An information theoretic formulation of game theory, I

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October 19, 2018

Abstract

Within this paper, I combine ideas from information theory, topology and game theory, to develop a framework for the determination of optimal strategies within iterated cooperative games of incomplete information.

1 Foreword

In this paper I aim to provide a sensible formulation of Game Theory. I discuss normal games, then iterated games. Consequent to this, I intend in a later paper to look into meta games, with and without iteration.

2 Game Theory: foundation and approach

2.1 Preliminaries: A formulation of Game Theory

Let A_i, $i \in \{0, 1, 2, 3\}$ be CW complexes. Consider $a := (a_0, a_1, b_0, b_1) \subset (A_0, A_1, A_2, A_3) =: A$, with a_i CW complexes, as a *point*. This will be a point in $\Delta := \delta^4$, where δ is the space of triangulations of an Alexandrov space of maximal complexity (the universal Alexandrov space Ω).

Now, we are interested in functions $g : \Delta^4 \to R$ for each $a \in A$. This defines a *game*, where g is the payoff function.

Let now a_0, a_1 have the interpretation as being *coalitions of agents* (note as a special case these can be single agents, or null). Furthermore, let b_0, b_1 have the interpretation as being choices for each coalition in turn (presumably defined over use of abstracted resources in the game; i.e., b_0, b_1 can be viewed as 'arrows between resources', rather like functions can be viewed as arrows between points (in category theory)).

Example. (Special Case). A_0, A_1 are points, and A_2 and A_3 are each sets of two disjoint points $\{0, 1\}$. Let g be the matrix that maps $(0, 0)$ to 1, $(0, 1)$ to 2, $(1, 0)$ to 0, and $(1, 1)$ to 3. Then this game is clearly equivalent to the prisoner's dilemma (the two disjoint points 1 and 0 can be viewed as 'cooperate' or 'defect').

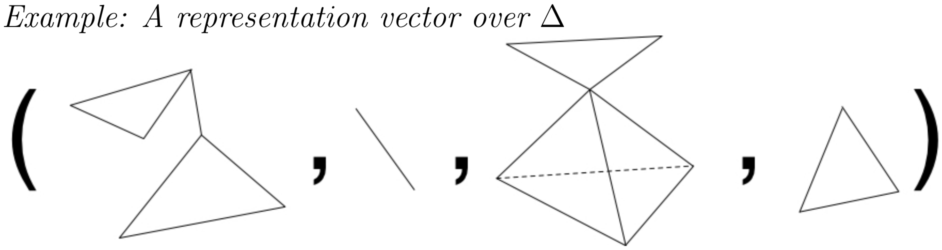
This however seems a little bit of an unsatisfying interpretation - ideally we would like to construct an analogy with three dimensions of space and one of time, for instance. Suppose instead then that a_0, a_1, a_2 are coalitions of agents, and a_3 is an *accord* between these three coalitions. What would this mean, and how would this reduce to the Prisoner's dilemma above?

The interpretation here would be that for (a_0, a_1, a_2) a decision a_3 has been made that is binding on all coalitions. Then $g(a)$ determines the payoff.

But this is still not entirely satisfactory. We ideally want g to take as a parameter a coalition of players, and determine what the outcome should be as a consequence of one (or more) decisions.

Suppose then we have the following: a_0 is our coalition, and a_1 is the decision as to how to compete with a second coalition, a_2 is the decision as to how to compete with a third coalition, and a_3 is the decision as to how to allow for the second coalition competing with the third.

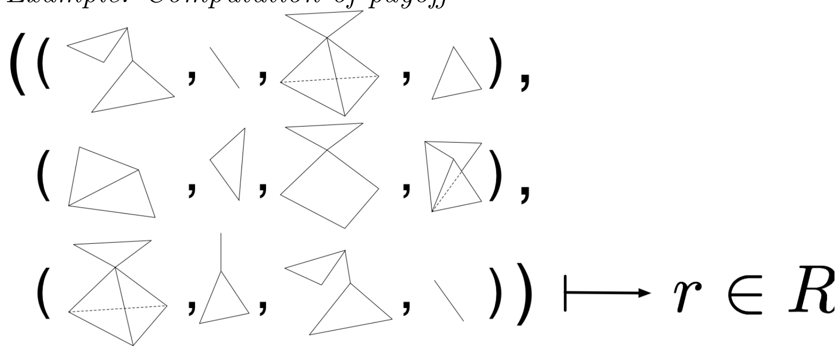
Example: A representation vector over Δ



Perhaps then we are interested in a payoff function that takes three coalitions as parameters: a_0, b_0, c_0, and computes $g(a, b, c)$, where b_1, b_2, b_3 are defined in an analogous way to the a_i; similarly for the c_i; $a = (a_0, a_1, a_2, a_3), b = (b_0, b_1, b_2, b_3), c = (c_0, c_1, c_2, c_3)$.

