Introducing: second-order permutation and corresponding second-order permutation factorial

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JANUARY 2019

1 Abstract

In this study we answer questions that have to do with finding out the total number of ways of arranging a finite set of symbols or objects directly under a single line constraint set of finite symbols such that common symbols between the two sets do not repeat on the vertical positions. We go further to study the outcomes when both sets consist of the same symbols and when they consist of different symbols. We identify this form of permutation as 'second-order permutation' and show that it has a corresponding unique factorial which plays a prominent role in most of the results obtained. This has been discovered by examining many practical problems which give rise to the emergence of second-order permutation. Upon examination of these problems, it becomes clear that a directly higher order of permutation exist. Hence this study aims at equipping mathematics scholars, educators and researchers with the necessary background knowledge and framework for incorporating second-order permutation into the field of combinatorial mathematics.

Keywords
Second-order Permutation, Permutation and Combination, Latin square, Bassey triangle, Factorial.

2 Introduction

According to Dike and Oyah[1], permutation is the different arrangements which can be made out of a given number of objects/symbols by taking some or all at a time.

However, in this study we introduce a more complex form of permutation known as "second-order permutation". Second-order permutation involves permutation carried out under the influence of a single line external constraint set of the same size as the permutation set but with the condition that common elements between the two sets do not repeat on the vertical positions.

Second-order permutation has a corresponding unique factorial denoted by the symbol $n!^{2'}$, this symbol has an apostrophe sign attached to the index number and should not be confused for $(n!)^2$. This second-order permutation symbol might just be the newest mathematical symbol and this study makes a case for its universal adoption and acceptance. By applying the method of mathematical induction, it will be shown how to derive the general notation for $n!^{2'}$ and its various real life applications.

Conditional cases of second-order permutation where different symbols make up the constraint set and permutation set is analyzed in detail in this paper. In addition, we explore the relationship between second-order permutation factorial ($n!^{2'}$) and what should now be known as first-order permutation factorial (n!) using a new mathematical triangle known as Bassey triangle.

Previous work on second-order permutation took the form of enumerating the size of Latin squares [4].

A central problem in the theory of Latin squares is to determine how many Latin squares of each size exist. This problem is difficult because there is no definite formula for determining the total number of reduced Latin squares of all orders[24]. This explains the reason why the actual size of Latin squares has only been known to the order 11 [3]. However, in this study, we show that this problem can be solved for all Latin squares in the category of (2,n) using the second-order permutation factorial $n!^{2'}$ along with the already known factorial n!. Hence, by so doing we provide an insight on the way to go in attempting to solve this problem on a larger scale which is to stimulate more research in the area of higher order permutations.

Therefore, what this study aims to achieve is to establish a basic framework upon which we can prove the existence of second-order permutation problems and to provide a systematic mathematical approach for tackling such problems including real life applications and its application to the Latin square theory.

3 Permutation

The different arrangements which can be made out of a given number of objects/symbols by taking some or all at a time are called permutations.

For instance:
All permutations/arrangements made with the letters ABC taking all at a time are ABC, ACB, BAC, BCA, CAB and CBA.
Instead of listing them like this, we can work it out.
There are 3 choices for the first alphabet and for each choice there are a further 2 choices for the second alphabet. That is, there are 3 x 2 = 6 choices of first and second alphabets combined. The third alphabet is then the one that is left.
So, how many four-digit numbers can be constructed using the numerals 1234 once each?
Like we did previously; the first numeral can be selected in one of 4 ways, each selection leaving 3 ways to select the second numeral. So there are:
4 x 3 = 12ways of selecting the first two numerals.
Each combination of the first two numerals leaves two ways to select the third numeral. So there are now:
4 x 3 x 2 = 24ways of selecting the first three numerals.
The last numeral is the one that is left.
This enumeration gives us a pattern such that if we have "n" different items then we can form:
n(n-1)(n-2)x.............. x2x1.
This type of product of decreasing natural numbers is known as factorials. [23, 2]

3.1 The Factorial Notation

Let n be a positive integer. Then, the continued product of first n natural numbers is called factorial n, denoted by n!.
Hence, n! = n(n-1)(n-2)(n-3)..............3.2.1.
For instance:
5! = 5 x 4 x 3 x 2 x 1 = 120
4! = 4 x 3 x 2 x 1 = 24

4 Second-order permutation

Permutations involving two finite sets of the same size containing the same elements or in some cases not the same elements, where one set is a constraint set while the other set is the permutation set. The aim is to determine the number of valid arrangements of the permutation set that is possible under the influence of the constraint set. Validity is determined on second-order

permutations when objects common to both sets do not repeat on a corresponding vertical line.
The two sets are usually positioned in rows such that elements in corresponding positions are all in line with each other. The constraint set elements are usually fixed while the permutation set elements are arranged to find out the total number of valid arrangements that will result from such permutation under the influence of the constraint set.
For example:

1. A B C——constraint set
   B C A——permutation set(valid)

2. A B C——constraint set
   C B A——permutation set(invalid)

The permutation set in 1, is valid because none of the elements are repeated on any of the vertical lines while the permutation set in 2, is invalid because of the repetition of the element B, on the second column.

