Expansions of maximum and minimum from Generalized Maxwell distribution [*]

[a]Jianwen Huang   [a]Xinling Liu   [b]Jianjun Wang

[a]School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China
[a]School of Artificial Intelligence, Southwest University, Chongqing, 400715, China

Abstract. Generalized Maxwell distribution is an extension of the classic Maxwell distribution. In this paper, we concentrate on the joint distributional asymptotics of normalized maxima and minima. Under optimal normalizing constants, asymptotic expansions of joint distribution and density for normalized partial maxima and minima are established. These expansions are used to educe speeds of convergence of joint distribution and density of normalized maxima and minima tending to its corresponding ultimate limits. Numerical analysis are provided to support our results.

Keywords. Asymptotic expansion; generalized Maxwell distribution; maxima and minima; speed of convergence.

1   Introduction

Let independent and identically distributed (i.i.d.) random variables $\{X_i, \ i \geq 1\}$ with marginal generalized Maxwell distribution F_k (abbreviated as $F_k \sim \mathrm{GMD}(k)$), where $k > 0$ is the shape parameter. The probability density function (pdf) of $\mathrm{GMD}(k)$ is given by [1]

$$f_k(x) = \frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)}x^{2k}\exp\left(-\frac{x^{2k}}{2\sigma^2}\right), \ x \geq 0, \tag{1.1}$$

where σ is a positive parameter and $\Gamma(\cdot)$ denotes the Gamma function. For $k = 1$, $\mathrm{GMD}(k)$ degenerates to the classical Maxwell distribution.

In the last few years, asymptotic behaviors regarding the normalized maxima of $\mathrm{GMD}(k)$ have been discussed in the literature. Huang and Chen [2] established the Mills' type inequality and Mills' type ratio of $\mathrm{GMD}(k)$, and demonstrated that

$$\lim_{n\to\infty}\mathbb{P}(M_n \leq u(x, b_n)) = \Lambda(x) \tag{1.2}$$

with

$$u(x, b_n) = k^{-1}\sigma^2 b_n^{1-2k}x + b_n \tag{1.3}$$

and the normalizing constants b_n is determined by

$$1 - F_k(b_n) = \frac{1}{n}, \tag{1.4}$$

where $M_n = \max\{X_i,\ 1 \le i \le n\}$ represents the partial maximum of $\{X_i,\ i \ge 1\}$ and $\Lambda(x) = \exp(-e^{-x})$ stands for the Gumbel distribution. Additionally, Huang and Wang [11] gained the asymptotic expansions for distribution as follows:

$$b_n^{2k} \left[b_n^{2k}(\mathbb{P}(M_n \le u(x, b_n)) - \Lambda(x)) - l_k(x)\Lambda(x) \right] \to \left[w_k(x) + \frac{1}{2} l_k^2(x) \right] \Lambda(x) \qquad (1.5)$$

as $n \to \infty$, where $l_k(x)$ and $w_k(x)$ are respectively given by

$$l_k(x) = 2^{-1}k^{-1}\sigma^2[(2k-1)x^2 - 2x]e^{-x} \qquad (1.6)$$

and

$$w_k(x) = -24^{-1}k^{-2}\sigma^4[3(2k-1)^2x^4 - 4(2k+1)(2k-1)x^3 + 24x^2 - 48kx]e^{-x}. \qquad (1.7)$$

Related work concerning properties of GMD(k) can be found in [3] [4] [5] and [6]. GMD(k) has received applications in a variety of areas, see, e.g., establishing the discretized equilibrium distributions of the lattice Boltzmann method [7], describing the stress relaxation behavior of food matrices [8], being used to derive a different class of multiple relaxation-time lattice Boltzmann models [9], and modeling the linear viscoelastic fluids [10].

For the basic properties, which determine the max-domain of attractions the distribution attributes, see [12] [13] [14] and [15]. In extreme value theory, asymptotics for the joint distribution of maxima and minima is an interesting topic in recent literature. For sequence of univariate random variables under conventional assumptions, see [16], [17] and [18]. For bivariate Gaussian arrays under relaxation assumptions, see [19]. For dissimilar improved assumptions, see [20] and [21]. Let $m_n = \min\{X_i,\ 1 \le i \le n\}$ denote the partial minimum of $\{X_i,\ i \ge 1\}$. The target of this paper is to establish asymptotic expansions for distribution and density of $(M_n,\ m_n)$ for GMD(k), from which we can respectively educe the speeds of convergence of the distribution and density from normalized $(M_n,\ m_n)$ to the distribution and density of the associating extreme value distribution.

The contents of this paper are constructed as follows. Section 2 provides the main results, and numerical analysis is given in Section 3. A few necessary auxiliary lemmas and the proofs of the main results are presented in Section 4.

2 Main results

In this section, we present the main results. Proposition 2.1 provides the asymptotic expansions of a distribution for the minimum of the GMD(k). Theorem 2.1 displays the asymptotic distribution of the joint distribution of normalized $(M_n,\ m_n)$. The asymptotic expansion for its distribution is showed in Theorem 2.2, and the associating asymptotic density expansion is revealed in Theorem 2.3.

Proposition 2.1. *Let $\{X_i,\ i \ge 1\}$ be an i.i.d. random sequence with common distribution function F_k obeying the GMD(k). Set $m_n = \min\{X_i,\ 1 \le i \le n\}$. Then,*

$$b_n^{2k} \left\{ b_n^{2k} \left[\mathbb{P}(m_n \le v(y, b_n)) - (1 - \Lambda(-y)) \right] + l_k(-y)\Lambda(-y) \right\}$$

$$\to - \left[w_k(-y) + \frac{1}{2} l_k^2(-y) \right] \Lambda(-y), \qquad (2.1)$$

as $n \to \infty$, where

$$v(y, b_n) = k^{-1}\sigma^2 b_n^{1-2k} y - b_n, \tag{2.2}$$

$l_k(\cdot)$, $w_k(\cdot)$ *and* b_n *are respectively given by (1.6), (1.7) and (1.4).*

Remark 2.1. *Noting that $b_n^{-2k} = O((\log n)^{-1})$ in view of (1.4), it follows from Proposition 2.1 that the convergence rate of the distribution of normalized minima under the normalized constant b_n converging to its extreme value distribution is the same order as $O((\log n)^{-1})$.*

In the following result, we establish the limiting distribution of the joint distribution of normalized $(M_n, \, m_n)$ from i.i.d. random variables.

