# TOPOLOGICAL GEOMETRODYNAMICS: PHYSICS AS INFINITE-DIMENSIONAL GEOMETRY

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# Preface

This book belongs to a series of online books summarizing the recent state Topological Geometrodynamics (TGD) and its applications. TGD can be regarded as a unified theory of fundamental interactions but is not the kind of unified theory as so called GUTs constructed by graduate students at seventies and eighties using detailed recipes for how to reduce everything to group theory. Nowadays this activity has been completely computerized and it probably takes only a few hours to print out the predictions of this kind of unified theory as an article in the desired format. TGD is something different and I am not ashamed to confess that I have devoted the last 32 years of my life to this enterprise and am still unable to write The Rules.

I got the basic idea of Topological Geometrodynamics (TGD) during autumn 1978, perhaps it was October. What I realized was that the representability of physical space-times as 4-dimensional surfaces of some higher-dimensional space-time obtained by replacing the points of Minkowski space with some very small compact internal space could resolve the conceptual difficulties of general relativity related to the definition of the notion of energy. This belief was too optimistic and only with the advent of what I call zero energy ontology the understanding of the notion of Poincare invariance has become satisfactory.

It soon became clear that the approach leads to a generalization of the notion of space-time with particles being represented by space-time surfaces with finite size so that TGD could be also seen as a generalization of the string model. Much later it became clear that this generalization is consistent with conformal invariance only if space-time is 4-dimensional and the Minkowski space factor of imbedding space is 4-dimensional.

It took some time to discover that also the geometrization of also gauge interactions and elementary particle quantum numbers could be possible in this framework: it took two years to find the unique internal space providing this geometrization involving also the realization that family replication phenomenon for fermions has a natural topological explanation in TGD framework and that the symmetries of the standard model symmetries are much more profound than pragmatic TOE builders have believed them to be. If TGD is correct, main stream particle physics chose the wrong track leading to the recent deep crisis when people decided that quarks and leptons belong to same multiplet of the gauge group implying instability of proton.

There have been also longstanding problems.

- Gravitational energy is well-defined in cosmological models but is not conserved. Hence the conservation of the inertial energy does not seem to be consistent with the Equivalence Principle. Furthermore, the imbeddings of Robertson-Walker cosmologies turned out to be vacuum extremals with respect to the inertial energy. About 25 years was needed to realize that the sign of the inertial energy can be also negative and in cosmological scales the density of inertial energy vanishes: physically acceptable universes are creatable from vacuum. Eventually this led to the notion of zero energy ontology which deviates dramatically from the standard ontology being however consistent with the crossing symmetry of quantum field theories. In this framework the quantum numbers are assigned with zero energy states located at the boundaries of so called causal diamonds defined as intersections of future and past directed light-cones. The notion of energy-momentum becomes length scale dependent since one has a scale hierarchy for causal diamonds. This allows to understand the non-conservation of energy as apparent. Equivalence Principle generalizes and has a formulation in terms of coset representations of Super-Virasoro algebras providing also a justification for p-adic thermodynamics.

- From the beginning it was clear that the theory predicts the presence of long ranged classical electro-weak and color gauge fields and that these fields necessarily accompany classical electromagnetic fields. It took about 26 years to gain the maturity to admit the obvious: these fields are classical correlates for long range color and weak interactions assignable to dark matter. The only possible conclusion is that TGD physics is a fractal consisting of an entire hierarchy of fractal copies of standard model physics. Also the understanding of electro-weak massivation and screening of weak charges has been a long standing problem, and 32 years was needed to discover that what I call weak form of electric-magnetic duality gives a satisfactory solution of the problem and provides also surprisingly powerful insights to the mathematical structure of quantum TGD.

I started the serious attempts to construct quantum TGD after my thesis around 1982. The original optimistic hope was that path integral formalism or canonical quantization might be enough to construct the quantum theory but the first discovery made already during first year of TGD was that these formalisms might be useless due to the extreme non-linearity and enormous vacuum degeneracy of the theory. This turned out to be the case.

- It took some years to discover that the only working approach is based on the generalization of Einstein's program. Quantum physics involves the geometrization of the infinite-dimensional "world of classical worlds" (WCW) identified as 3-dimensional surfaces. Still few years had to pass before I understood that general coordinate invariance leads to a more or less unique solution of the problem and implies that space-time surfaces are analogous to Bohr orbits. Still a coupled of years and I discovered that quantum states of the Universe can be identified as classical spinor fields in WCW. Only quantum jump remains the genuinely quantal aspect of quantum physics.

- During these years TGD led to a rather profound generalization of the space-time concept. Quite general properties of the theory led to the notion of many-sheeted space-time with sheets representing physical subsystems of various sizes. At the beginning of 90s I became dimly aware of the importance of p-adic number fields and soon ended up with the idea that p-adic thermodynamics for a conformally invariant system allows to understand elementary particle massivation with amazingly few input assumptions. The attempts to understand p-adicity from basic principles led gradually to the vision about physics as a generalized number theory as an approach complementary to the physics as an infinite-dimensional spinor geometry of WCW approach. One of its elements was a generalization of the number concept obtained by fusing real numbers and various p-adic numbers along common rationals. The number theoretical trinity involves besides p-adic number fields also quaternions and octonions and the notion of infinite prime.

- TGD inspired theory of consciousness entered the scheme after 1995 as I started to write a book about consciousness. Gradually it became difficult to say where physics ends and consciousness theory begins since consciousness theory could be seen as a generalization of quantum measurement theory by identifying quantum jump as a moment of consciousness and by replacing the observer with the notion of self identified as a system which is conscious as long as it can avoid entanglement with environment. "Everything is conscious and consciousness can be only lost" summarizes the basic philosophy neatly. The idea about p-adic physics as physics of cognition and intentionality emerged also rather naturally and implies perhaps the most dramatic generalization of the space-time concept in which most points of p-adic space-time sheets are infinite in real sense and the projection to the real imbedding space consists of discrete set of points. One of the most fascinating outcomes was the observation that the entropy based on p-adic norm can be negative. This observation led to the vision that life can be regarded as something in the intersection of real and p-adic worlds. Negentropic entanglement has interpretation as a correlate for various positively colored aspects of conscious experience and means also the possibility of strongly correlated states stable under state function reduction and different from the conventional bound states and perhaps playing key role in the energy metabolism of living matter.

