Propositional Logic Without The Deduction Theorem

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Abstract

In propositional logic, given a set of axioms, we can derive formulas. Here we present the derivations of some formulas without the use of the Deduction Theorem. The derivations are presented compactly with only few referrals to other theorems. Most textbooks in this subject avoid this kind of approach.

1. Introduction

For well-formed formulas $\mathscr{A}, \mathscr{B}, \mathscr{C},$ a set of axioms is as follows:

(*L*1) $\mathscr{A} \to (\mathscr{B} \to \mathscr{A})$

(*L*2) $(\mathscr{A} \to (\mathscr{B} \to \mathscr{C})) \to ((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C}))$

(*L*3) $(\sim\mathscr{A} \to \sim\mathscr{B}) \to (\mathscr{B} \to \mathscr{A})$.

The rule of inference is *modus ponens* abbreviated by MP.

2. Results

(*T*1) $\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B})$

Proof. [1] p. 32.

(*T*2) $\mathscr{A} \to \mathscr{A}$

Proof. [1] p. 32.

(*T*3) $\sim\sim\mathscr{A} \to \mathscr{A}$

Proof.

(1)	$\sim\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim\sim\sim\mathscr{A})$	(*T*1)
(2)	$(\sim\mathscr{A} \to \sim\sim\sim\mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{A})$	(*L*3)
(3)	$((\sim\mathscr{A} \to \sim\sim\sim\mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{A})) \to (\sim\sim\mathscr{A} \to ((\sim\mathscr{A} \to \sim\sim\sim\mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{A})))$	(*L*1)
(4)	$\sim\sim\mathscr{A} \to ((\sim\mathscr{A} \to \sim\sim\sim\mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{A}))$	(2), (3) MP
(5)	$(\sim\sim\mathscr{A} \to ((\sim\mathscr{A} \to \sim\sim\sim\mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{A}))) \to$ $((\sim\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim\sim\sim\mathscr{A})) \to (\sim\sim\mathscr{A} \to (\sim\sim\mathscr{A} \to \mathscr{A})))$	(*L*2)
(6)	$(\sim\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim\sim\sim\mathscr{A})) \to (\sim\sim\mathscr{A} \to (\sim\sim\mathscr{A} \to \mathscr{A}))$	(4), (5) MP
(7)	$\sim\sim\mathscr{A} \to (\sim\sim\mathscr{A} \to \mathscr{A})$	(1), (6) MP
(8)	$(\sim\sim\mathscr{A} \to (\sim\sim\mathscr{A} \to \mathscr{A})) \to ((\sim\sim\mathscr{A} \to \sim\sim\mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{A}))$	(*L*2)
(9)	$\sim\sim\mathscr{A} \to \sim\sim\mathscr{A}$	(*T*2)
(10)	$(\sim\sim\mathscr{A} \to \sim\sim\mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{A})$	(7), (8) MP
(11)	$\sim\sim\mathscr{A} \to \mathscr{A}$	(9), (10) MP

(*T*4) $\mathscr{A} \to \sim\sim\mathscr{A}$

Proof.

(1)	$\sim\sim\sim\mathscr{A} \to \sim\mathscr{A}$	(*T*3)
(2)	$(\sim\sim\sim\mathscr{A} \to \sim\mathscr{A}) \to (\mathscr{A} \to \sim\sim\mathscr{A})$	(*L*3)
(3)	$\mathscr{A} \to \sim\sim\mathscr{A}$	(1), (2) MP

(*T*5) $(\mathscr{A} \to \mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A})$

Proof.

