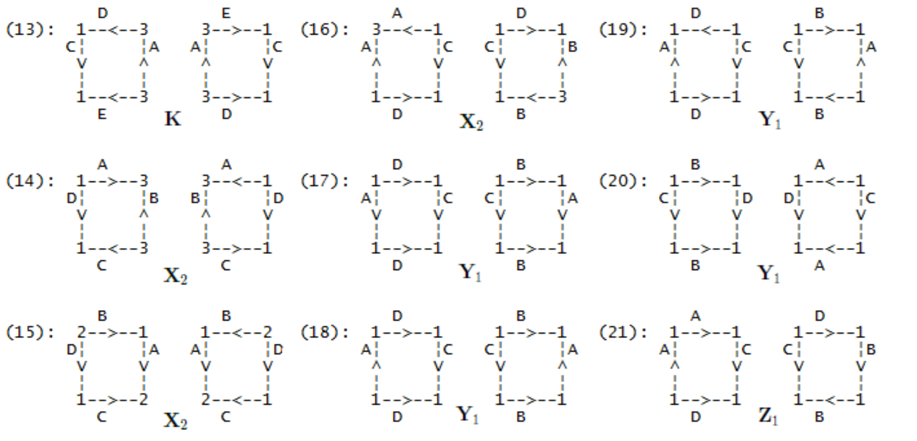

Figure 3: Solid Strip Configuration Boundaries for Classes 13-21