- One of the latest threads in the evolution of ideas is only slightly more than six years old. Learning about the paper of Laurent Nottale about the possibility to identify planetary orbits as Bohr orbits with a gigantic value of gravitational Planck constant made once again possible to see the obvious. Dynamical quantized Planck constant is strongly suggested by quantum classical correspondence and the fact that space-time sheets identifiable as quantum coherence regions can have arbitrarily large sizes. During summer 2010 several new insights about the mathematical structure and interpretation of TGD emerged. One of these insights was the realization that the postulated hierarchy of Planck constants might follow from the basic structure of quantum TGD. The point is that due to the extreme non-linearity of the classical action principle the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is one-to-many and the natural description of the situation is in terms of local singular covering spaces of the imbedding space. One could speak about effective value of Planck constant coming as a multiple of its minimal value. The implications of the hierarchy of Planck constants are extremely far reaching so that the significance of the reduction of this hierarchy to the basic mathematical structure distinguishing between TGD and competing theories cannot be under-estimated.

From the point of view of particle physics the ultimate goal is of course a practical construction recipe for the S-matrix of the theory. I have myself regarded this dream as quite too ambitious taking into account how far reaching re-structuring and generalization of the basic mathematical structure of quantum physics is required. It has indeed turned out that the dream about explicit formula is unrealistic before one has understood what happens in quantum jump. Symmetries and general physical principles have turned out to be the proper guide line here. To give some impressions about what is required some highlights are in order.

- With the emergence of zero energy ontology the notion of S-matrix was replaced with M-matrix which can be interpreted as a complex square root of density matrix representable as a diagonal and positive square root of density matrix and unitary S-matrix so that quantum theory in zero energy ontology can be said to define a square root of thermodynamics at least formally.

- A decisive step was the strengthening of the General Coordinate Invariance to the requirement that the formulations of the theory in terms of light-like 3-surfaces identified as 3-surfaces at which the induced metric of space-time surfaces changes its signature and in terms of space-like 3-surfaces are equivalent. This means effective 2-dimensionality in the sense that partonic 2-surfaces defined as intersections of these two kinds of surfaces plus 4-D tangent space data at partonic 2-surfaces code for the physics. Quantum classical correspondence requires the coding of the quantum numbers characterizing quantum states assigned to the partonic 2-surfaces to the geometry of space-time surface. This is achieved by adding to the modified Dirac action a measurement interaction term assigned with light-like 3-surfaces.

- The replacement of strings with light-like 3-surfaces equivalent to space-like 3-surfaces means enormous generalization of the super conformal symmetries of string models. A further generalization of these symmetries to non-local Yangian symmetries generalizing the recently discovered Yangian symmetry of $\mathcal{N} = 4$ supersymmetric Yang-Mills theories is highly suggestive. Here the replacement of point like particles with partonic 2-surfaces means the replacement of conformal symmetry of Minkowski space with infinite-dimensional super-conformal algebras. Yangian symmetry provides also a further refinement to the notion of conserved quantum numbers allowing to define them for bound states using non-local energy conserved currents.

- A further attractive idea is that quantum TGD reduces to almost topological quantum field theory. This is possible if the Kähler action for the preferred extremals defining WCW Kähler function reduces to a 3-D boundary term. This takes place if the conserved currents are so called Beltrami fields with the defining property that the coordinates associated with flow lines extend to single global coordinate variable. This ansatz together with the weak form of electric-magnetic duality reduces the Kähler action to Chern-Simons term with the condition that the 3-surfaces are extremals of Chern-Simons action subject to the constraint force defined by the weak form of electric magnetic duality. It is the latter constraint which prevents the trivialization of the theory to a topological quantum field theory. Also the identification of the Kähler function of WCW as Dirac determinant finds support as well as the description of the scattering amplitudes in terms of braids with interpretation in terms of finite measurement resolution coded to the basic structure of the solutions of field equations.

- In standard QFT Feynman diagrams provide the description of scattering amplitudes. The beauty of Feynman diagrams is that they realize unitarity automatically via the so called Cutkosky rules. In contrast to Feynman's original beliefs, Feynman diagrams and virtual particles are taken only as a convenient mathematical tool in quantum field theories. QFT approach is however plagued by UV and IR divergences and one must keep mind open for the possibility that a genuine progress might mean opening of the black box of the virtual particle.

  In TGD framework this generalization of Feynman diagrams indeed emerges unavoidably. Light-like 3-surfaces replace the lines of Feynman diagrams and vertices are replaced by 2-D partonic

2-surfaces. Zero energy ontology and the interpretation of parton orbits as light-like "wormhole throats" suggests that virtual particle do not differ from on mass shell particles only in that the four- and three- momenta of wormhole throats fail to be parallel. The two throats of the wormhole defining virtual particle would contact carry on mass shell quantum numbers but for virtual particles the four-momenta need not be parallel and can also have opposite signs of energy. Modified Dirac equation suggests a number theoretical quantization of the masses of the virtual particles. The kinematic constraints on the virtual momenta are extremely restrictive and reduce the dimension of the sub-space of virtual momenta and if massless particles are not allowed (IR cutoff provided by zero energy ontology naturally), the number of Feynman diagrams contributing to a particular kind of scattering amplitude is finite and manifestly UV and IR finite and satisfies unitarity constraint in terms of Cutkosky rules. What is remarkable that fermionic propagatos are massless propagators but for on mass shell four-momenta. This gives a connection with the twistor approach and inspires the generalization of the Yangian symmetry to infinite-dimensional super-conformal algebras.

What I have said above is strongly biased view about the recent situation in quantum TGD and I have left all about applications to the introductions of the books whose purpose is to provide a bird's eye of view about TGD as it is now. This vision is single man's view and doomed to contain unrealistic elements as I know from experience. My dream is that young critical readers could take this vision seriously enough to try to demonstrate that some of its basic premises are wrong or to develop an alternative based on these or better premises. I must be however honest and tell that 32 years of TGD is a really vast bundle of thoughts and quite a challenge for anyone who is not able to cheat himself by taking the attitude of a blind believer or a light-hearted debunker trusting on the power of easy rhetoric tricks.

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Neither TGD nor these books would exist without the help and encouragement of many people. The friendship with Heikki and Raija Haila and their family have been kept me in contact with the everyday world and without this friendship I would not have survived through these lonely 32 years most of which I have remained unemployed as a scientific dissident. I am happy that my children have understood my difficult position and like my friends have believed that what I am doing is something valuable although I have not received any official recognition for it.

During last decade Tapio Tammi has helped me quite concretely by providing the necessary computer facilities and being one of the few persons in Finland with whom to discuss about my work. I have had also stimulating discussions with Samuli Penttinen who has also helped to get through the economical situations in which there seemed to be no hope. The continual updating of fifteen online books means quite a heavy bureaucracy at the level of bits and without a systemization one ends up with endless copying and pasting and internal consistency is soon lost. Pekka Rapinoja has offered his help in this respect and I am especially grateful for him for my Python skills. Also Matti Vallinkoski has helped me in computer related problems.