Theorem 2.1. *Let $\{X_i, \, i \geq 1\}$ denote i.i.d. random variables with common distribution function $GMD(k)$ F_k. Set $M_n = \bigvee_{i=1}^n X_i$ and $m_n = \bigwedge_{i=1}^n X_i$ respectively indicate the maxima and the minima. We have*

$$\mathbb{P}(M_n \leq u(x, b_n), \, m_n \leq v(y, b_n)) \to \Lambda(x)(1 - \Lambda(-y)), \tag{2.3}$$

as $n \to \infty$, where $u(x, b_n)$ and $v(y, b_n)$ are respectively determined by (1.3) and (2.2) with the normalizing constant b_n defined by (1.4).

Remark 2.2. *A combination of (1.2) and (2.1), Theorem 2.1 demonstrates that M_n and m_n are asymptotically independent.*

In the following result, we present asymptotic expansion of a distribution for $(M_n, \, m_n)$, by which one derives the corresponding speed of convergence of the joint distribution to its limit. Denote $l_k(x, y) = l_k(x) + l_k(-y)$ and $w_k(x, y) = w_k(x) + w_k(-y)$.

Theorem 2.2. *With $u(x, b_n)$ and $v(y, b_n)$ defined by (1.3) and (2.2), we get*

$$b_n^{2k} \Bigg\{ b_n^{2k} \left[\mathbb{P}(M_n \leq u(x, b_n), \, m_n \leq v(y, b_n)) - \Lambda(x)(1 - \Lambda(-y)) \right]$$

$$- l_k(x)\Lambda(x) + l_k(x, y)\Lambda(x)\Lambda(-y) \Bigg\}$$

$$\to \left[w_k(x) + \frac{1}{2}l_k^2(x) \right] \Lambda(x) - \left[w_k(x, y) + \frac{1}{2}l_k^2(x, y) \right] \Lambda(x)\Lambda(-y) \tag{2.4}$$

as $n \to \infty$, where $l_k(\cdot)$ and $w_k(\cdot)$ are respectively provided by (1.6) and (1.7).

Remark 2.3. *By (1.4), one can easily check that $b_n^{2k} = O(\log n)$. Accordingly, Theorem 2.2 clears that the speed of convergence of $\mathbb{P}(M_n \leq u(x, b_n), \, m_n \leq v(y, b_n))$ tending to its ultimate limit is proportional to $(\log n)^{-1}$.*

Theorem 2.2 could be used to deduce asymptotic expansions for the density function of $(M_n, \, m_n)$. Let

$$g_n(x, y) = k^{-2}\sigma^4 b_n^{2-4k} n(n-1) \left[F_k(u(x, b_n)) - F_k(v(y, b_n)) \right]^{n-2} f_k(u(x, b_n)) f_k(v(y, b_n)) \tag{2.5}$$

stand for the pdf of the normalized $(M_n, \, m_n)$, and let

$$\Delta_n(g_n, g; x, y) = g_n(x, y) - g(x, y),$$

3

where $g(x,y) = \Lambda(x)\Lambda(-y)e^{-x}e^{y}$ is the joint density function of $\Lambda(x)\Lambda(-y)$. By means of Proposition 2.5 in [22], it results in $\lim_{n\to\infty} \Delta_n(g_n, g; x, y) = 0$.

In the following result, we establish the higher order expansion of the probability density for normalized $(M_n,\ m_n)$, from which we obtain the relevant convergence speed of the joint density function to its limiting density.

Theorem 2.3. *For $u(x, b_n)$ and $v(y, b_n)$ determined by (1.3) and (2.2), we get*

$$b_n^{2k}\left\{b_n^{2k}\Delta_n(g_n, g; x, y) - C_1(x,y)\Lambda(x)\Lambda(-y)e^{-x}e^{y}\right\} \to C_2(x,y)\Lambda(x)\Lambda(-y)e^{-x}e^{y},$$

as $n \to \infty$, where

$$C_1(x,y) = l_k(x,y) - \sigma^2\left[\frac{2k-1}{2k}(x^2+y^2) - 2(x-y) + \frac{2}{k}\right]$$

and

$$
\begin{aligned}
C_2(x,y) =& \sigma^4\left\{\frac{1}{k}\left[\frac{(2k-1)^2}{8k}(x^4+y^4) - \frac{(2k+1)(4k-1)}{3k}(x^3-y^3) + \frac{(2k+1)(2k-1)}{2k}(x^2+y^2)\right.\right. \\
& \left.\left. - 2(x-y) + 4\right] + \left(\frac{2k-1}{2k}x^2 - 2x + \frac{1}{k}\right)\left(\frac{2k-1}{2k}y^2 + 2y + \frac{1}{k}\right)\right\} \\
& - \sigma^2\left[\frac{2k-1}{2k}(x^2+y^2) - 2(x-y) + \frac{2}{k}\right]l_k(x,y) + w_k(x,y) + \frac{1}{2}l_k^2(x,y).
\end{aligned}
$$

Remark 2.4. *Observing that $b_n^{2k} = O(\log n)$, Theorem 2.3 evidences that the speed of convergence of the joint pdf of the normalized $(M_n,\ m_n)$ tending to its ultimate limit is the same order as $O((\log n)^{-1})$.*

