The Hyperbolic Menelaus Theorem in The Poincaré Disc Model of Hyperbolic Geometry

Abstract. In this note, we present the hyperbolic Menelaus theorem in the Poincaré disc of hyperbolic geometry.

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1. Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-euclidian geometry. Menelaus of Alexandria was a Greek mathematician and astronomer, the first to recognize geodesics on a curved surface as natural analogs of straight lines. Here, in this study, we present a proof of Menelaus's theorem in the Poincaré disc model of hyperbolic geometry. The well-known Menelaus theorem states that if l is a line not through any vertex of a triangle ABC such that l meets BC in D, CA in E, and AB in F, then $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$ [1]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by A. Johnson [2], N. A. Court [3], C. Coşniţă [4], A. Ungar [5].

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a,b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot gyr[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$ for all points $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u},\mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u},\mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a},\mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Theorem 1. ***(The law of gyrosines in Möbius gyrovector spaces).*** *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and with gyroangles α, β, and γ at the vertices A, B, and C. Then $\frac{a_\gamma}{\sin\alpha} = \frac{b_\gamma}{\sin\beta} = \frac{c_\gamma}{\sin\gamma}$, where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$* [6, *p. 267*].

Definition 2. *The hyperbolic distance function in D is defined by the equation*

$$d(a,b) = |a \ominus b| = \left|\frac{a-b}{1-\overline{a}b}\right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

For further details we refer to the recent book of A.Ungar [5].

2. Main results

In this section we prove the Menelaus's theorem in the Poincaré disc model of hyperbolic geometry.

Theorem 3. (***The Menelaus's Theorem for Hyperbolic Gyrotriangle***) *If l is an gyroline not through any vertex of an gyrotriangle ABC such that l meets BC in D, CA in E, and AB in F, then*

$$\frac{(AF)_\gamma}{(BF)_\gamma}\cdot\frac{(BD)_\gamma}{(CD)_\gamma}\cdot\frac{(CE)_\gamma}{(AE)_\gamma} = 1.$$

Proof. In function of the position of the gyroline l intersect internally a side of ABC triangle and the other two externally (See Figure 1), or the line l intersect all three sides externally (See Figure 2).

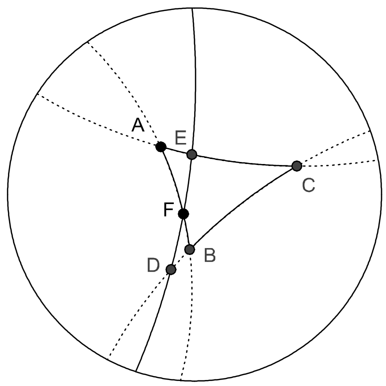

Figure 1

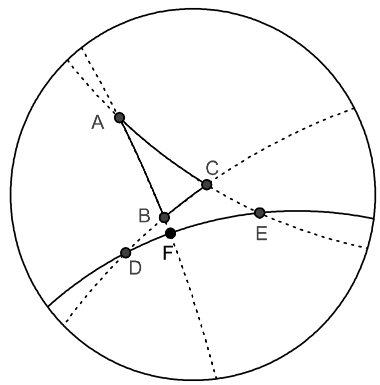

Figure 2

If we consider the first case, the law of gyrosines (See Theorem 1), gives for the gyrotriangles AEF, BFD, and CDE, respectively

$$\frac{(AE)_\gamma}{(AF)_\gamma} = \frac{\sin \widehat{AFE}}{\sin \widehat{AEF}}, \tag{1}$$

$$\frac{(BF)_\gamma}{(BD)_\gamma} = \frac{\sin \widehat{FDB}}{\sin \widehat{DFB}}, \tag{2}$$

and

$$\frac{(CD)_\gamma}{(CE)_\gamma} = \frac{\sin \widehat{DEC}}{\sin \widehat{EDC}}, \tag{3}$$

where $\sin \widehat{AFE} = \sin \widehat{DFB}$, $\sin \widehat{EDC} = \sin \widehat{FDB}$, and $\sin \widehat{AEF} = \sin \widehat{DEC}$, since gyroangles $\widehat{AEF}$ and $\widehat{DEC}$ are suplementary. Hence, by (1), (2) and (3), we have

$$\frac{(AE)_\gamma}{(AF)_\gamma} \cdot \frac{(BF)_\gamma}{(BD)_\gamma} \cdot \frac{(CD)_\gamma}{(CE)_\gamma} = \frac{\sin \widehat{AFE}}{\sin \widehat{AEF}} \cdot \frac{\sin \widehat{FDB}}{\sin \widehat{DFB}} \cdot \frac{\sin \widehat{DEC}}{\sin \widehat{EDC}} = 1, \tag{4}$$

the conclusion follows. The second case is treated similar to the first. ■

Naturally, one may wonder whether the converse of the Menelaus theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 4. (***Converse of Menelaus's Theorem for Hyperbolic Gyrotriangle***) *If D lies on the gyroline BC, E on CA, and F on AB such that*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1, \tag{5}$$

then D, E, and F are collinear.

Proof. Relabelling if necessary, we may assume that the gyropoint D lies beyond B on BC. If E lies between C and A, then the gyroline ED cuts the gyroside AB, at F' say.

Applying Menelaus's theorem to the gyrotriangle ABC and the gyroline $E - F' - D$, we get

$$\frac{(AF')_\gamma}{(BF')_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1 \tag{6}$$

From (5) and (6), we get $\frac{(AF)_\gamma}{(BF)_\gamma} = \frac{(AF')_\gamma}{(BF')_\gamma}$. This equation holds for $F = F'$. Indeed, if we take $x := |\ominus A \oplus F'|$ and $c := |\ominus A \oplus B|$, then we get $c \ominus x = |\ominus F' \oplus B|$. For $x \in (-1, 1)$ define

$$f(x) = \frac{x}{1 - x^2} : \frac{c \ominus x}{1 - (c \ominus x)^2}. \tag{7}$$

Because $c \ominus x = \frac{c-x}{1-cx}$, then $f(x) = \frac{x(1-c^2)}{(c-x)(1-cx)}$. Since the following equality holds

$$f(x) - f(y) = \frac{c(1 - c^2)(1 - xy)}{(c - x)(1 - cx)(c - y)(1 - cy)}(x - y), \tag{8}$$

we get $f(x)$ is an injective function and this implies $F = F'$, so D, E, F are collinear.

There are still two possible cases. The first is if we suppose that the gyropoint F lies on the gyroside AB, then the gyrolines DF cuts the gyrosegment AC in the gyropoint E'. The second possibility is that E is not on the gyroside AC, E lies beyond C. Then DE cuts the gyroline AB in the gyropoint F'. In each case a similar application of Menelaus gives the result. ■

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