The collaboration with Lian Sidorov was extremely fruitful and she also helped me to survive economically through the hardest years. The participation to CASYS conferences in Liege has been an important window to the academic world and I am grateful for Daniel Dubois and Peter Marcer for making this participation possible. The discussions and collaboration with Eduardo de Luna and Istvan Dienes stimulated the hope that the communication of new vision might not be a mission impossible after all. Also blog discussions have been very useful. During these years I have received innumerable email contacts from people around the world. In particualr, I am grateful for Mark McWilliams and Ulla Matfolk for providing links to possibly interesting web sites and articles. These contacts have helped me to avoid the depressive feeling of being some kind of Don Quixote of Science and helped me to widen my views: I am grateful for all these people.

In the situation in which the conventional scientific communication channels are strictly closed it is important to have some loop hole through which the information about the work done can at

least in principle leak to the publicity through the iron wall of the academic censorship. Without any exaggeration I can say that without the world wide web I would not have survived as a scientist nor as individual. Homepage and blog are however not enough since only the formally published result is a result in recent day science. Publishing is however impossible without a direct support from power holders- even in archives like arXiv.org.

Situation changed for five years ago as Andrew Adamatsky proposed the writing of a book about TGD when I had already got used to the thought that my work would not be published during my life time. The Prespacetime Journal and two other journals related to quantum biology and consciousness - all of them founded by Huping Hu - have provided this kind of loop holes. In particular, Dainis Zeps, Phil Gibbs, and Arkadiusz Jadczyk deserve my gratitude for their kind help in the preparation of an article series about TGD catalyzing a considerable progress in the understanding of quantum TGD. Also the viXra archive founded by Phil Gibbs and its predecessor Archive Freedom have been of great help: Victor Christianto deserves special thanks for doing the hard work needed to run Archive Freedom. Also the Neuroquantology Journal founded by Sultan Tarlaci deserves a special mention for its publication policy. And last but not least: there are people who experience as a fascinating intellectual challenge to spoil the practical working conditions of a person working with something which might be called unified theory: I am grateful for the people who have helped me to survive through the virus attacks, an activity which has taken roughly one month per year during the last half decade and given a strong hue of grey to my hair.

For a person approaching his sixty year birthday it is somewhat easier to overcome the hard feelings due to the loss of academic human rights than for an inpatient youngster. Unfortunately the economic situation has become increasingly difficult during the twenty years after the economic depression in Finland which in practice meant that Finland ceased to be a constitutional state in the strong sense of the word. It became possible to depose people like me from the society without fear about public reactions and the classification as dropout became a convenient tool of ridicule to circumvent the ethical issues. During last few years when the right wing has held the political power this trend has been steadily strengthening. In this kind of situation the concrete help from individuals has been and will be of utmost importance. Against this background it becomes obvious that this kind of work is not possible without the support from outside and I apologize for not being able to mention all the people who have helped me during these years.

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# Introduction

## 1.1 Basic Ideas of TGD

The basic physical picture behind TGD was formed as a fusion of two rather disparate approaches: namely TGD is as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model.

### 1.1.1 Background

*T(opological) G(eometro)D(ynamics)* is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [K1]. The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last twenty-three years for the realization of this dream and this has resulted in seven online books about TGD and eight online books about TGD inspired theory of consciousness and of quantum biology.

Quantum T(opological) G(eometro)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness and of quantum biology have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

For few years ago the discussions with Tony Smith initiated a fourth thread which deserves the name 'TGD as a generalized number theory'. The basic observation was that classical number fields might allow a deeper formulation of quantum TGD. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and extremely fruitful revision of the basic views about what the final form and physical content of quantum TGD might be. Together with the vision about the fusion of p-adic and real physics to a larger coherent structure these sub-threads fused to the "physics as generalized number theory" th

A further thread emerged from the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and certainly possible in TGD framework. The identification of hierarchy of Planck constants whose values TGD "predicts" in terms of dark matter hierarchy would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

Every updating of the books makes me frustrated as I see how badly the structure of the representation reflects my bird's eye of view as it is at the moment of updating. At this time I realized that the chronology based identification of the threads is quite natural but not logical and it is much more logical to see p-adic physics, the ideas related to classical number fields, and infinite primes as sub-threads of a thread which might be called "physics as a generalized number theory". In the following I adopt this view. This reduces the number of threads to four! I am not even sure about the number of threads! Be patient!

TGD forces the generalization of physics to a quantum theory of consciousness, and represent TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations. The seven online books [K60, K43, K37, K34, K44, K51, K50] about TGD and eight online books about TGD inspired theory of consciousness and of quantum biology [K55, K7, K41, K6, K22, K27, K30, K49] are warmly recommended to the interested reader.

### 1.1.2 TGD as a Poincare invariant theory of gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_\times CP_2$, where $M^4$ denotes Minkowski space and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [A69, A47, A63, A45].

The identification of the space-time as a submanifold [A43, A67] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity.

It soon however turned out that submanifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of $CP_2$ explains electro-weak and color quantum numbers. The different H-chiralities of $H$-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the $CP_2$ spinor connection, Killing vector fields of $CP_2$ and of $H$-metric to four-surface define classical electro-weak, color gauge fields and metric in $X^4$.

### 1.1.3 TGD as a generalization of the hadronic string model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

This decay vertex does not however correspond to a direct generalization of trouser vertex of string models. Indeed, the important difference between TGD and string models is that the analogs of string world sheet diagrams do not describe particle decays but the propagation of particles via different routes. Particle reactions are described by generalized Feynman diagrams for which 3-D light-like surface describing particle propagating join along their ends at vertices. As 4-manifolds the space-time surfaces are therefore singular like Feynman diagrams as 1-manifolds.

### 1.1.4 Fusion of the two approaches via a generalization of the space-time concept

The problem is that the two approaches to TGD seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there could be "vapor phase" that is a "gas" of particle like 3-surfaces (counterpart of the "baby universies" of GRT) and the nonconservation of energy in GRT corresponds to the transfer of energy between the topological condensate and vapor phase.

What one obtains is what I have christened as many-sheeted space-time. One particular aspect is topological field quantization meaning that various classical fields assignable to a physical system correspond to space-time sheets representing the classical fields to that particular system. One can speak of the field body of a particular physical system. Field body consists of topological light rays, and electric and magnetic flux quanta. In Maxwell's theory system does not possess this kind of field identity. The notion of magnetic body is one of the key players in TGD inspired theory of consciousness and quantum biology.