4.1 Types of second-order permutation

Second-order permutation usually occur in two general forms namely:

1. Homogeneous form
2. Non-homogeneous form

4.1.1 Homogeneous form

This form occurs when the same type of elements makes up the constraint set and the permutation set respectively.
See examples below:

1.

A	B	C
C	A	B

2.

RED	BLUE	GREEN
BLUE	GREEN	RED

4.1.2 Non-Homogeneous form

This form occurs when both constraint and permutation sets contain elements that are not common to each other.
See examples below.

1.

A	B	C
C	D	A

2.

RED	BLUE
GREEN	RED

Non-homogeneous forms can also occur when both sets contain entirely different elements. In such cases, the constraint set has no influence over the permutation set.
See examples below.

1.

P	Q	R
F	G	H

2.

YELLOW	BLUE	PURPLE
GREEN	RED	BROWN

4.2 Derivation of the general formula for second-order permutation

In order to derive the general formula for second-order permutations, we will examine three second-order permutation cases.
Case 1: A B C —– (constraint set)
Our task is to find the total number of valid sets from elements (ABC) that will fit the constraint above.
ABC can be arranged in 3 x 2 ways without considering the constraint set. See fig 1.

A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
A	B	C	A	C	B	B	C	A	B	A	C	C	A	B	C	B	A

Fig 1: Arrangement of "ABC" without considering the constraint set.

If we now take into consideration the constraint set, we will only have 2 valid permutation sets: (BCA) and (CAB). See fig 2.

A	B	C	A	B	C
B	C	A	C	A	B

Fig 2: Showing the 2 valid sets.

Therefore, the total number of ways of arranging 3 elements under the same 3 elements to obtain valid sets is 2 ways.

Case 2:

A B —– (constraint set)

Our task is to find the total number of valid sets from elements (AB) that will fit the constraint above. AB can be arranged in 2 x 1 ways without considering the constraint set. See fig 3.

A	B	A	B
A	B	B	A

Fig 3: Arrangement of "AB" without considering the constraint set.

If we now take into consideration the constraint set, we will only have 1 valid set: (BA). See fig 4.

A	B
B	A

Fig 4: Showing the only valid set.

Therefore, the total number of ways of arranging 2 elements under the same 2 elements to obtain valid sets is 1 way.

Case 3:

A —– (Constraint set)

Our task is to find the total number of valid sets from element (A) that will fit the constraint above. A can be arranged in 1 way without considering the constraint. See fig 5.

A
A

Fig 5. Arrangement of "A" without considering the constraint set.

If we now take into consideration the constraint set, we will have no valid set. This is because, A under A, is invalid. Therefore, the total number of ways of arranging 1 element under the same 1 element to obtain valid sets is 0.

Let us consider case 3: single element (A)

$$1 - 1 = 0$$

Case 2: double element (AB)

$$2 - 1 = 2(1 - 1) + 1 = 1$$

Case 1: triple element (ABC)

$$6 - 4 = 3\{2(1 - 1) + 1\} - 1 = 2$$

Can you see the pattern developing here?
It means, if we have a constraint set (ABCD)
We will obtain the following valid sets:

$$4\{3[2(1 - 1) + 1] - 1\} + 1 = 9$$

Let us count to confirm if the answer above is correct.
<u>A B C D</u>
A B C D
A B D C
A C B D
A C D B
A D C B
A D B C
B A C D
B A D C valid 1
B C D A valid 2
B C A D
B D A C valid 3
B D C A
C A B D
C A D B valid 4
C B D A
C B A D
C D A B valid 5
C D B A valid 6
D A B C valid 7
D A C B
D B C A
D B A C
D C A B valid 8
D C B A valid 9
This confirms the validity of our answer, meaning that there are 9 valid sets of (ABCD) that can be arranged under (ABCD).
The result obtained here is very exciting because we can now apply the same

method for (ABCDE) thus:

$$5\{4[3(2(1-1)+1)-1]+1\}-1=44$$

Therefore, there are 44 ways of arranging (ABCDE) under any order of (ABCDE).
We can now use this relationship to formulate a general term thus: If we have "n" different elements, then we obtain a general term:

$$n\{n-1(n-2(n-3(\dots\dots\dots\dots(2(1-1)+1)-1)+1\}\dots\dots\dots\dots \pm 1$$

4.3 Second-Order Permutation Factorial ($n!^{2'}$)

The second-order permutation factorial is denoted by the symbol $n!^{2'}$.
Note: We add the apostrophe sign to the index number to distinguish it from $(n!)^2$ or $n!^2$, which has been used in other works as the square of n!.

$$n!^{2'} = n(n-1(n-2(n-3(\dots\dots(2(1-1)+1)-1)+1)\dots\dots \pm 1 \quad (1)$$

Therefore.
The second-order permutation of 1 is given by:
$1!^{2'} = 1-1=0$
The second-order permutation of 2 is given by:
$2!^{2'} = 2(1-1)+1=1$
The second-order permutation of 3 is given by:
$3!^{2'} = 3\{2(1-1)+1\}-1=2$
The second-order permutation of 4 is given by:
$4!^{2'} = 4\{3[2(1-1)+1]-1\}+1=9$