3   Numerical analysis

In this section, numerical research is presented to demonstrate the precision of higher order expansions for the joint distribution and the joint density of normalized $(M_n,\ m_n)$. It follows from (4.10) and (2.5) that the real values of the joint distribution of $(M_n,\ m_n)$ are equal to $\mathbb{P}(M_n \le u(x, b_n),\ m_n \le v(y, b_n)) = F_k^n(u(x, b_n)) - [F_k(u(x, b_n)) - F_k(v(y, b_n))]^n$ and the real values of the joint density of $(M_n,\ m_n)$ are provided by (2.5). Let $S_i(x, y)$ and $T_i(x, y)$, $i = 1, 2, 3$, respectively represent the first-order, the second-order and the third-order asymptotics of the joint distribution and the joint density of normalized $(M_n,\ m_n)$. Due to Theorems 2.2 and 2.3, we can obtain

$$
\begin{aligned}
S_1(x,y) &= \Lambda(x) - \Lambda(x)\Lambda(-y), \\
S_2(x,y) &= S_1(x,y) + b_n^{-2k}[l_k(x)\Lambda(x) - l_k(x,y)\Lambda(x)\Lambda(-y)], \\
S_3(x,y) &= S_2(x,y) + b_n^{-4k}\left\{\left[w_k(x) + \frac{1}{2}l_k^2(x)\right]\Lambda(x) - \left[w_k(x,y) + \frac{1}{2}l_k^2(x,y)\right]\Lambda(x)\Lambda(-y)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
T_1(x,y) &= \Lambda(x)\Lambda(-y)e^{-x}e^{y}, \\
T_2(x,y) &= T_1(x,y) + C_1(x,y)\Lambda(x)\Lambda(-y)e^{-x}e^{y}b_n^{-2k},
\end{aligned}
$$

4

$$T_3(x,y) = T_2(x,y) + C_2(x,y)\Lambda(x)\Lambda(-y)e^{-x}e^{y}b_n^{-4k},$$

where b_n is defined in (1.4).

In order to compare the precision of real values with that of its corresponding asymptotics, let $\Delta_i = |\mathbb{P}(M_n \leq u(x,b_n), \ m_n \leq v(y,b_n)) - S_i(x,y)|$ and $\Theta_i = |g_n(x,y) - T_i(x,y)|, \ i = 1,2,3$, respectively stand for the absolute errors. We use Matlable to compute the real values and the asmyptotics of the joint distribution and the joint density of normalized $(M_n, \ m_n)$. To demonstrate the precision of all asymptotics with varying n, we then plot the real values and its asymptotics of the joint cdf of the joint pdf with fixed (x,y). Figures 1 and 2 indicate that the larger n the better of all asymptotics.
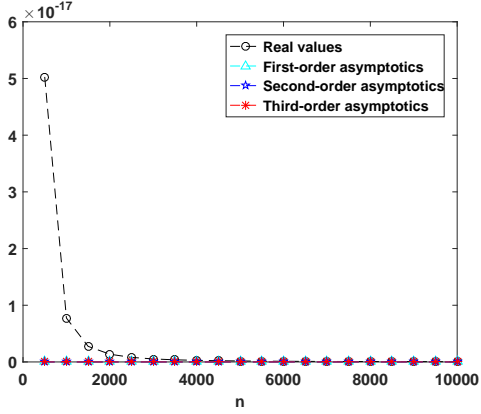
(a) $(x,y) = (5,5), \ \sigma = 1.$
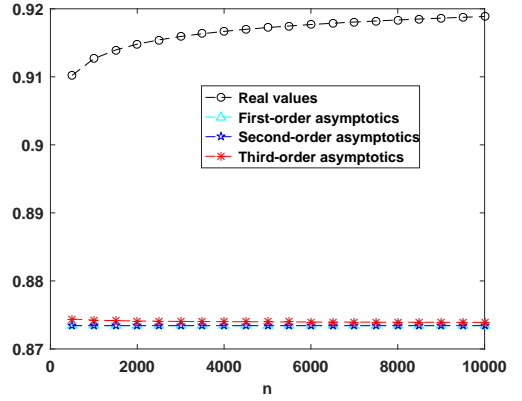
(b) $(x,y) = (2,6), \ \sigma = 1.$
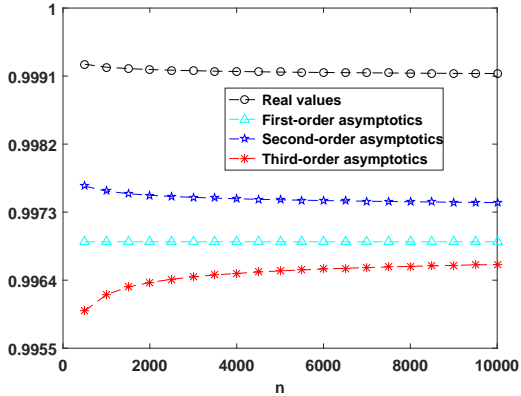
(c) $(x,y) = (6,2), \ \sigma = 1.$
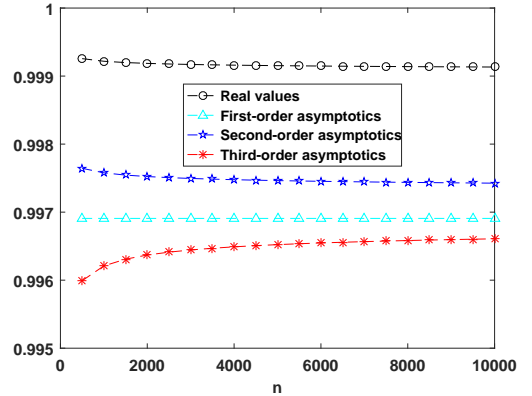
(d) $(x,y) = (6,2), \ \sigma = 15.$

Figure 1: Real values and its approximations of the joint pdf of $(M_n, \ m_n)$ with fixed (x,y) and $k = 1$.