This picture became more detailed with the advent of zero energy ontology (ZEO). The basic notion of ZEO is causal diamond ($CD$) identified as the Cartesian product of $CP_2$ and of the intersection of future and past directed light-cones and having scale coming as an integer multiple of $CP_2$ size is fundamental. $CD$s form a fractal hierarchy and zero energy states decompose to products of positive and negative energy parts assignable to the opposite boundaries of $CD$ defining the ends of the space-time surface. The counterpart of zero energy state in positive energy ontology is in terms of initial and final states of a physical event, say particle reaction.

General Coordinate Invariance allows to identify the basic dynamical objects as space-like 3-surfaces at the ends of space-time surface at boundaries of $CD$: this means that space-time surface is analogous to Bohr orbit. An alternative identification is as light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian and interpreted as lines of generalized Feynman diagrams. Also the Euclidian 4-D regions would have similar interpretation. The requirement that the two interpretations are equivalent, leads to a strong form of General Coordinate Invariance. The outcome is effective 2-dimensionality stating that the partonic 2-surfaces identified as intersections of the space-like ends of space-time surface and light-like wormhole throats are the fundamental objects. That only effective 2-dimensionality is in question is due to the effects caused by the failure of strict determinism of Kähler action. In finite length scale resolution these effects can be neglected below UV cutoff and above IR cutoff. One can also speak about strong form of holography.

There is a further generalization of the space-time concept inspired by p-adic physics forcing a generalization of the number concept through the fusion of real numbers and various p-adic number fields. Also the hierarchy of Planck constants forces a generalization of the notion of space-time.

A very concise manner to express how TGD differs from Special and General Relativities could be following. Relativity Principle (Poincare Invariance), General Coordinate Invariance, and Equivalence Principle remain true. What is new is the notion of sub-manifold geometry: this allows to realize Poincare Invariance and geometrize gravitation simultaneously. This notion also allows a geometrization of known fundamental interactions and is an essential element of all applications of TGD ranging from Planck length to cosmological scales. Sub-manifold geometry is also crucial in the applications of TGD to biology and consciousness theory.

The worst objection against TGD is the observation that all classical gauge fields are expressible in terms of four imbedding space coordinates only- essentially $CP_2$ coordinates. The linear superposition of classical gauge fields taking place independently for all gauge fields is lost. This would be a catastrophe without many-sheeted space-time. Instead of gauge fields, only the effects such as gauge forces are superposed. Particle topologically condenses to several space-time sheets simultaneously and experiences the sum of gauge forces. This transforms the weakness to extreme economy: in a typical unified theory the number of primary field variables is countered in hundreds if not thousands, now it is just four.

## 1.2 The threads in the development of quantum TGD

The development of TGD has involved several strongly interacting threads: physics as infinite-dimensional geometry; TGD as a generalized number theory, the hierarchy of Planck constants interpreted in terms of dark matter hierarchy, and TGD inspired theory of consciousness. In the following these threads are briefly described.

### 1.2.1 Quantum TGD as spinor geometry of World of Classical Worlds

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and have been the following ones:

1. Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude in the configuration space $CH$ consisting of all possible 3-surfaces in $H$. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included. Particle reactions are identified as topology changes [A59, A70, A71]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \rightarrow B+C$. Classically this corresponds to a path of configuration space leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

2. During years this naive and very rough vision has of course developed a lot and is not anymore quite equivalent with the original insight. In particular, the space-time correlates of Feynman graphs have emerged from theory as Euclidian space-time regions and the strong form of General Coordinate Invariance has led to a rather detailed and in many respects un-expected visions. This picture forces to give up the idea about smooth space-time surfaces and replace space-time surface with a generalization of Feynman diagram in which vertices represent the failure of manifold property. I have also startd introduced the word "world of classical worlds" (WCW) instead of rather formal "configuration space". I hope that "WCW" does not induce despair in the reader having tendency to think about the technicalities involved!

3. WCW is endowed with metric and spinor structure so that one can define various metric related differential operators, say Dirac operator, appearing in the field equations of the theory. The most ambitious dream is that zero energy states correspond to a complete solution basis for the Dirac operator of WCW so that this classical free field theory would dictate M-matrices which form orthonormal rows of what I call U-matrix. Given M-matrix in turn would decompose to a product of a hermitian density matrix and unitary S-matrix.

   M-matrix would define time-like entanglement coefficients between positive and negative energy parts of zero energy states (all net quantum numbers vanish for them) and can be regarded as a hermitian quare root of density matrix multiplied by a unitary S-matrix. Quantum theory would be in well-defined sense a square root of thermodynamics. The orthogonality and hermiticity of the complex square roots of density matrices commuting with $S$-matrix means that they span infinite-dimensional Lie algebra acting as symmetries of the S-matrix. Therefore quantum TGD would reduce to group theory in well-defined sense: its own symmetries would define the symmetries of the theory. In fact the Lie algebra of Hermitian M-matrices extends to Kac-Moody type algebra obtained by multiplying hermitian square roots of density matrices with powers of the S-matrix. Also the analog of Yangian algebra involving only non-negative powers of S-matrix is possible.

4. By quantum classical correspondence the construction of WCW spinor structure reduces to the second quantization of the induced spinor fields at space-time surface. The basic action is so called modified Dirac action in which gamma matrices are replaced with the modified gamma matrices defined as contractions of the canonical momentum currents with the imbedding space gamma matrices. In this manner one achieves super-conformal symmetry and conservation of fermionic currents among other things and consistent Dirac equation. This modified gamma matrices define as anticommutators effective metric, which might provide geometrization for some basic observables of condensed matter physics. The conjecture is that Dirac determinant for the modified Dirac action gives the exponent of Kähler action for a preferred extremal as vacuum functional so that one might talk about bosonic emergence in accordance with the prediction that the gauge bosons and graviton are expressible in terms of bound states of fermion and antifermion.

The evolution of these basic ideas has been rather slow but has gradually led to a rather beautiful vision. One of the key problems has been the definition of Kähler function. Kähler function is Kähler action for a preferred extremal assignable to a given 3-surface but what this preferred extremal is? The obvious first guess was as absolute minimum of Kähler action but could not be proven to be right or wrong. One big step in the progress was boosted by the idea that TGD should reduce to almost topological QFT in which braids wold replace 3-surfaces in finite measurement resolution, which could be inherent property of the theory itself and imply discretization at partonic 2-surfaces with discrete points carrying fermion number.