4.4 Deduction

$$1!^{2'} = (0!^{2'})-1 \quad (2)$$

$$2!^{2'} = 2(1!^{2'})+1 \quad (3)$$

$$3!^{2'} = 3(2!^{2'})-1 \quad (4)$$

$$n!^{2'} = n(n-1)!^{2'} \pm 1 \quad (5)$$

Note: In equation (5), the (±) sign is (+) when "n" is an even integer and (-) when "n" is an odd integer.
from (2)

$$0!^{2'} = \frac{1!^{2'}+1}{1} \quad (6)$$

from (3)

$$1!^{2'} = \frac{2!^{2'} - 1}{2} \tag{7}$$

from (4)

$$2!^{2'} = \frac{3!^{2'} + 1}{3} \tag{8}$$

$$n!^{2'} = \frac{(n+1)!^{2'} \pm 1}{n+1} \tag{9}$$

Theorem 1.

$$n!^{2'} + (n+1)!^{2'} = \frac{(n+2)!^{2'}}{n+1} \tag{10}$$

Proof. equating (3) and (8)

$$2(1!^{2'}) + 1 = \frac{3!^{2'} + 1}{3} \tag{11}$$

$$1!^{2'} = \frac{3!^{2'} + 1}{6} - \frac{1}{2} \tag{12}$$

$$n!^{2'} = \frac{(n+2)!^{2'} - 1}{(n+1)(n+2)} - \frac{1}{n+1} \tag{13}$$

sum (12) and (8)

$$1!^{2'} + 2!^{2'} = \frac{3!^{2'} + 1}{6} - \frac{1}{2} + \frac{3!^{2'} + 1}{3} = \frac{3(3!^{2'})}{6} \tag{14}$$

$$1!^{2'} + 2!^{2'} = \frac{3!^{2'}}{2} \tag{15}$$

□

Remark 1. *Interestingly, Theorem 1 also holds for permutations without external constraint. For example:*

$$1! + 2! = 1 + 2 = \frac{3!}{2}$$

$$2! + 3! = 2 + 6 = \frac{4!}{3}$$

Therefore

$$n! + (n+1)! = \frac{(n+2)!}{n+1}$$

Theorem 2. *Prove that:*

$$0!^{2'} = 1 \tag{16}$$

Proof. According to theorem 1.

$$0!^{2'} + 1!^{2'} = \frac{2!^{2'}}{1}$$

$$0!^{2'} = \frac{2!^{2'}}{1} - 1!^{2'}$$

Therefore

$$0!^{2'} = 1 - 0 = 1$$

□

4.5 Applications

1. A marriage counselor is to hold a counseling session with 4 couples in his office. She plans to sit the husbands on the first row of 4 chairs while the wives are to sit on the second row of 4 chairs. However, he doesn't want any of the wives to sit directly behind their husbands. In how many ways can she achieve this position?
   Solution
   Since the constraint set is not ordered, it implies that the husbands can be positioned on the first row in 4! Ways = 4 x 3 x 2 x 1 = 24 ways.
   In each of the 24 ways the husbands can sit, the wives can be positioned in $4!^{2'} = 4\{3[2(1-1)+1]-1\}+1 = 9$ ways on the second row such that no wife sits directly behind her husband.
   Hence the marriage counselor can achieve her target position in 24 x 9 ways = 216 ways.

2. If the counselor plans to position the husbands on the front row on first-come first-sit basis, how many ways can she still achieve her objective of not sitting wives directly behind their husbands on the second row.
   Solution.
   Here the constraint set is ordered because they will be positioned on first-come first-sit basis, meaning that there is one way of positioning the husbands on the front row.

Wives on the second row can be positioned in $4!^{2'} = 4\{3[2(1-1)+1]-1\}+1 = 9 ways.$
Therefore, the marriage counselor can achieve her target in 9 x 1 ways = 9 ways.

3. An examination officer is given the task of positioning six sets of twins for two examinations. He is to position them on two rows of six chairs each. The eldest of each set of twins are to sit on the front row in the alphabetical order of their initials for the first examination and the roles are reversed for the second examination. What is the total number of ways he can position them for the two papers?
   Solution
   Total number of ways for the two exams = number of ways for exam 1 + number of ways for exam 2
   For exam 1: the constraint set is ordered hence there is one way of positioning older twins on the front row according to their initials.
   Number of ways for positioning the 6 younger twins not to sit directly behind their twin brothers is
   $6!^{2'} = 6\{5[4(3(2(1-1)+1)-1)+1]-1\}+1 = 265$
   For exam 2: the roles are reversed hence constraint set is ordered. Number of ways for positioning the 6 older twins not to sit directly behind their twin brothers is.
   $6!^{2'} = 6\{5[4(3(2(1-1)+1)-1)+1]-1\}+1 = 265$
   Therefore, total number of ways for the two exams = 265 + 265 = 530 ways.

4. An artist intends to paint a wall in his room with 4 different colors. To do this he must divide the wall surface into equal horizontal and vertical segments. He plans to use the colors red, blue, green and yellow in that order for the first row. For subsequent rows he doesn't want any of the colors on the constraint set to repeat on the columns and each combination of colors on the rows must be unique. If he must divide the wall surface into 4 equal vertical segments, how many equal horizontal segments will he require in order to achieve his aim?
   Solution.
   The constraint set is ordered and is equal to 1 row.
   The next thing we have to find is the number of rows that can be placed under the constraint set to form valid sets.
   The number equals $4!^{2'} = 4\{3[2(1-1)+1]-1\}+1 = 9$
   Hence the artist requires 9 + 1 = 10 equal horizontal segments. See fig 6.