To evidence the precision of all asymptotics with varying x and y, we respectively plot the absolute errors of the joint cdf and the joint pdf with fixed k, σ and $n = 500$ in Figure 3 and Figure 4.
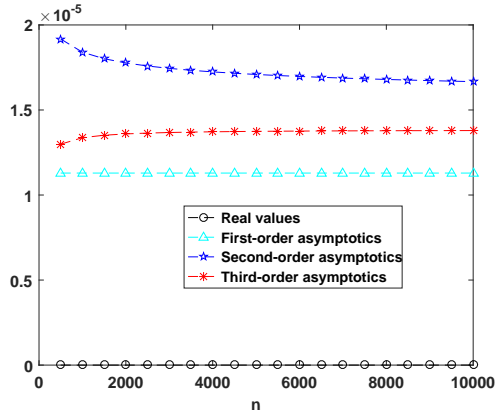
According to Figures 3 and 4, we have the following findings: (i) When x increasingly enlarges, all asymptotics of the joint cdf are always closer to its real values except some special cases. (ii) All asymptotics of the joint pdf are more closer to its real values as x and y becomes large. (iii) The first-order asymptotics may be better except some special cases.
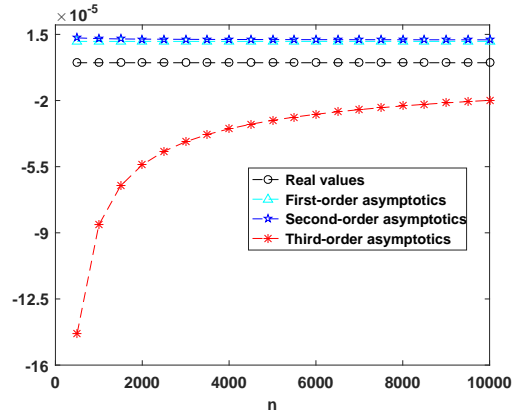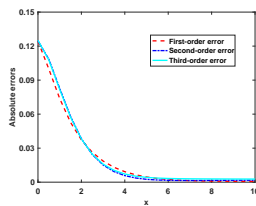
5
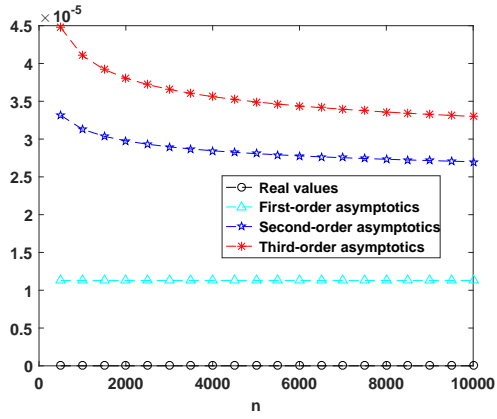
(a) $k = 0.5$.
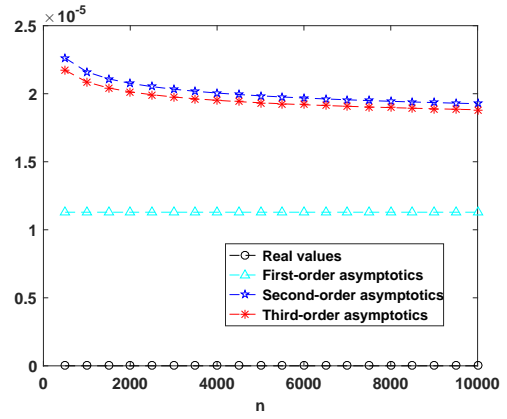
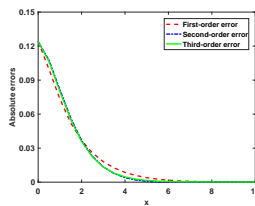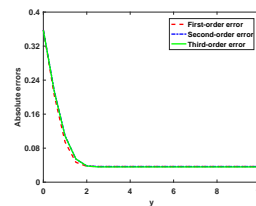(b) $k = 1.0$.

(c) $k = 1.5$.

(d) $k = 6.0$.

Figure 2: Real values and its approximations of the joint pdf of $(M_n, \ m_n)$ with fixed $(x, y) = (2, 6)$ and $\sigma = 1$.
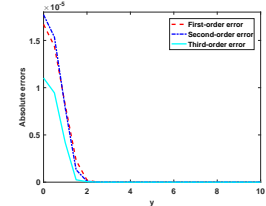

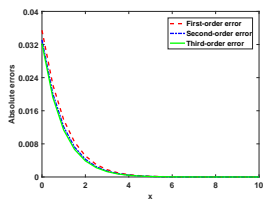(a) $k = 1, \ y = 2$.
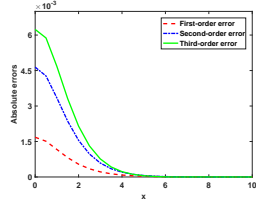
(b) $k = 1, \ y = 10$.
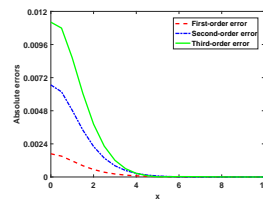
(c) $k = 1, \ x = 2$.

(d) $k = 1, \ x = 10$.

Figure 3: The absolute errors between real values and its approximations of the joint cdf with $n = 500$.
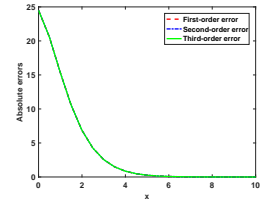
6

(a) $k = 1$, $y = 2$. (b) $k = 2$, $y = 2$. (c) $k = 4$, $y = 2$. (d) $k = 1$, $y = 10$.