1. TGD as almost topological QFT vision suggests that Kähler action for preferred extremals reduces to Chern-Simons term assigned with space-like 3-surfaces at the ends of space-time (recall the notion of causal diamond ($CD$)) and with the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. Minkowskian and Euclidian regions would give at wormhole throats the same contribution apart from coefficients and in Minkowskian regions the $\sqrt{g_4}$ factor would be imaginary so that one would obtain sum of real term identifiable as Kähler function and imaginary term identifiable as the ordinary action giving rise to interference effects and stationary phase approximation central in both classical and quantum field theory. Imaginary contribution - the presence of which I realized only after 33 years of TGD - could also havetopological interpretation as a Morse function. On physical side the emergence of Euclidian space-time regions is something completely new and leads to a dramatic modification of the ideas about black hole interior.

2. The manner to achieve the reduction to Chern-Simons terms is simple. The vanishing of Coulombic contribution to Kähler action is required and is true for all known extremals if one makes a general ansatz about the form of classical conserved currents. The so called weak form of electric-magnetic duality defines a boundary condition reducing the resulting 3-D terms to Chern-Simons terms. In this manner almost topological QFT results. But only "almost" since the Lagrange multiplier term forcing electric-magnetic duality implies that Chern-Simons action for preferred extremals depends on metric.

3. A further quite recent hypothesis inspired by effective 2-dimensionality is that Chern-Simons terms reduce to a sum of two 2-dimensional terms. An imaginary term proportional to the total area of Minkowskian string world sheets and a real tem proportional to the total area of partonic 2-surfaces or equivalently strings world sheets in Euclidian space-time regions. Also the equality of the total areas of strings world sheets and partonic 2-surfaces is highly suggestive and would realize a duality between these two kinds of objects. String world sheets indeed emerge naturally for the proposed ansatz defining preferred extremals. Therefore Kähler action would have very stringy character apart from effects due to the failure of the strict determinism meaning that radiative corrections break the effective 2-dimensionality.

### 1.2.2 TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. The fourth thread deserves the name 'TGD as a generalized number theory'. It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of hyper-counterparts of classical number fields identified as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product, and the notion of infinite prime.

#### p-Adic TGD and fusion of real and p-adic physics to single coherent whole

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzless and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired 'Universe as Computer' vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

There were many interpretational and technical questions crying for a definite answer.

1. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *The Physics*? Should one perform p-adicization also at the level of the configuration space of 3-surfaces? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations.

2. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that clever inventions rather than real discoveries might be in question. Quite recently I however learned that the problem of making sense about p-adic integration has been for decades central problem in the frontier of mathematics and a lot of profound work has been done along same intuitive lines as I have proceeded in TGD framework. The basic idea is certainly the notion of algebraic continuation from the world of rationals belonging to the intersection of real world and various p-adic worlds.

Despite these frustrating uncertainties, the number of the applications of the poorly defined p-adic physics growed steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

The key challenge is to fuse various p-adic physics and real physics to single larger structures. This has inspired a proposal for a generalization of the notion of number field by fusing real numbers and various p-adic number fields and their extensions along rationals and possible common algebraic numbers. This leads to a generalization of the notions of imbedding space and space-time concept and one can speak about real and p-adic space-time sheets. The quantum dynamics should be such that it allows quantum transitions transforming space-time sheets belonging to different number fields to each other. The space-time sheets in the intersection of real and p-adic worlds are of special interest and the hypothesis is that living matter resides in this intersection. This leads to surprisingly detailed predictions and far reaching conjectures. For instance, the number theoretic generalization of entropy concept allows negentropic entanglement central for the applications to living matter.

The basic principle is number theoretic universality stating roughly that the physics in various number fields can be obtained as completion of rational number based physics to various number fields. Rational number based physics would in turn describe physics in finite measurement resolution and cognitive resolution. The notion of finite measurement resolution has become one of the basic principles of quantum TGD and leads to the notions of braids as representatives of 3-surfaces and inclusions of hyper-finite factors as a representation for finite measurement resolution.

### The role of classical number fields

The vision about the physical role of the classical number fields relies on the notion of number theoretic compactifiction stating that space-time surfaces can be regarded as surfaces of either $M^8$ or $M^4 \times CP_2$. As surfaces of $M^8$ identifiable as space of hyper-octonions they are hyper-quaternionic or co-hyper-quaternionic- and thus maximally associative or co-associative. This means that their tangent space is either hyper-quaternionic plane of $M^8$ or an orthogonal complement of such a plane. These surface can be mapped in natural manner to surfaces in $M^4 \times CP_2$ [K54] provided one can assign to each point of tangent space a hyper-complex plane $M^2(x) \subset M^4$. One can also speak about $M^8 - H$ duality.

This vision has very strong predictive power. It predicts that the extremals of Kähler action correspond to either hyper-quaternionic or co-hyper-quaternionic surfaces such that one can assign to tangent space at each point of space-time surface a hyper-complex plane $M^2(x) \subset M^4$. As a consequence, the $M^4$ projection of space-time surface at each point contains $M^2(x)$ and its orthogonal complement. These distributions are integrable implying that space-time surface allows dual slicings defined by string world sheets $Y^2$ and partonic 2-surfaces $X^2$. The existence of this kind of slicing was earlier deduced from the study of extremals of Kähler action and christened as Hamilton-Jacobi structure. The physical interpretation of $M^2(x)$ is as the space of non-physical polarizations and the plane of local 4-momentum.

One can fairly say, that number theoretical compactification is responsible for most of the understanding of quantum TGD that has emerged during last years. This includes the realization of Equivalence Principle at space-time level, dual formulations of TGD as Minkowskian and Euclidian string model type theories, the precise identification of preferred extremals of Kähler action as extremals for which second variation vanishes (at least for deformations representing dynamical symmetries) and thus providing space-time correlate for quantum criticality, the notion of number theoretic braid implied by the basic dynamics of Kähler action and crucial for precise construction of quantum TGD as almost-topological QFT, the construction of configuration space metric and spinor structure in terms of second quantized induced spinor fields with modified Dirac action defined by Kähler action realizing automatically the notion of finite measurement resolution and a connection with inclusions of hyper-finite factors of type $II_1$ about which Clifford algebra of configuration space represents an example.

The two most important number theoretic conjectures relate to the preferred extremals of Kähler action. The general idea is that classical dynamics for the preferred extremals of Kähler action should reduce to number theory: space-time surfaces should be either associative or co-associative in some sense.

1. The first meaning for associativity (co-associativity) would be that tangent (normal) spaces of space-time surfaces are quaternionic in some sense and thus associative. This can be formulated in terms of octonionic representation of the imbedding space gamma matrices possible in dimension $D = 8$ and states that induced gamma matrices generate quaternionic sub-algebra at each space-time point. It seems that induced rather than modified gamma matrices must be in question.