Red	Blue	Green	Yellow
Blue	Green	Yellow	Red
Blue	Red	Yellow	Green
Blue	Yellow	Red	Green
Green	Yellow	Red	Blue
Green	Red	Yellow	Blue
Green	Yellow	Blue	Red
Yellow	Green	Blue	Red
Yellow	Green	Red	Blue
Yellow	Red	Blue	Green

Fig 6: The appearance of the wall.

5 Conditional second-order permutation

Conditional second-order permutation exist in situations where the elements in the constraint set are not exactly the same as the elements in the permutation set. In that case the aim is to arrange the permutation set such that the common elements in the two sets do not repeat on any of the vertical lines; also, the constraint set has no effect over any element in the permutation set that is not contained in the constraint set.
Example.
ABC – constraint set
BAD
only 2 elements in the constraint set are contained in the permutation set. Our task is to figure out the number of ways BAD can be arranged under ABC such that common elements A and B will not be repeated on any vertical line.
We can represent this problem as ${}^{n}E_r$ where n is the number of elements in the constraint set and r is the number of elements in the permutation set that is not in the constraint set.
Let r and n be positive integers such that 0 ≤ r ≤ n (n ≥ r, r ≥ 0)
Then, the number of all arrangements of the permutation set including r under the influence of the constraint set n is denoted by ${}^{n}E_r$.
Hence, in the example above, we are to solve for ${}^{3}E_1$.
Note:

1. When r = 0, it implies that
   ${}^{n}E_0 = n!^{2'}$ (all elements in constraint set are contained in permutation

set)

2. When r = n, it implies that
$^{n}E_n = n!$ (all elements in permutation set are not contained in constraint set)

Putting them all together according to their respective orders, we obtain the shape of a triangle. See fig 7.

$^{0}E_0$						
$^{1}E_0$	$^{1}E_1$					
$^{2}E_0$	$^{2}E_1$	$^{2}E_2$				
$^{3}E_0$	$^{3}E_1$	$^{3}E_2$	$^{3}E_3$			
$^{4}E_0$	$^{4}E_1$	$^{4}E_2$	$^{4}E_3$	$^{4}E_4$		
$^{5}E_0$	$^{5}E_1$	$^{5}E_2$	$^{5}E_3$	$^{5}E_4$	$^{5}E_5$	
$^{6}E_0$	$^{6}E_1$	$^{6}E_2$	$^{6}E_3$	$^{6}E_4$	$^{6}E_5$	$^{6}E_6$

Fig 7: Triangle showing the several representations of $^{n}E_r$.
This triangle can be built-up in this order to infinity; however, the major aim is to evaluate the cell values.
By inputting the values we already know into the triangle.
The triangle becomes. See fig 8.

1						
0	1					
1	$^{2}e_1$	2				
2	$^{3}e_1$	$^{3}e_2$	6			
9	$^{4}e_1$	$^{4}e_2$	$^{4}e_3$	24		
44	$^{5}e_1$	$^{5}e_2$	$^{5}e_3$	$^{5}e_4$	120	
265	$^{6}e_1$	$^{6}e_2$	$^{6}e_3$	$^{6}e_4$	$^{6}e_5$	720

Fig 8: Triangle with values of $n!^{2'}$ and n! inputted.
By manually counting $^{2}E_1$, $^{3}E_1$, and $^{4}E_1$.
The following relationship can be observed:
$^{0}E_0 + {}^{1}E_0 = {}^{1}E_1$
$^{1}E_0 + {}^{2}E_0 = {}^{2}E_1$
$^{2}E_0 + {}^{3}E_0 = {}^{3}E_1$
This implies that; by adding corresponding values on column 1 we obtain the values on column 2.
Therefore.

$^1E_1 = 1 + 0 = 1$
$^2E_1 = 0 + 1 = 1$
$^3E_1 = 1 + 2 = 3$
$^4E_1 = 2 + 9 = 11$
$^5E_1 = 9 + 44 = 53$
$^6E_1 = 44 + 265 = 309$
This kind of relationship is observed on all the columns of the triangle; hence, a general assumption of the values on the triangle can be made thus:

Theorem 3. *The values of two corresponding cells on a column sum up to give the value of the cell adjacent to the bottom cell of the two corresponding cells of the triangle. The theorem can be represented pictorially as seen on fig 9.*

A	
B	$A + B$

Fig 9.

Mathematically.

$$^nE_r +^{n+1} E_r =^{n+1} E_{r+1} \tag{17}$$

By applying theorem 3, we can complete the triangle as seen on fig 10.

1						
0	1					
1	1	2				
2	3	4	6			
9	11	14	18	24		
44	53	64	78	96	120	
265	309	362	426	504	600	720

Fig 10. Bassey triangle

The resulting triangle is known as BASSEY triangle, it highlights the relationship between n! and $n!^{2'}$.
It can be clearly seen from the triangle the transformation that occurs as $n!^{2'}$ moves towards n! on each row of the triangle.
Notice that the values on the first cells of each row are all $n!^{2'}$ while the values on the last cells of each row are all n!