(e) $k = 2$, $y = 10$. (f) $k = 4$, $y = 10$. (g) $k = 1$, $x = 2$. (h) $k = 2$, $x = 2$.

(i) $k = 4$, $x = 2$. (j) $k = 1$, $x = 10$. (k) $k = 2$, $x = 10$. (l) $k = 4$, $x = 10$.

Figure 4: The absolute errors between real values and its approximations of the joint pdf with $n = 500$.

4 The proofs

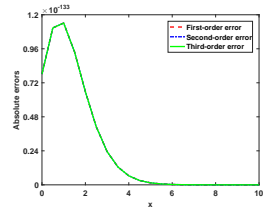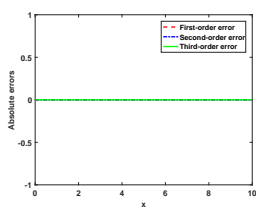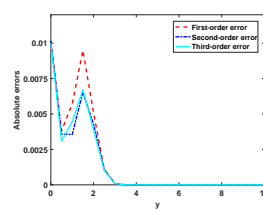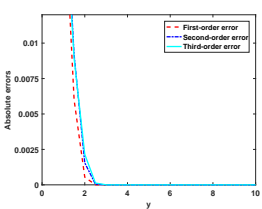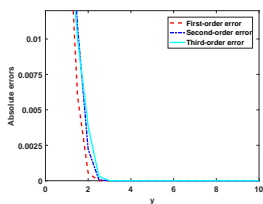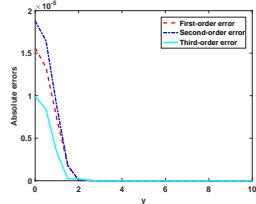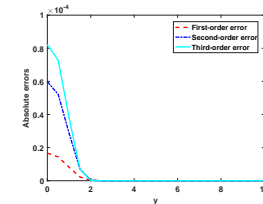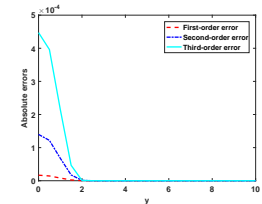In this section, we present the proofs of main results. In order to prove the main results, we need some necessary lemmas. Lemmas 1 and 2 are given by [11]. The results are expressed as follows.

Lemma 4.1. *Let $F_k(x)$ and $f_k(x)$ respectively represent the cdf and the pdf of GMD(k), for large x, then we get*

$$1 - F_k(x) = f_k(x)\frac{\sigma^2}{k}x^{1-2k}\left[1 + \frac{\sigma^2}{k}x^{-2k} + \frac{1-2k}{k^2}\sigma^4 x^{-4k} + O(x^{-6k})\right]$$

$$= \frac{\exp(-\frac{1}{2\sigma^2})}{2^{\frac{k}{2}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})}\left[1 + \frac{\sigma^2}{k}x^{-2k} + \frac{1-2k}{k^2}\sigma^4 x^{-4k} + O(x^{-6k})\right]\exp\left[-\int_1^x \frac{\tilde{g}(t)}{f(t)}dt\right] \quad (4.1)$$

where $f(t) = k^{-1}\sigma^2 x^{1-2k}$ and $\tilde{g}(t) = 1 - k^{-1}\sigma^2 x^{-2k}$.

Lemma 4.2. *For $u(x, b_n)$ determined by (1.3) with normalizing constant b_n given by (1.4), we have*

$$b_n^{2k}\left\{b_n^{2k}\left[n\log F_k(u(x, b_n)) + e^{-x}\right] - l_k(x)\right\} \to w_k(x), \quad (4.2)$$

where $l_k(x)$ and $w_k(x)$ are respectively provided by (1.6) and (1.7).

Lemma 4.3. *Let $h_k(x, y) = n\log[F_k(u(x, b_n)) - F_k(v(y, b_n))] + e^{-x} + e^y$ with $u(x, b_n)$ and $v(y, b_n)$ determined by (1.3) and (2.2). Then,*

$$b_n^{2k}\left[b_n^{2k}h_k(x, y) - l_k(x, y)\right] \to w_k(x, y), \quad (4.3)$$

as $n \to \infty$, where $l_k(x, y)$ and $w_k(x, y)$ are provided by Theorem 2.2.

Proof. It is easy to verify that

$$F_k(v(y, b_n)) = 1 - F_k(u(-y, b_n)). \quad (4.4)$$

Then,

$$\lim_{n\to\infty} b_n^{2k}h_k(x, y)$$
$$= \lim_{n\to\infty} nb_n^{2k}\left\{\log\left[1 - (1 - F_k(u(x, b_n)) + F_k(v(y, b_n)))\right] + n^{-1}(e^{-x} + e^y)\right\}$$
$$\overset{(a)}{=} \lim_{n\to\infty} nb_n^{2k}\left\{-\left[1 - (1 - F_k(u(x, b_n)) + F_k(v(y, b_n)))\right]\right.$$
$$\left. - \frac{1}{2}\left[1 - (1 - F_k(u(x, b_n)) + F_k(v(y, b_n)))\right]^2(1 + o(1)) + n^{-1}(e^{-x} + e^y)\right\}$$
$$= \lim_{n\to\infty} nb_n^{2k}\left\{-\left[1 - F_k(u(x, b_n))\right] - \frac{1}{2}\left[1 - F_k(u(x, b_n))\right]^2(1 + o(1)) + n^{-1}e^{-x}\right.$$
$$- F_k(v(y, b_n)) - \frac{1}{2}F_k(v(y, b_n))^2(1 + o(1)) + n^{-1}e^y$$
$$\left. - F_k(v(y, b_n))\left[1 - F_k(u(x, b_n))\right](1 + o(1))\right\}$$

$$\overset{(b)}{=} \lim_{n\to\infty} \left\{ b_n^{2k}[n\log F_k(u(x,b_n)) + e^{-x}] + b_n^{2k}[n\log F_k(u(-y,b_n)) + e^y] + O(n^{-1}b_n^{2k}) \right\}$$

$$\overset{(c)}{=} l_k(x) + l_k(-y)$$

$$:= l_k(x,y), \tag{4.5}$$

where (a) and (b) follow from $\log(1-x) = -x - x^2/2 - x^3/3 + o(x^3)$, $x \to 0$, and (c) is due to Lemma 4.2 and (4.4).