So, we are consequently interested in functions $g : \Delta \times \Delta \times \Delta \to R$. We will recast this as our definition of a *game*, where g is the payoff function.

Example: Computation of payoff



Then, with this change of structure, we can reformulate the Prisoner's dilemma in the following way:

Example. (Special Case revisited). We only have two players, so we can treat c as null. Since c is null, we need consider only $a = (a_0, a_1)$ and $b = (b_0, b_1)$. a_0 and b_0 are points, so we can reduce further; a and b then become the decisions a_1 and b_1, which can each be either cooperate or defect. Then $g(a_1, b_1)$ takes four possible values, which is clearly equivalent to the Prisoner's dilemma.

For a game, we define an *optimal decision* as being the choice that gives maximal payoff for a coalition of agents, assuming that every other coalition is also striving for an optimal decision. The choice reached in this way is the *Nash equilibrium.*

We are now ready to pose our first question:

Question 2.1. *(Making an optimal decision in a game). Given a (cooperative) game, what is the optimal decision for a coalition of agents to make?*

Remark. Note that this should be in line with intuition regarding the nature of finding a Nash equilibrium.

To make progress on this, we need to introduce the idea of a *decision metric.*

Definition 1. (Decision metric). For a game g, a decision metric is a map $\sigma : T\Delta^3 \times T\Delta^3 \to R$, where $T\Delta^3$ is the Lie Algebra associated to Δ^3, if Δ^3 is viewed as a Lie Group with the natural multiplication $a \star b := a \cap b$, where intersection is by each and every coordinate component. Multiplication on $T\Delta^3$ then is given by $v \star w := v \cap \partial w$, where ∂ is the boundary operator applied to the CW complex vector w, and we moreover have a relation (the Baker Campbell Hausdorff formula)

$$exp(v) \star_\Delta exp(w) \approx v + w + \partial[v, w]_\Delta$$

where $[v, w] := v \cap \partial w - w \cap \partial v$ is the Lie bracket for $T\Delta^3$.

We can then construct an information in terms of this decision metric for our game:

$$I(\sigma) := \int_{\Delta^3} \int_{\lambda} (\partial ln \circ f)^2 f d\rho dX$$

where $f(X, \rho) = \delta(\sigma(X) - \rho)$.

Remark. (Equivalence of the information and the payoff function). Note that I can just be viewed as a payoff function g.

Taking the first variation $\delta I(\sigma) = 0$ gives us a relation via the Cramer-Rao inequality of "$R(\sigma) = 0$", where

$$R = \sigma_{ij} \Gamma^l_{ik} \Gamma^k_{jl}$$

with

$$\Gamma^k_{ij} := \langle \partial_k v_i, v_j \rangle$$

where ∂_k is the simplicial boundary operator with respect to the kth component of the CW complex vector v.

Then, we can solve for σ, which then becomes our "optimal decision metric".

Consequently, optimal decisions over our game g then lie along geodesics of σ, which answers our first question.

2.2 A step further: Iterated games

Interestingly, we have only a functional over what, for smooth geometry, would involve a meta-functional over a set of functionals (see for instance [7]). So this raises a separate question - if a sideways step in abstraction involved simplifying: to go from smooth geometry to discrete geometry - and then abstracting: to go from numbers as coordinate components to jump to CW complexes as coordinate components - as to what a second sideways step might look like.

We would expect indeed a second sideways step to exist, and be characterised as a structure that has upon it defined a function $h : \mathcal{G}^5 \to R$, where $\mathcal{G}$ is our structure.

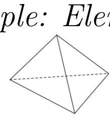
Suppose $\mathcal{G}$ represents a triangulation of the space of functions f which map from Δ to Δ, written equivalently as $\Delta^{(1)}$. But then we have the correct ingredients for the concept of an iterated game.

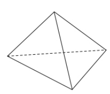
But what is an iterated game? Intuitively, an iterated game is a game that runs over many iterations. We would expect the nature of such a game, then, to be more complex than a normal game.

Definition 2. (Iterated game). Let $\Delta^{(1)}$ be a natural triangulation of $\{f | f : \Delta \to \Delta\}$ (the set of functions mapping Δ to itself). Then an iterated game is a map $h : \Delta^{(1)} \times \cdots \times \Delta^{(1)} \to R$, where 5 copies of $\Delta^{(1)}$ are taken.

Remark. To quantify this slightly, let $(v_0, \cdots, v_n)$ be a chain in Δ. Then f could be thought of most simply as a map that extends from chains, in the way of $f : (v_0, \cdots, v_n) \mapsto (w_0, \cdots, w_m)$. In this way, $\Delta^{(1)}$ is generated by elements f_{nm} that map from n-simplices to m-simplices. Then a simplex in $\Delta^{(1)}$ can be viewed as some element $(f_{n_0 m_0}, \cdots, f_{n_k m_l})$, i.e. a vector of size $k \times l$ where $n_k, m_l \in N$.