5.0.1 Application of Bassey triangle

1. How many ways can a British girl, a Swedish girl and a Chilean girl be paired at the same time with a British Boy, a Jamaican boy and a Nigerian boy, if a boy and girl from the same country cannot be paired.
   Solution.
   B S C.......... Constraint set
   J B N.......... permutation set

   $$n = 3, r = 2$$

   Hence, we are to find ${}^3E_2$ on the triangle.
   check answer on the 4th row and 3rd column.
   ${}^3E_2 = 4$

B	S	C	B	S	C	B	S	C	B	S	C
J	B	N	J	N	B	N	B	J	N	J	B

   Fig 11: Shows total number of pairings.

2. How many ways can the alphabets that make up the word "freak" be arranged under the word "claim", if the common alphabets must not be directly under each other.
   Solution.
   C L A I M.......... Constraint set
   F R E A K.......... permutation set

   $$n = 5, r = 4$$

   Hence, we are to find ${}^5E_4$ on the triangle.
   check answer on the 6th row and 5th column.

   $${}^5E_4 = 96$$

5.1 Mathematical evaluation of $({}^{n}E_r)$

It is possible to directly evaluate ${}^{n}E_r$ without having to construct Bassey triangle or counting directly. First; we refer to theorem 1, which states that:

$$n!^{2'} + (n+1)!^{2'} = \frac{(n+2)!^{2'}}{n+1}$$

. Let's denote this as a_n

Example 1.

$$0!^{2'} + 1!^{2'} = \frac{2!^{2'}}{1} = a_0$$

Example 2.

$$1!^{2'} + 2!^{2'} = \frac{3!^{2'}}{2} = a_1$$

Example 3.

$$2!^{2'} + 3!^{2'} = \frac{4!^{2'}}{3} = a_2$$

and so on.

By applying Theorem 3 from the first column to the last column in each of the rows of Bassey triangle, we arrive at the following mathematical relationships.

$$1! =^{1} E_1 = \frac{1(2!^{2'})}{1}$$

$$2! =^{2} E_2 = \frac{1(2!^{2'})}{1} + \frac{1(3!^{2'})}{2}$$

$$3! =^3 E_3 = \frac{1(2!^{2'})}{1} + \frac{2(3!^{2'})}{2} + \frac{1(4!^{2'})}{3}$$

$$4! =^4 E_4 = \frac{1(2!^{2'})}{1} + \frac{3(3!^{2'})}{2} + \frac{3(4!^{2'})}{3} + \frac{1(5!^{2'})}{4}$$

$$5! =^5 E_5 = \frac{1(2!^{2'})}{1} + \frac{4(3!^{2'})}{2} + \frac{6(4!^{2'})}{3} + \frac{4(5!^{2'})}{4} + \frac{1(6!^{2'})}{5}$$

$$6! =^6 E_6 = \frac{1(2!^{2'})}{1} + \frac{5(3!^{2'})}{2} + \frac{10(4!^{2'})}{3} + \frac{10(5!^{2'})}{4} + \frac{5(6!^{2'})}{5} + \frac{1(7!^{2'})}{6}$$

Notice that the coefficients are all combinations, same as what obtains on Pascal triangle. Re-writing the relationship in terms of a_n and ${}^nC_r$, produces the following result.

$1! =^1 E_1 =^0 C_0 a_0$

$2! =^2 E_2 =^1 C_1 a_0 +^1 C_0 a_1$

$3! =^3 E_3 =^2 C_2 a_0 +^2 C_1 a_1 +^2 C_0 a_2$

$4! =^4 E_4 =^3 C_3 a_0 +^3 C_2 a_1 +^3 C_1 a_2 +^3 C_0 a_3$

$5! =^5 E_5 =^4 C_4 a_0 +^4 C_3 a_1 +^4 C_2 a_2 +^4 C_1 a_3 +^4 C_0 a_4$

$6! =^6 E_6 =^5 C_5 a_0 +^5 C_4 a_1 +^5 C_3 a_2 +^5 C_2 a_3 +^5 C_1 a_4 +^5 C_0 a_5$

From the relationships above, it can be deduced that:

$$^nE_r =^{r-1} C_{r-1} a_{n-r} + {}^{r-1}C_{r-2} a_{n-r+1} + {}^{r-1}C_{r-3} a_{n-r+2} + \ldots\ldots {}^{r-1}C_{r-r} a_{\{(n-r)+(r-1)\}} \tag{18}$$

Example 4. *Find* ${}^5E_3$

$$\begin{aligned} {}^5E_3 &= {}^{3-1}C_{3-1} a_{5-3} +^{3-1} C_{3-2} a_{5-3+1} +^{3-1} C_{3-3} a_{5-3+2} \\ &= {}^2C_2 a_2 +^2 C_1 a_3 +^2 C_0 a_4 \end{aligned}$$

$$
\begin{aligned}
&= \frac{(2!)}{(2!0!)}\frac{4!^{2'}}{3} + \frac{(2!)}{(1!1!)}\frac{5!^{2'}}{4} + \frac{(2!)}{(0!2!)}\frac{6!^{2'}}{5} \\
&= \frac{(1)}{(1)}\frac{9}{3} + \frac{(2)}{(1)}\frac{44}{4} + \frac{(1)}{(1)}\frac{265}{5} \\
&= 3 + 22 + 53 = 78
\end{aligned}
$$

This answer can be confirmed on Bassey triangle.