Likewise, by Lemma 4.2, we can obtain

$$\lim_{n\to\infty} b_n^{2k}\left[b_n^{2k}h_k(x,y) - l_k(x,y) \right]$$

$$= \lim_{n\to\infty} \left\{ b_n^{2k}\left\{ b_n^{2k}\left[n\log F_k(u(x,b_n)) + e^{-x}\right] - l_k(x) \right\} \right.$$

$$\left. + b_n^{2k}\left\{ b_n^{2k}\left[n\log F_k(u(-y,b_n)) + e^y\right] - l_k(-y) \right\} + O(n^{-1}b_n^{-4k}) \right\}$$

$$= w_k(x) + w_k(-y)$$

$$:= w_k(x,y). \tag{4.6}$$

The proof is complete.

$\square$

Proof of Proposition 2.1. It is not hard to check that $m_n = -\max_{1\le i\le n}\{-X_i\}$ with $-X_i$ following GMD(k). Combining with (1.2) and (1.3), it implies that

$$\lim_{n\to\infty} \mathbb{P}(m_n \le v(y,b_n)) = 1 - \Lambda(-y), \tag{4.7}$$

where

$$v(y,b_n) = k^{-1}\sigma^2 b_n^{1-2k}y - b_n.$$

By (4.4), Lemmas 4.1 and 4.2 and analogous discussion for the proof of Theorem 2.1 [11], the desired result can be derived.

$\square$

Proof of Theorem 2.1. By (1.2), (4.7) together with Theorem 1.8.2 [23], the claim is deduced.

$\square$

Proof of Theorem 2.2. By (4.5), it follows that

$$h_k(x,y) \to 0 \quad \text{and} \quad \left| \sum_{j=3}^{\infty} \frac{h_k^{j-3}(x,y)}{j!} \right| \le \exp(|h_k(x,y)|) \to 1, \tag{4.8}$$

as $n \to \infty$. By (4.5), (4.6) and (4.8), we get

$$\lim_{n\to\infty} b_n^{2k}\left\{ b_n^{2k}\left[\mathbb{P}(M_n \le u(x,b_n), \ m_n > v(y,b_n)) - \Lambda(x)\Lambda(-y)\right] - l_k(x,y)\Lambda(x)\Lambda(-y) \right\}$$

$$= \lim_{n\to\infty} b_n^{2k}\left\{ b_n^{2k}\left[\exp(h_k(x,y)) - 1\right] - l_k(x,y) \right\}\Lambda(x)\Lambda(-y)$$

$$\overset{(a)}{=} \lim_{n\to\infty} \left\{ b_n^{2k}[b_n^{2k}h_k(x,y) - l_k(x,y)] + b_n^{4k}h_k^2(x,y)\left[\frac{1}{2} + h_k(x,y)\sum_{j=3}^{\infty} \frac{h_k^{j-3}(x,y)}{j!} \right] \right\}\Lambda(x)\Lambda(-y)$$

9

$$= \left[w_k(x,y) + \frac{1}{2} l_k^2(x,y) \right] \Lambda(x)\Lambda(-y), \tag{4.9}$$

where (a) is because of $e^x = 1 + x + x^2(1 + O(x))$, $x \to 0$. By (1.5) and (4.9), we get

$$\begin{aligned}
&\mathbb{P}(M_n \le u(x,b_n),\ m_n \le v(y,b_n)) \\
&= \mathbb{P}(M_n \le u(x,b_n)) - \mathbb{P}(M_n \le u(x,b_n),\ m_n > v(y,b_n)) \\
&= \Lambda(x)(1 - \Lambda(-y)) + b_n^{-2k}[l_k(x)\Lambda(x) - l_k(x,y)\Lambda(x)\Lambda(-y)] \\
&\quad + b_n^{-4k}\left\{ \left[w_k(x) + \frac{1}{2} l_k^2(x) \right] \Lambda(x) - \left[w_k(x,y) + \frac{1}{2} l_k^2(x,y) \right] \Lambda(x)\Lambda(-y) \right\} + O(b_n^{-6k}), \tag{4.10}
\end{aligned}$$

as n is large. We complete the proof.

$\square$

Proof of Theorem 2.3. It is known that $n(1 - F_k(u(x,b_n))) \to e^{-x}$ for large n implies $1 - F_k(u(x,b_n)) = O(n^{-1})$, and similarly, $F_k(v(y,b_n)) = O(n^{-1})$. Therefore,

$$\begin{aligned}
&[F_k(u(x,b_n)) - F_k(v(y,b_n))]^{n-2} \\
&= \frac{[F_k(u(x,b_n)) - F_k(v(y,b_n))]^n}{\{1 - \{1 - [F_k(u(x,b_n)) - F_k(v(y,b_n))]\}\}^2} \\
&\overset{(a)}{=} (1 + O(n^{-1}))[F_k(u(x,b_n)) - F_k(v(y,b_n))]^n \\
&\overset{(b)}{=} \Lambda(x)\Lambda(-y) + b_n^{-2k}l_k(x,y)\Lambda(x)\Lambda(-y) + b_n^{-4k}\left[w_k(x,y) + \frac{1}{2} l_k^2(x,y) \right] \Lambda(x)\Lambda(-y) + O(b_n^{-6k})
\end{aligned}$$
$$\tag{4.11}$$

for large n, where (a) is from $1/(1-x) = 1 + x + x^2 + x^3(1 + o(1))$, $x \to 0$ and (b) is thanks to (4.9).