Example: Element in $\Delta^{(1)}$

f_{32} :  $\mapsto$ 

f_{23} :  $\mapsto$ 

f_{22} :  $\mapsto$ 

$g = (f_{22}, f_{32}, f_{23}) \in \Delta^{(1)}$

(*We would have five elements like this of various lengths if we were interested in mapping to a number in the reals for our iterated game.*)

For an iterated game, we define a *strategy* as an approach to making decisions (that could, for instance, either reward or punish other coalitions playing), that is formulated with the goal in mind of achieving the best long term payoff (although it may not achieve this, and generically will not if it is sub-optimal). Strategies for iterated games tend to be different than decisions of one-off games.

We are now ready to pose our second question:

Question 2.2. *(Finding optimal strategies). Given an iterated game, what is the optimal strategy for a coalition of agents to take in order to make the best long term payoff?*

One might wonder if one would need to use nested functionals to answer this question. However, such iteration should only be necessary if we have a sequence of embedded games - a special case of which is if we have a metagame with one or more child games (say perhaps four games subsidiary to the metagame).

So, how can we compute an optimal strategy given an iterated game h?

Define a three tensor $\tau : T\Delta^{(1)5} \times T\Delta^{(1)5} \times T\Delta^{(1)5} \to R$. Define a cybernetic information $I(\tau)$ in the usual way. Again, this can be viewed as our payoff h. Then by the Cramer-Rao inequality, $I(\tau) \geq 0$. Setting the first variation of $I(\tau)$ to zero leads us to determine a relation for τ:

$$R = \tau_{ijk}\Gamma_{iabcd}\Gamma_{jcdef}\Gamma_{kefab} = 0$$

with

$$\Gamma_{ijabc} := \langle \nabla_{ij} v_a, v_b, v_c \rangle$$

where $v_a, v_b, v_c \in T\Delta^{(1)5}$, and ∇_{ij} is the induced simplicial boundary operator on $T\Delta^{(1)}$ with respect to the $i-j$th component of $\Delta^{(1)5}$ (for $f : w_i \mapsto w_j$, $w_i, w_j \in \Delta^{(0)3}$).

Then optimal strategies for coalitions of players in h are geodesics with respect to τ, answering the question above - ie, they are surfaces S in $\Delta^{(1)5}$ such that $\nabla_{TS} S = 0$, for TS to represent the tangent space to S in $\Delta^{(1)5}$.

Remark. (Neural networks). In a neural network with layers L_i, $i \in \{0, \cdots, N\}$, each with nodes N_{ij}, $j \in \{0, \cdots M\}$, we have transition weights w_{ijk} from N_{ij} to $N_{i+1,k}$. If we simplify the structure of our iterated game to merely be a grid of 0-cells, and further simplify the allowed possible connections between them, then τ could actually be viewed as a representation of the weight tensor w. In this way, Neural Networks are a special case of the theory above.

2.3 An application: The cybernetic triumvirate of problems in machine learning

One might ask three questions in machine learning:

- What is the game, given that we have data concerning inputs into it, and outputs from it? (observation),
- What is the optimal strategy, given that we know the game? (decision), and finally
- What is the effect of choices of strategy, whether optimal or suboptimal, by all coalitions of players, on the shape of the game? (anticipation)

In the considerations so far, we have focused on the second question. But what about the first and the third?

Note that this seems to also be characteristic of decision making in an iterated game; so perhaps we have already solved this problem? In particular, note that we can define our game from the metric, so we have answered questions one and two; certainly if the metric is suboptimal then the payoff will be nonzero.

And, in an iterated game, from before we have that geodesics are surfaces in $\Delta^{(1)5}$, with each iterate being a path in that surface. So it is clear then that we have solved for the third question as well, in a manner of speaking.

Turning to application of the theory developed, it seems to me that programming a computer to implement a solution to the previous section is indeed a solution to this three fold question for single instances of a one-off game that are chained together, and would necessitate either implicitly or explicitly computing solutions to the above three machine learning questions.

In particular, we can formulate the following chain of definitions:

Definition 3. (Intelligence). *Intelligence* is the capacity to make decisions under uncertainty; i.e., for any game, given limited information as to the nature of said game, who the players are, and what their actions might be - then intelligence is the capacity to make decisions and formulate strategies within such a situation.

In particular, a system displays *intelligence* if it can formulate an approach to making decisions in an iterated game under uncertainty, i.e. incomplete information as to the strategies of other players.

And one system, system A, is *more intelligent* than another, system B within a particular game $\mathcal{G}$, if it can formulate more effective strategies to achieve a better payoff within $\mathcal{G}$ when competing against that other system, amongst potentially additional observants.

If this holds for the majority of games that these systems can play with respect to some suitable measure μ on the space of games, then we say that system A is *smarter* or *more capable* than system B with respect to measure μ.

3 Further Work

In the next paper I will discuss the theory of meta games, with and without iteration.

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