Example 5. *Find* ${}^{6}E_{2}$

$$
\begin{aligned}
{}^{6}E_{2} &= {}^{2-1}C_{2-1}a_{6-2} + {}^{2-1}C_{2-2}a_{6-2+1} \\
&= {}^{1}C_{1}a_{4} + {}^{1}C_{0}a_{5} \\
&= \frac{(1!)}{(1!0!)}\frac{6!^{2'}}{5} + \frac{(1!)}{(0!1!)}\frac{7!^{2'}}{6} \\
&= \frac{(1)}{(1)}\frac{265}{5} + \frac{(1)}{(1)}\frac{1854}{6} \\
&= 53 + 309 = 362
\end{aligned}
$$

Example 6. *How many ways can A B C D be arranged under A B C D, if B must always be directly under A.*
Solution.
Because B must always be under A, we now have to evaluate the total number of ways of arranging DAC under BCD such that the common elements do not repeat. See fig 12.

A	B	C	D
B	D	A	C

Fig 12.

The problem becomes ${}^{3}E_{1}$

$$
\begin{aligned}
{}^{3}E_{1} &= {}^{1-1}C_{1-1}a_{3-1} \\
&= {}^{0}C_{0}a_{2} \\
&= \frac{(1)}{(1)}\frac{9}{3} \\
&= 3
\end{aligned}
$$

5.2 Transposed Bassey Triangle

Bassey triangle is transposed when the contents of all the diagonals are moved to the corresponding columns as shown in figure 13 below.

${}^0E_0$						
${}^1E_1$	${}^1E_0$					
${}^2E_2$	${}^2E_1$	${}^2E_0$				
${}^3E_3$	${}^3E_2$	${}^3E_1$	${}^3E_0$			
${}^4E_4$	${}^4E_3$	${}^4E_2$	${}^4E_1$	${}^4E_0$		
${}^5E_5$	${}^5E_4$	${}^5E_3$	${}^5E_2$	${}^5E_1$	${}^5E_0$	
${}^6E_6$	${}^6E_5$	${}^6E_4$	${}^6E_3$	${}^6E_2$	${}^6E_1$	${}^6E_0$

Fig 13: Transposed Bassey triangle showing the new positions of ${}^nE_r$.
The resulting triangle with values inputted is shown in figure 14 below.

1						
1	0					
2	1	1				
6	4	3	2			
24	18	14	11	9		
120	96	78	64	53	44	
720	600	504	426	362	309	265

Fig 14: Transposed Bassey triangle.
The following relationship is observed along the columns.

$$^1E_1 -^0 E_0 =^1 E_0$$

$$^2E_1 -^1 E_0 =^2 E_0$$

In general.

$$^nE_r -^{n-1} E_{r-1} =^n E_{r-1} \tag{19}$$

Also.
$(n+1)! - n! = n(n!)$
we denote this as b_n
Therefore.
$1! - 0! = 0(0!)$ is denoted as b_0

$2! - 1! = 1(1!)$ is denoted as b_1
$3! - 2! = 2(2!)$ is denoted as b_2
and so on.

By applying equation (20) from the first column to the last column in each of the rows of the triangle, we arrive at the following mathematical relationships.

$$1!^{2'} =^1 E_0 =^0 C_0 b_0$$

$$2!^{2'} =^2 E_0 =^1 C_0 b_1 -^1 C_1 b_0$$

$$3!^{2'} =^3 E_0 =^2 C_0 b_2 -^2 C_1 b_1 +^2 C_2 b_0$$

$$4!^{2'} =^4 E_0 =^3 C_0 b_3 -^3 C_1 b_2 +^3 C_2 b_1 -^3 C_3 b_0$$

$$5!^{2'} =^5 E_0 =^4 C_0 b_4 -^4 C_1 b_3 +^4 C_2 b_2 -^4 C_3 b_1 +^4 C_4 b_0$$

$$6!^{2'} =^6 E_0 =^5 C_0 b_5 -^5 C_1 b_4 +^5 C_2 b_3 -^5 C_3 b_2 +^5 C_4 b_1 -^5 C_5 b_0$$

From the relationships above, it can be deduced that:

$$^nE_r =^{r-1} C_{r-1} b_{n-r} -^{r-1} C_{r-2} b_{n-r+1} +^{r-1} C_{r-3} b_{n-r+2} - \ldots\ldots ^{r-1}C_{r-r} b_{\{(n-r)+(r-1)\}} \tag{20}$$

This relationship offers an alternative means of evaluating the values of ^nE_r

Theorem 4.

$$\frac{^nE_0}{^{n-1}E_1} = n - 1 \tag{21}$$

Proof.

$$\frac{n!^{2'}}{^0C_0 a_{n-1-1}} = \frac{n!^{2'}}{a_{n-2}}$$

$$\begin{aligned} a_{n-2} &= \frac{(n+2-2)!^{2'}}{n-1} \\ &= \frac{n!^{2'}}{n-1} \end{aligned}$$

Therefore.

$$\frac{(n!^{2'})(n-1)}{n!^{2'}} = n - 1$$

□

So, according to the theorem.

$$\frac{{}^{2}E_0}{{}^{1}E_1} = 2 - 1$$
$$\frac{{}^{3}E_0}{{}^{2}E_1} = 3 - 1$$

This theorem is useful in computing the total number of home and away league fixtures in league format competitions.