(1)	$(\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B}))$	(*L*1)
(2)	$(\sim\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B}))$	(*L*2)
(3)	$((\sim\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B}))) \to$ $((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B}))))$	(*L*1)

(4) $(\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B})))$ (2), (3) MP

(5) $((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B})))) \to$
$(((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B}))) \to ((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B}))))$ (*L*2)

(6) $((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to (\mathscr{A} \to \mathscr{B}))) \to ((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B})))$ (4), (5) MP

(7) $(\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B}))$ (1), (6) MP

(8) $\sim\sim\mathscr{A} \to \mathscr{A}$ (*T*3)

(9) $(\sim\sim\mathscr{A} \to \mathscr{A}) \to ((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{A}))$ (*L*1)

(10) $(\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{A})$ (8), (9) MP

(11) $((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \mathscr{A}) \to (\sim\sim\mathscr{A} \to \mathscr{B}))) \to$
$(((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{A})) \to ((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{B})))$ (*L*2)

(12) $((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{A})) \to ((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{B}))$ (7), (11) MP

(13) $(\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{B})$ (10), (12) MP

(14) $\mathscr{B} \to \sim\sim\mathscr{B}$ (*T*4)

(15) $(\mathscr{B} \to \sim\sim\mathscr{B}) \to (\sim\sim\mathscr{A} \to (\mathscr{B} \to \sim\sim\mathscr{B}))$ (*L*1)

(16) $\sim\sim\mathscr{A} \to (\mathscr{B} \to \sim\sim\mathscr{B})$ (14), (15) MP

(17) $(\sim\sim\mathscr{A} \to (\mathscr{B} \to \sim\sim\mathscr{B})) \to ((\sim\sim\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}))$ (*L*2)

(18) $(\sim\sim\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B})$ (16), (17) MP

(19) $((\sim\sim\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B})) \to ((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B})))$ (*L*1)

(20) $(\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}))$ (18), (19) MP

(21) $((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}))) \to$
$(((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{B})) \to ((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B})))$ (*L*2)

(22) $((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \mathscr{B})) \to ((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}))$ (20), (21) MP

(23) $(\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B})$ (13), (22) MP

(24) $(\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A})$ (*L*3)

(25) $((\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A})) \to ((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A})))$ (*L*1)

(26) $(\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A}))$ (24), (25) MP

(27) $((\mathscr{A} \to \mathscr{B}) \to ((\sim\sim\mathscr{A} \to \sim\sim\mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A}))) \to$
$(((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B})) \to ((\mathscr{A} \to \mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A})))$ (*L*2)

(28) $((\mathscr{A} \to \mathscr{B}) \to (\sim\sim\mathscr{A} \to \sim\sim\mathscr{B})) \to ((\mathscr{A} \to \mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A}))$ (26), (27) MP

(29) $(\mathscr{A} \to \mathscr{B}) \to (\sim\mathscr{B} \to \sim\mathscr{A})$ (23), (28) MP

(*T*6) $\mathscr{B} \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C}))$

Proof.

(1) $\mathscr{B} \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{B})$ (*L*1)

(2) $(\mathscr{B} \to \mathscr{C}) \to (\mathscr{B} \to \mathscr{C})$ (*T*2)

(3) $((\mathscr{B} \to \mathscr{C}) \to (\mathscr{B} \to \mathscr{C})) \to (((\mathscr{B} \to \mathscr{C}) \to \mathscr{B}) \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}))$ (*L*2)

(4) $((\mathscr{B} \to \mathscr{C}) \to \mathscr{B}) \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C})$ (2), (3) MP

(5) $(((\mathscr{B} \to \mathscr{C}) \to \mathscr{B}) \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C})) \to (\mathscr{B} \to (((\mathscr{B} \to \mathscr{C}) \to \mathscr{B}) \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C})))$ (*L*1)

(6) $\mathscr{B} \to (((\mathscr{B} \to \mathscr{C}) \to \mathscr{B}) \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}))$ (4), (5) MP

(7) $(\mathscr{B} \to (((\mathscr{B} \to \mathscr{C}) \to \mathscr{B}) \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}))) \to$
$((\mathscr{B} \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{B})) \to (\mathscr{B} \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C})))$ (*L*2)

(8) $(\mathscr{B} \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{B})) \to (\mathscr{B} \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}))$ (6), (7) MP

(9) $\mathscr{B} \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C})$ (1), (8) MP

(10) $((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}) \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C}))$ (*T*5)

(11) $(((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}) \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C}))) \to (\mathscr{B} \to (((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}) \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C}))))$ (*L*1)

(12) $\mathscr{B} \to (((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}) \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C})))$ (10), (11) MP