2. Second meaning for associative (co-associativity) would be following. In the case of complex numbers the vanishing of the real part of real-analytic function defines a 1-D curve. In octnionic case one can decompose octonion to sum of quaternion and quaternion multiplied by an octonionic imaginary unit. Quaternionicity could mean that space-time surfaces correspond to the vanishing of the imaginary part of the octonion real-analytic function. Co-quaternionicity would be defined in an obvious manner. Octonionic real analytic functions form a function field closed also with respect to the composition of functions. Space-time surfaces would form the analog of function field with the composition of functions with all operations realized as algebraic operations for space-time surfaces. Co-associaty could be perhaps seen as an additional feature making the algebra in question also co-algebra.

3. The third conjecture is that these conjectures are equivalent.

### Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory. The work with Riemann hypothesis led to further ideas.

After the realization that infinite primes can be mapped to polynomials representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of rational (hyper-)quaternions and (hyper-)octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

What is especially satisfying is that p-adic and real regions of the space-time surface could emerge automatically as solutions of the field equations. In the space-time regions where the solutions of field equations give rise to in-admissible complex values of the imbedding space coordinates, p-adic solution can exist for some values of the p-adic prime. The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to 'mind stuff', the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably an extremely brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

### 1.2.3 Hierarchy of Planck constants and dark matter hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

#### Dark matter as large $\hbar$ phase

D. Da Rocha and Laurent Nottale [E1] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). $v_0$ is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of $v_0$ seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests astrophysical systems are not only quantum systems at larger space-time sheets but correspond to a gigantic value of gravitational Planck constant. The gravitational (ordinary) Schrödinger equation would provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [K47] .

TGD predicts correctly the value of the parameter $v_0$ assuming that cosmic strings and their decay remnants are responsible for the dark matter. The harmonics of $v_0$ can be understood as corresponding to perturbations replacing cosmic strings with their n-branched coverings so that tension becomes $n^2$-fold: much like the replacement of a closed orbit with an orbit closing only after $n$ turns. $1/n$-sub-harmonic would result when a magnetic flux tube split into n disjoint magnetic flux tubes. Also a model for the formation of planetary system as a condensation of ordinary matter around quantum coherent dark matter emerges [K47] .

The values of Planck constants postulated by Nottale are gigantic and it is natural to assign them to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta). The magnetic energy of these flux quanta would correspond to dark energy and magnetic tension would give rise to negative "pressure" forcing accelerate cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside magnetic flux tubes identifiable as dark energy.

### Hierarchy of Planck constants from the anomalies of neuroscience biology

The quantal effects of ELF em fields on vertebrate brain have been known since seventies. ELF em fields at frequencies identifiable as cyclotron frequencies in magnetic field whose intensity is about 2/5 times that of Earth for biologically important ions have physiological effects and affect also behavior. What is intriguing that the effects are found only in vertebrates (to my best knowledge). The energies for the photons of ELF em fields are extremely low - about $10^{-10}$ times lower than thermal energy at physiological temperatures- so that quantal effects are impossible in the framework of standard quantum theory. The values of Planck constant would be in these situations large but not gigantic.

This inspired the hypothesis that these photons correspond to so large value of Planck constant that the energy of photons is above the thermal energy. The proposed interpretation was as dark photons and the general hypothesis was that dark matter corresponds to ordinary matter with non-standard value of Planck constant. If only particles with the same value of Planck constant can appear in the same vertex of Feynman diagram, the phases with different value of Planck constant are dark relative to each other. The phase transitions changing Planck constant can however make possible interactions between phases with different Planck constant but these interactions do not manifest themselves in particle physics. Also the interactions mediated by classical fields should be possible. Dark matter would not be so dark as we have used to believe.

Also the anomalies of biology support the view that dark matter might be a key player in living matter.

### Does the hierarchy of Planck constants reduce to the vacuum degeneracy of Kähler action?

This starting point led gradually to the recent picture in which the hierarchy of Planck constants is postulated to come as integer multiples of the standard value of Planck constant. Given integer multiple $\hbar = n\hbar_0$ of the ordinary Planck constant $\hbar_0$ is assigned with a multiple singular covering of the imbedding space [K18]. One ends up to an identification of dark matter as phases with non-standard value of Planck constant having geometric interpretation in terms of these coverings providing generalized imbedding space with a book like structure with pages labelled by Planck constants or integers characterizing Planck constant. The phase transitions changing the value of Planck constant would correspond to leakage between different sectors of the extended imbedding space. The question is whether these coverings must be postulated separately or whether they are only a convenient auxiliary tool.

The simplest option is that the hierarchy of coverings of imbedding space is only effective. Many-sheeted coverings of the imbedding space indeed emerge naturally in TGD framework. The huge vacuum degeneracy of Kähler action implies that the relationship between gradients of the imbedding space coordinates and canonical momentum currents is many-to-one: this was the very fact forcing to give up all the standard quantization recipes and leading to the idea about physics as geometry of the "world of classical worlds". If one allows space-time surfaces for which all sheets corresponding to the same values of the canonical momentum currents are present, one obtains effectively many-sheeted covering of the imbedding space and the contributions from sheets to the Kähler action are identical. If all sheets are treated effectively as one and the same sheet, the value of Planck constant is an integer multiple of the ordinary one. A natural boundary condition would be that at the ends of space-time at future and past boundaries of causal diamond containing the space-time surface, various branches co-incide. This would raise the ends of space-time surface in special physical role.

### Dark matter as a source of long ranged weak and color fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of bio-chemistry and bio-nuclear physics. A possible solution of the matter antimatter asymmetry is based on the identification of also antimatter as dark matter.

### 1.2.4 TGD as a generalization of physics to a theory consciousness

General coordinate invariance forces the identification of quantum jump as quantum jump between entire deterministic quantum histories rather than time=constant snapshots of single history. The new view about quantum jump forces a generalization of quantum measurement theory such that observer becomes part of the physical system. Thus a general theory of consciousness is unavoidable outcome. This theory is developed in detail in the books [K55, K7, K41, K6, K22, K27, K30, K49] .

#### Quantum jump as a moment of consciousness

The identification of quantum jump between deterministic quantum histories (configuration space spinor fields) as a moment of consciousness defines microscopic theory of consciousness. Quantum jump involves the steps

$$\Psi_i \rightarrow U\Psi_i \rightarrow \Psi_f \quad ,$$

where $U$ is informational "time development" operator, which is unitary like the S-matrix characterizing the unitary time evolution of quantum mechanics. $U$ is however only formally analogous to Schrödinger time evolution of infinite duration although there is *no* real time evolution involved. It is not however clear whether one should regard U-matrix and S-matrix as two different things or not: $U$-matrix is a completely universal object characterizing the dynamics of evolution by self-organization whereas S-matrix is a highly context dependent concept in wave mechanics and in quantum field theories where it at least formally represents unitary time translation operator at the limit of an infinitely long interaction time. The S-matrix understood in the spirit of superstring models is however something very different and could correspond to U-matrix.