Example 7. *The total number of home and away matches in a 5 team league =*

$$\frac{5(44)}{11}$$

Example 8. *The total number of home and away matches in a 6 team league =*

$$\frac{6(265)}{53}$$

The total number of home and away matches in "n" team league =

$$\frac{n({}^{n}E_0)}{{}^{n-1}E_1} = n(n-1)$$

The total number of pairings =

$$\frac{n(n-1)}{2}$$

6 The Size of all (2,n) Latin rectangles ($L_{2,n}$)

Definition 1. *Latin square is a natural generalization of a permutation. More concretely, a n^{th} order Latin square is an n by n grid in which the numbers 1,2......n(often called symbols) are each used exactly once in each row and column[3].*

We readily know that there are n! permutation and so it comes as a surprise that the number of Latin squares of order n is only known up to n = 11 [4]. If we let L_n be the number of n^{th} order Latin squares, then the best known bounds of L_n are very far apart. For example, van Lint and Wilson [22] give upper and lower bounds which differ asymptotically by a factor of n^n.
Here we aim to show that the number of all distinct (2,n) Latin rectangles

can be solved. This is made possible by the discovery of the general term for $n!^{2'}$. This would give us a good idea on how to approach the problem generally and open up a wide area of further research on higher-order permutations.
Example of Latin squares.

1.

A	B
B	A

Fig 15: Order 2 Latin square

2.

A	B	C
B	C	A
C	A	B

Fig 16: Order 3 Latin square

Definition 2. *Latin rectangle is a k x n array L, with symbols such that each row and column contains only distinct symbols. If k = n then L is a Latin square of order n[4].*

Latin rectangles can also be described as incomplete Latin squares. For example (2,3) Latin rectangle is an incomplete order 3 Latin square. (3,4), (2,4) are all incomplete forms of an order 4 Latin square and so on. All (2,n) Latin squares are incomplete except (2,2).

A	B	C
B	C	A

Fig 17: (2,3) Latin rectangle
L_k, n denotes the number of distinct Latin rectangles.

Definition 3. *A Latin square of order n is said to be reduced if its first row and first column are in the standard order 0, 1,....,n-1.*

Example.

0	1	2
1	2	0
2	0	1

Figure 18: Reduced Latin square of order 3.

The use of the term "reduced" goes back at least to MacMahon[6], and was adopted, for example by Fisher and Yates[7], Denes and Keedwell[8, 9] and Laywine and Mullen[10]. Euler[11] instead used the term "regular square". Some authors use "normalized" [14, 12], "standardized" [15], "standard" or "in standard form"[13] in place of what we call "reduced." Similarly, our definition of "normalized" also has some alternative names; for example "standardized" [16] "in the standard form" [17], "semi-normalised" [12] and "reduced" [19, 18], which can be confusing. Some authors avoid this problem by not assigning names to reduced or normalized Latin squares, for example [20, 21, 22, 5].

R_n denotes the number of distinct reduced latin squares of order n.

$R_{k,n}$ denotes the number of distinct reduced k x n Latin rectangles.

Theorem 5.

$$L_{2,n} = n!(n!^{2'}) \tag{22}$$

Proof. If we define $L_{2,n}$ as the total number of distinct Latin rectangles with 2 rows and n columns such that $n \geq 1$. Then the total number of arrangement(s) of row 1 $= n!$; hence, for every single arrangement of row 1, there are $n!^{2'}$ arrangement(s) of row 2 under row 1. Therefore, total number of arrangement(s) of row 2 under the total number of arrangements of row $1 = n!(n!^{2'})$ $\square$

Example 9. $L_{2,2} = 2!(2!^{2'}) = 2(1) = 2$

Example 10. $L_{2,3} = 3!(3!^{2'}) = 6(2) = 12$

Example 11. $L_{2,4} = 4!(4!^{2'}) = 24(9) = 216$

Example 12. $L_{2,12} = 12!(12!^{2'}) = 479001600(176214841) = 84407190782745600$

Here we see for the first time the value of $L_{2,12}$ that has not been counted exhaustively using a computer.