(13) $(\mathscr{B} \to (((\mathscr{B} \to \mathscr{C}) \to \mathscr{C}) \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C})))) \to$
$((\mathscr{B} \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C})) \to (\mathscr{B} \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C}))))$ (*L*2)

(14) $(\mathscr{B} \to ((\mathscr{B} \to \mathscr{C}) \to \mathscr{C})) \to (\mathscr{B} \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C})))$ (12), (13) MP

(15) $\mathscr{B} \to (\sim\mathscr{C} \to \sim(\mathscr{B} \to \mathscr{C}))$ (9), (14) MP

(*T*7) $(\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to (\mathscr{A} \to \mathscr{B})$

Proof.

(1)	$\mathscr{A} \to \mathscr{A}$	($T2$)
(2)	$(\mathscr{A} \to \mathscr{A}) \to ((\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to (\mathscr{A} \to \mathscr{A}))$	($L1$)
(3)	$(\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to (\mathscr{A} \to \mathscr{A})$	(1), (2) MP
(4)	$(\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to ((\mathscr{A} \to \mathscr{A}) \to (\mathscr{A} \to \mathscr{B}))$	($L2$)
(5)	$((\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to ((\mathscr{A} \to \mathscr{A}) \to (\mathscr{A} \to \mathscr{B}))) \to$ $(((\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to (\mathscr{A} \to \mathscr{A})) \to ((\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to (\mathscr{A} \to \mathscr{B})))$	($L2$)
(6)	$((\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to (\mathscr{A} \to \mathscr{A})) \to ((\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to (\mathscr{A} \to \mathscr{B}))$	(4), (5) MP
(7)	$(\mathscr{A} \to (\mathscr{A} \to \mathscr{B})) \to (\mathscr{A} \to \mathscr{B})$	(3), (6) MP

(*T*8) $(\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A})$

Proof.

(1)	$(\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to \sim\mathscr{B})$	($T5$)
(2)	$\sim\mathscr{B} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))$	($T6$)
(3)	$(\sim\mathscr{B} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to (\sim\mathscr{A} \to (\sim\mathscr{B} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))))$	($L1$)
(4)	$\sim\mathscr{A} \to (\sim\mathscr{B} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))$	(2), (3) MP
(5)	$(\sim\mathscr{A} \to (\sim\mathscr{B} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))) \to ((\sim\mathscr{A} \to \sim\mathscr{B}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))))$	($L2$)
(6)	$(\sim\mathscr{A} \to \sim\mathscr{B}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))$	(4), (5) MP
(7)	$((\sim\mathscr{A} \to \sim\mathscr{B}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))) \to$ $((\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to \sim\mathscr{B}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))))$	($L1$)
(8)	$(\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to \sim\mathscr{B}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))))$	(6), (7) MP
(9)	$((\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to \sim\mathscr{B}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))))) \to$ $(((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to \sim\mathscr{B})) \to ((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))))$	($L2$)
(10)	$((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to \sim\mathscr{B})) \to ((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))))$	(8), (9) MP
(11)	$(\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))$	(1), (10) MP
(12)	$(\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))$	($T7$)
(13)	$((\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to$ $((\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))))$	($L1$)
(14)	$(\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))$	(12), (13) MP
(15)	$((\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))) \to$ $(((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))) \to ((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))))$	($L2$)
(16)	$((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))) \to ((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})))$	(14), (15) MP
(17)	$(\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))$	(11), (16) MP
(18)	$(\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A})$	($L3$)
(19)	$((\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A})) \to$ $((\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A})))$	($L1$)
(20)	$(\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A}))$	(18), (19) MP
(21)	$((\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A})) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A}))) \to$ $(((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to ((\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A})))$	($L2$)
(22)	$((\mathscr{B} \to \mathscr{A}) \to (\sim\mathscr{A} \to \sim(\sim\mathscr{B} \to \mathscr{A}))) \to ((\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A}))$	(20), (21) MP
(23)	$(\mathscr{B} \to \mathscr{A}) \to ((\sim\mathscr{B} \to \mathscr{A}) \to \mathscr{A})$	(17), (22) MP

3. References

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