The requirement that quantum jump corresponds to a measurement in the sense of quantum field theories implies that each quantum jump involves localization in zero modes which parameterize also the possible choices of the quantization axes. Thus the selection of the quantization axes performed by the Cartesian outsider becomes now a part of quantum theory. Together these requirements imply that the final states of quantum jump correspond to quantum superpositions of space-time surfaces which are macroscopically equivalent. Hence the world of conscious experience looks classical. At least formally quantum jump can be interpreted also as a quantum computation in which matrix $U$ represents unitary quantum computation which is however not identifiable as unitary translation in time direction and cannot be 'engineered'.

Can one say anything about the unitary process? Zero energy states correspond in positive energy ontology to physical events and break time reversal invariance. This because either the positive or negative energy part of the state is prepared whereas the second end of $CD$ corresponds to a superposition of (negative/positive energy) states with varying particle numbers and single particle quantum numbers just as in ordinary particle physics experiment. State function reduction must change the roles of the ends of $CD$s. Therefore $U$-matrix should correspond to the unitary matrix relating zero energy state basis prepared at different ends of $CD$ and state function reduction would be equivalent with state preparation.

The basic objection is that the arrow of geometric time alternates at imbedding space level but we know that arrow of time is universal. What one can say about the arrow of time at space-time level? Quantum classical correspondence requires that quantum mechanical irreversibility corresponds to irreversibility at space-time level. If the observer is analogous to an inhabitant of Flatland gaining information only about space-time surface, he or she is not able to discover that the arrow of time alternates at the level of imbedding space. The inhabitant of a folded bath towel is not able to observer the folding of the towel! Only by observing systems for which the imbedding space arrow of time is opposite, observer can discover the alternation. Living systems indeed behave as if they would contain space-time sheets with opposite arrow of geometric time (self-organization). Phase conjugate light beam is second example of this.

#### The notion of self

The concept of self is absolutely essential for the understanding of the macroscopic and macro-temporal aspects of consciousness. Self corresponds to a subsystem able to remain un-entangled under the sequential informational 'time evolutions' $U$. Exactly vanishing entanglement is practically impossible in ordinary quantum mechanics and it might be that 'vanishing entanglement' in the condition for self-property should be replaced with 'subcritical entanglement'. On the other hand, if space-time decomposes into p-adic and real regions, and if entanglement between regions representing physics in different number fields vanishes, space-time indeed decomposes into selves in a natural manner.

It is assumed that the experiences of the self after the last 'wake-up' sum up to single average experience. This means that subjective memory is identifiable as conscious, immediate short term memory. Selves form an infinite hierarchy with the entire Universe at the top. Self can be also interpreted as mental images: our mental images are selves having mental images and also we represent mental images of a higher level self. A natural hypothesis is that self $S$ experiences the experiences of its subselves as kind of abstracted experience: the experiences of subselves $S_i$ are not experienced as such but represent kind of averages $\langle S_{ij} \rangle$ of sub-subselves $S_{ij}$. Entanglement between selves, most naturally realized by the formation of join along boundaries bonds between cognitive or material space-time sheets, provides a possible a mechanism for the fusion of selves to larger selves (for instance, the fusion of the mental images representing separate right and left visual fields to single visual field) and forms wholes from parts at the level of mental images.

An attractive possibility suggested by zero energy ontology is that the notions of self and quantum jump reduce to each other and that a fractal hierarchy of quantum jumps within quantum jumps is enough. $CD$s would serve as imbedding space correlates of selves and quantum jumps would be followed by cascades of state function reductions beginning from given $CD$ and proceeding downwards to the smaller scales (smaller $CD$s). State function reduction cascades could also take place in parallel branches of the quantum state. One ends up with concrete ideas about how the arrow of geometric time is induced from that of subjective time defined by the experiences induced by the sequences of quantum jumps for sub-selves of self. One ends also ends up with concrete ideas about how the localization of the contents of sensory experience and cognition to the upper boundaries of $CD$ could take place.

#### Relationship to quantum measurement theory

The third basic element relates TGD inspired theory of consciousness to quantum measurement theory. The assumption that localization occurs in zero modes in each quantum jump implies that the world of conscious experience looks classical. It also implies the state function reduction of the standard quantum measurement theory as the following arguments demonstrate (it took incredibly long time to realize this almost obvious fact!).

1. The standard quantum measurement theory a la von Neumann involves the interaction of brain with the measurement apparatus. If this interaction corresponds to entanglement between microscopic degrees of freedom $m$ with the macroscopic effectively classical degrees of freedom $M$ characterizing the reading of the measurement apparatus coded to brain state, then the reduction of this entanglement in quantum jump reproduces standard quantum measurement theory provide the unitary time evolution operator $U$ acts as flow in zero mode degrees of freedom and correlates completely some orthonormal basis of configuration space spinor fields in non-zero modes with the values of the zero modes. The flow property guarantees that the localization is consistent with unitarity: it also means 1-1 mapping of quantum state basis to classical variables (say, spin direction of the electron to its orbit in the external magnetic field).

2. Since zero modes represent classical information about the geometry of space-time surface (shape, size, classical Kähler field,...), they have interpretation as effectively classical degrees of freedom and are the TGD counterpart of the degrees of freedom $M$ representing the reading of the measurement apparatus. The entanglement between quantum fluctuating non-zero modes and zero modes is the TGD counterpart for the $m-M$ entanglement. Therefore the localization in zero modes is equivalent with a quantum jump leading to a final state where the measurement apparatus gives a definite reading.

This simple prediction is of utmost theoretical importance since the black box of the quantum measurement theory is reduced to a fundamental quantum theory. This reduction is implied by the replacement of the notion of a point like particle with particle as a 3-surface. Also the infinite-dimensionality of the zero mode sector of the configuration space of 3-surfaces is absolutely essential. Therefore the reduction is a triumph for quantum TGD and favors TGD against string models.