Employing the following Taylor's expansion

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + O(x^4),\ x \to 0,\ \alpha \in \mathbb{R},$$

and the fact in view of (1.4) that $b_n^{2k} \sim \log n$ for large n, we get

$$\begin{aligned}
&\left(\frac{\sigma^2}{k} b_n^{1-2k} x + b_n \right)^{2k} \\
&= b_n^{2k}\left(1 + \frac{\sigma^2}{k} b_n^{-2k} x \right)^{2k} \\
&= b_n^{2k}\left[1 + 2\sigma^2 b_n^{-2k} x + \frac{2k-1}{k}\sigma^4 b_n^{-4k} x^2 + \frac{(2k-1)(2k-2)}{3k^2}\sigma^6 b_n^{-6k} x^3 + O(b_n^{-8k}) \right]. \tag{4.12}
\end{aligned}$$

Applying (4.12), we get

$$\begin{aligned}
&\exp\left(-\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{k} b_n^{1-2k} x + b_n \right)^{2k} + \frac{1}{2\sigma^2} b_n^{2k} + x \right) \\
&= \exp\left(-\frac{2k-1}{2k}\sigma^2 b_n^{-2k} x^2 - \frac{(2k-1)(k-1)}{3k^2}\sigma^4 b_n^{-4k} x^3 + O(b_n^{-6k}) \right)
\end{aligned}$$

10

$$\stackrel{(a)}{=} 1 - \frac{2k-1}{2k}\sigma^2 b_n^{-2k}x^2 + \frac{(2k-1)\sigma^4}{k^2}\left(\frac{2k-1}{8}x^4 - \frac{k-1}{3}x^3\right)b_n^{-4k} + O(b_n^{-6k}), \tag{4.13}$$

where (a) follows from the fact that $e^x = 1 + x + x^2/2 + x^3/6 + O(x^4), \ x \to 0$.

Utilizing the following Taylor's expansion

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + O(x^4), \ x \to 0,$$

we get

$$\left[1 + \frac{\sigma^2}{k}b_n^{-2k} + \frac{1-2k}{k^2}\sigma^4 b_n^{-4k} + O(b_n^{-6k})\right]^{-1}$$
$$= 1 - \frac{\sigma^2}{k}b_n^{-2k} + \frac{2\sigma^4}{k}b_n^{-4k} + O(b_n^{-6k}). \tag{4.14}$$

Hence, by (1.4), (4.12), (4.13), (4.14) and Lemma 4.1, we get

$$\frac{\sigma^2}{k}\frac{nf_k(u(x,b_n))}{b_n^{2k-1}}e^x$$

$$= \frac{\sigma^2}{k}\frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)}\frac{(\frac{\sigma^2}{k}b_n^{1-2k}x + b_n)^{2k}\exp(-\frac{1}{2\sigma^2}(\frac{\sigma^2}{k}b_n^{1-2k}x + b_n)^{2k})}{b_n^{2k-1}(1-F_k(b_n))}e^x$$

$$= \frac{(\frac{\sigma^2}{k}b_n^{1-2k}x + b_n)^{2k}\exp(-\frac{1}{2\sigma^2}(\frac{\sigma^2}{k}b_n^{1-2k}x + b_n)^{2k} + x))}{b_n^{2k}\exp(-\frac{1}{2\sigma^2}b_n^{2k})}[1 + \frac{\sigma^2}{k}b_n^{-2k} + \frac{1-2k}{k^2}\sigma^4 b_n^{-4k} + O(b_n^{-6k})]^{-1}$$

$$= 1 - (\frac{2k-1}{2k}x^2 - 2x + \frac{1}{k})\sigma^2 b_n^{-2k} + \frac{\sigma^4}{k}[\frac{(2k-1)^2}{8k}x^4 - \frac{(2k+1)(4k-1)}{3k}x^3$$

$$+ \frac{(2k+1)(2k-1)}{2k}x^2 - 2x + 2]b_n^{-4k} + O(b_n^{-6k}) \tag{4.15}$$

and

$$\frac{\sigma^2}{k}\frac{nf_k(v(y,b_n))}{b_n^{2k-1}}e^{-y}$$

$$= \frac{\sigma^2}{k}\frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)}\frac{(\frac{\sigma^2}{k}b_n^{1-2k}(-y) + b_n)^{2k}\exp(-\frac{1}{2\sigma^2}(\frac{\sigma^2}{k}b_n^{1-2k}(-y) + b_n)^{2k})}{b_n^{2k-1}(1-F_k(b_n))}e^{-y}$$

$$= 1 - (\frac{2k-1}{2k}y^2 + 2y + \frac{1}{k})\sigma^2 b_n^{-2k} + \frac{\sigma^4}{k}[\frac{(2k-1)^2}{8k}y^4 + \frac{(2k+1)(4k-1)}{3k}y^3$$

$$+ \frac{(2k+1)(2k-1)}{2k}y^2 + 2y + 2]b_n^{-4k} + O(b_n^{-6k}) \tag{4.16}$$

for large n. Combining with (4.11), (4.15) and (4.16), we have

$$\Delta_n(g_n, g; x, y)$$

$$= \Lambda(x)\Lambda(-y)e^{-x}e^y\left\{(1-n^{-1})\frac{\frac{\sigma^2}{k}nf_k(u(x,b_n))e^x}{b_n^{2k-1}}\frac{\frac{\sigma^2}{k}nf_k(v(y,b_n))e^{-y}}{b_n^{2k-1}}\right.$$

$$\left.\times \frac{[F_k(u(x,b_n)) - F_k(v(y,b_n))]^{n-2}}{\Lambda(x)\Lambda(-y)} - 1\right\}$$

$$= \Lambda(x)\Lambda(-y)e^{-x}e^y\left\{\left[l_k(x,y) - \sigma^2\left[\frac{2k-1}{2k}(x^2+y^2) - 2(x-y) + \frac{2}{k}\right]\right]b_n^{-2k}\right.$$