Remark 2. *Clearly this relationship solves all cases of $L_{2,n}$. From here we can see that to solve all cases $L_{3,n}$ it is essential to find $n!^{3'}$. This is an area of further research.*

6.1 Reduced size of all (2,n) Latin rectangles ($R_{2,n}$)

Theorem 6.

$$R_{2,n} = \frac{n!^{2'}}{n-1} \tag{23}$$

Proof. From Bassey triangle, we observe that:

$$R_{2,n} =^{n-1} E_1 \tag{24}$$

Therefore.
$R_{2,2} =^1 E_1$
$R_{2,3} =^2 E_1$
$R_{2,4} =^3 E_2$

Applying equation (24) to theorem 4, we obtain:

$$\frac{^nE_0}{^{n-1}E_1} = \frac{n!^{2'}}{R_{2,n}} = n-1 \tag{25}$$

Therefore.

$$R_{2,n} = \frac{n!^2}{n-1} \tag{26}$$

□

$$R_{2,2} = \frac{2!^{2'}}{2-1} = \frac{1}{1} = 1$$

$$R_{2,3} = \frac{3!^{2'}}{3-1} = \frac{2}{2} = 1$$

$$R_{2,4} = \frac{4!^{2'}}{4-1} = \frac{9}{3} = 3$$

$$R_{2,12} = \frac{12!^{2'}}{12-1} = \frac{176214841}{11} = 16019531$$

We can also deduce from equation (25) that:

$$n!^{2'} = (n-1)R_{2,n}$$

Therefore, from theorem 5:

$$L_{2,n} = n!(n-1)R_{2,n} \tag{27}$$

This formula is almost the same as the already established formula for L_n.

$$L_n = n!(n-1)!R_n \tag{28}$$

Notice that the two equations (27) and (28) are equal when $n = 2$
Therefore.

$$R_n = R_{2,n} for (n = 2) \tag{29}$$

7 Higher Order Permutations

We have so far been able to find the general term for second-order permutation, however other higher orders of permutation exist, like third-order, fourth-order etc. Below is a table showing values of higher order permutations obtained by counting manually.

$5! = 120$	$4! = 24$	$3! = 6$	$2! = 2$	$1! = 1$
$5!^{2'} = 44$	$4!^{2'} = 9$	$3!^{2'} = 2$	$2!^{2'} = 1$	
$5!^{3'} = 12, 13$	$4!^{3'} = 2, 4$	$3!^{3'} = 1$		
$5!^{4'} = 2, 4$	$4!^{4'} = 1$			
$5!^{5'} = 1$				

Fig 19: Table showing higher permutation values.

From the table, it can be observed that the value of the permutation of a number which is the same as the order of permutation is 1. This implies that the first-order permutation of 1 is 1, the second-order permutation of 2 is 1, the third-order permutation of 3 is 1 and so on.

Proposition 1.

$$n!^{n'} = 1 \tag{30}$$

Therefore.

$$1! = 1$$

$$2!^{2'} = 1$$

$$3!^{3'} = 1$$

$$4!^{4'} = 1$$

and so on.
The implication of proposition 1, can be seen in the following corollary.

Corollary 1. *The size of all complete Latin squares above the first order is equal to the size of the largest corresponding incomplete Latin square.*
This implies that:

$$L_n = L_{(n-1),n} \tag{31}$$

for $(n \geq 2)$

Therefore.

$$L_2 = L_{1,2}$$

$$L_3 = L_{2,3}$$

Proof. $L_n = L_{(n-1),n}(n!^{n'})$
But from equation (30) $n!^{n'} = 1$
Therefore.

$$L_n = L_{(n-1),n}$$

□

Example 13.

$$L_2 = L_{1,2}(2!^{2'})$$

$$L_2 = L_{1,2}$$

Example 14.

$$L_3 = L_{2,3}(3!^{3'})$$

$$L_3 = L_{2,3}$$

8 Result Summary

n	$n!^{2'}$	$R_{2,n}$	$L_{2,n}$
1	0	0	0
2	1	1	2
3	2	1	12
4	9	3	216
5	44	11	5280
6	265	53	190800
7	1854	309	9344160
8	14833	2119	598066560
9	133496	16687	48443028480
10	1334961	148329	4844306476800
11	14684570	1468457	586161043776000
12	176214841	16019531	84407190782745600
13	2290792932	190899411	14264815236056985600
14	32071101049	2467007773	2795903786354347468800
15	481066515734	34361893981	629078351928420506112000

Fig 20: Values of $n!^{2'}$, $R_{2,n}$, and $L_{2,n}$ up to $n = 15$

9 Conclusion

This study makes a strong case for the existence of second-order permutation and its corresponding second-order permutation factorial ($n!^{2'}$). For the first time, we now know the general expression for $n!^{2'}$ and can now apply it to solve second-order permutation problems in real life.
The relationship between the second-order permutation factorial ($n!^{2'}$) and what should now be known as first-order permutation factorial (n!) has been clearly elaborated using a revolutionary mathematical triangle known as Bassey triangle. With this triangle, we could clearly see the mathematical process involved in the transformation from $n!^{2'}$ to n! and vice versa.
However, further research work is required to find the general expression for other higher order permutation forms from third-order and above. This work has shown that Latin squares are actually complex permutation structures which will be better understood as advances are made on the study of higher order permutations.
Thanks to this study, we have now cracked the Latin square code for accurately computing the sizes of all (2,n) Latin rectangles for the first time in the history of mathematics, this would not have been possible without the

discovery of $n!^{2'}$ and it's general expression.
It is my hope that this work has provided sufficient evidence to prove the existence of $n!^{2'}$ and for it to be adopted worldwide by mathematics scholars and educators as a valid mathematical symbol and phenomenon.

10 Acknowledgment

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.
My profound gratitude goes to Professor Umana Itaketo who personally supervised this work. To Mr. G. O. Bassey who made out time to vet and offer useful advise along with Engr Etimbuk Bassey.

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