11

$$+ \left\{ \sigma^4 \left\{ \frac{1}{k} \left[\frac{(2k-1)^2}{8k}(x^4+y^4) - \frac{(2k+1)(4k-1)}{3k}(x^3-y^3) + \frac{(2k+1)(2k-1)}{2k}(x^2+y^2) \right. \right. \right.$$

$$\left. \left. - 2(x-y) + 4 \right] + \left(\frac{2k-1}{2k}x^2 - 2x + \frac{1}{k} \right) \left(\frac{2k-1}{2k}x^2 - 2x + \frac{1}{k} \right) \right\}$$

$$\left. - \sigma^2 \left[\frac{2k-1}{2k}(x^2+y^2) - 2(x-y) + \frac{2}{k} \right] l_k(x,y) + w_k(x,y) + \frac{1}{2}l_k^2(x,y) \right\} b_n^{-4k} + O(b_n^{-6k}) \right\}$$

$$:= \Lambda(x)\Lambda(-y)e^{-x}e^y \left[C_1(x,y)b_n^{-2k} + C_2(x,y)b_n^{-4k} + O(b_n^{-6k}) \right]$$

for large n. We complete the proof.

$\square$

Acknowledgements

References

[1] Vodǎ V G. A modified Weibull hazard rate as generator of a generalized Maxwell distribution[J]. Math Rep, 2009, 11: 171-179.

[2] Huang J, Chen S. Tail behavior of the generalized Maxwell distribution[J]. Communications in Statistics-Theory and Methods, 2016, 45(14): 4230-4236.

[3] Nadarajah S, Cordeiro G M, Ortega E M M. The exponentiated Weibull distribution: a survey[J]. Statistical Papers, 2013, 54(3): 839-877.

[4] Liu C, Liu B. Convergence rate of extremes from Maxwell sample[J]. Journal of Inequalities and Applications, 2013, 2013(1): 477.

[5] Dar A A, Ahmed A, Reshi J A. Bayesian analisis of Maxwell-Boltzmann distribution under different loss functions and prior distributions[J]. Pakistan Journal of Statistics, 2017, 33(6).

[6] Huang J, Wang J, Luo G. On the rate of convergence of maxima for the generalized Maxwell distribution[J]. Statistics, 2017, 51(5): 1105-1117.

[7] Shim J W. Parametric lattice boltzmann method[J]. Journal of Computational Physics, 2017, 338: 240-251.

[8] Del Nobile M A, Chillo S, Mentana A, et al. Use of the generalized Maxwell model for describing the stress relaxation behavior of solid-like foods[J]. Journal of Food Engineering, 2007, 78(3): 978-983.

[9] Asinari P, Karlin I V. Generalized Maxwell state and H theorem for computing fluid flows using the lattice Boltzmann method[J]. Physical Review E, 2009, 79(3): 036703.

[10] Liu Q, Jian Y, Yang L. Time periodic electroosmotic flow of the generalized Maxwell fluids between two micro-parallel plates[J]. Journal of Non-Newtonian Fluid Mechanics, 2011, 166(9-10): 478-486.

[11] Huang J, Wang J. On asymptotic of extremes from generalized Maxwell distribution[J]. Bull. Korean Math. Soc., 2018, 55(3): 679-698.

[12] Liao X, Peng Z, Nadarajah S, et al. Rates of convergence of extremes from skew-normal samples[J]. Statistics & Probability Letters, 2014, 84: 40-47.

[13] Lin F, Peng Z, Yu K. Convergence rate of extremes for the generalized short-tailed symmetric distribution[J]. Bull. Korean Math. Soc, 2016, 53(5): 1549-1566.

[14] Du L, Chen S. Asymptotic properties for distributions and densities of extremes from generalized gamma distribution[J]. Journal of the Korean Statistical Society, 2016, 45(2): 188-198.

[15] Liao X, Peng Z, Nadarajah S. Tail behavior and limit distribution of maximum of logarithmic general error distribution[J]. Communications in Statistics-Theory and Methods, 2014, 43(24): 5276-5289.

[16] Leadbetter M R, Lindgren G, Rootzén H. Extremes and related properties of random sequences and processes[M]. Springer Science & Business Media, 2012.

[17] Hashorva E, Weng Z. Maxima and minima of complete and incomplete stationary sequences[J]. Stochastics An International Journal of Probability and Stochastic Processes, 2014, 86(5): 707-720.

[18] Liao X, Xiong Q, Weng Z. Joint distributional expansions of maxima and minima from skew-normal samples[J]. Communications in Statistics-Theory and Methods, 2019, DOI: 10.1080/03610926.2019.1623256

[19] Hashorva E, Weng Z. Limit laws for extremes of dependent stationary Gaussian arrays[J]. Statistics & Probability Letters, 2013, 83(1): 320-330.

[20] Liao X, Peng Z. Asymptotics for the maxima and minima of Hsler-Reiss bivariate Gaussian arrays[J]. Extremes, 2015, 18(1): 1-14.

[21] Lu Y, Peng Z. Maxima and minima of independent and non-identically distributed bivariate Gaussian triangular arrays[J]. Extremes, 2017, 20(1): 187-198.

[22] Resnick S I. Extreme values, regular variation and point processes[M]. Springer, 1987.

[23] Leadbetter M R, Lindgren G, Rootzén H. Extremes and related properties of random sequences and processes[M]. New York: Springer Verlag. 1983.