Standard quantum measurement theory involves also the notion of state preparation which reduces to the notion of self measurement. Each localization in zero modes is followed by a cascade of self measurements leading to a product state. This process is obviously equivalent with the state preparation process. Self measurement is governed by the so called Negentropy Maximization Principle (NMP) stating that the information content of conscious experience is maximized. In the self measurement the density matrix of some subsystem of a given self localized in zero modes (after ordinary quantum measurement) is measured. The self measurement takes place for that subsystem of self for which the reduction of the entanglement entropy is maximal in the measurement. In p-adic context NMP can be regarded as the variational principle defining the dynamics of cognition. In real context self measurement could be seen as a repair mechanism allowing the system to fight against quantum thermalization by reducing the entanglement for the subsystem for which it is largest (fill the largest hole first in a leaking boat).

### Selves self-organize

The fourth basic element is quantum theory of self-organization based on the identification of quantum jump as the basic step of self-organization [K45] . Quantum entanglement gives rise to the generation of long range order and the emergence of longer p-adic length scales corresponds to the emergence of larger and larger coherent dynamical units and generation of a slaving hierarchy. Energy (and quantum entanglement) feed implying entropy feed is a necessary prerequisite for quantum self-organization. Zero modes represent fundamental order parameters and localization in zero modes implies that the sequence of quantum jumps can be regarded as hopping in the zero modes so that Haken's classical theory of self organization applies almost as such. Spin glass analogy is a further important element: self-organization of self leads to some characteristic pattern selected by dissipation as some valley of the "energy" landscape.

Dissipation can be regarded as the ultimate Darwinian selector of both memes and genes. The mathematically ugly irreversible dissipative dynamics obtained by adding phenomenological dissipation terms to the reversible fundamental dynamical equations derivable from an action principle can be understood as a phenomenological description replacing in a well defined sense the series of reversible quantum histories with its envelope.

### Classical non-determinism of Kähler action

The fifth basic element are the concepts of association sequence and cognitive space-time sheet. The huge vacuum degeneracy of the Kähler action suggests strongly that the absolute minimum space-time is not always unique. For instance, a sequence of bifurcations can occur so that a given space-time branch can be fixed only by selecting a finite number of 3-surfaces with time like(!) separations on the orbit of 3-surface. Quantum classical correspondence suggest an alternative formulation. Space-time surface decomposes into maximal deterministic regions and their temporal sequences have interpretation a space-time correlate for a sequence of quantum states defined by the initial (or final) states of quantum jumps. This is consistent with the fact that the variational principle selects preferred extremals of Kähler action as generalized Bohr orbits.

In the case that non-determinism is located to a finite time interval and is microscopic, this sequence of 3-surfaces has interpretation as a simulation of a classical history, a geometric correlate for contents of consciousness. When non-determinism has long lasting and macroscopic effect one can identify it as volitional non-determinism associated with our choices. Association sequences relate closely with the cognitive space-time sheets defined as space-time sheets having finite time duration and psychological time can be identified as a temporal center of mass coordinate of the cognitive space-time sheet. The gradual drift of the cognitive space-time sheets to the direction of future force by the geometry of the future light cone explains the arrow of psychological time.

### p-Adic physics as physics of cognition and intentionality

The sixth basic element adds a physical theory of cognition to this vision. TGD space-time decomposes into regions obeying real and p-adic topologies labelled by primes $p = 2, 3, 5, ....$ p-Adic regions obey the same field equations as the real regions but are characterized by p-adic non-determinism since the functions having vanishing p-adic derivative are pseudo constants which are piecewise constant functions. Pseudo constants depend on a finite number of positive pinary digits of arguments just like numerical predictions of any theory always involve decimal cutoff. This means that p-adic space-time regions are obtained by gluing together regions for which integration constants are genuine constants. The natural interpretation of the p-adic regions is as cognitive representations of real physics. The freedom of imagination is due to the p-adic non-determinism. p-Adic regions perform mimicry and make possible for the Universe to form cognitive representations about itself. p-Adic physics space-time sheets serve also as correlates for intentional action.

A more more precise formulation of this vision requires a generalization of the number concept obtained by fusing reals and p-adic number fields along common rationals (in the case of algebraic extensions among common algebraic numbers). This picture is discussed in [K53] . The application this notion at the level of the imbedding space implies that imbedding space has a book like structure with various variants of the imbedding space glued together along common rationals (algebraics). The implication is that genuinely p-adic numbers (non-rationals) are strictly infinite as real numbers so that most points of p-adic space-time sheets are at real infinity, outside the cosmos, and that the projection to the real imbedding space is discrete set of rationals (algebraics). Hence cognition and intentionality are almost completely outside the real cosmos and touch it at a discrete set of points only.

This view implies also that purely local p-adic physics codes for the p-adic fractality characterizing long range real physics and provides an explanation for p-adic length scale hypothesis stating that the primes $p \simeq 2^k$, $k$ integer are especially interesting. It also explains the long range correlations and short term chaos characterizing intentional behavior and explains why the physical realizations of cognition are always discrete (say in the case of numerical computations). Furthermore, a concrete quantum model for how intentions are transformed to actions emerges.

The discrete real projections of p-adic space-time sheets serve also space-time correlate for a logical thought. It is very natural to assign to p-adic pinary digits a $p$-valued logic but as such this kind of logic does not have any reasonable identification. p-Adic length scale hypothesis suggest that the $p = 2^k - n$ pinary digits represent a Boolean logic $B^k$ with $k$ elementary statements (the points of the $k$-element set in the set theoretic realization) with $n$ taboos which are constrained to be identically true.

#### p-Adic and dark matter hierarchies and hierarchy of moments of consciousness

Dark matter hierarchy assigned to a spectrum of Planck constant having arbitrarily large values brings additional elements to the TGD inspired theory of consciousness.

1. Macroscopic quantum coherence can be understood since a particle with a given mass can in principle appear as arbitrarily large scaled up copies (Compton length scales as $\hbar$). The phase transition to this kind of phase implies that space-time sheets of particles overlap and this makes possible macroscopic quantum coherence.

2. The space-time sheets with large Planck constant can be in thermal equilibrium with ordinary ones without the loss of quantum coherence. For instance, the cyclotron energy scale associated with EEG turns out to be above thermal energy at room temperature for the level of dark matter hierarchy corresponding to magnetic flux quanta of the Earth's magnetic field with the size scale of Earth and a successful quantitative model for EEG results [K16] .

Dark matter hierarchy leads to detailed quantitative view about quantum biology with several testable predictions [K16] . The general prediction is that Universe is a kind of inverted Mandelbrot fractal for which each bird's eye of view reveals new structures in long length and time scales representing scaled down copies of standard physics and their dark variants. These structures would correspond to higher levels in self hierarchy. This prediction is consistent with the belief that 75 per cent of matter in the universe is dark.

*1. Living matter and dark matter*

Living matter as ordinary matter quantum controlled by the dark matter hierarchy has turned out to be a particularly successful idea. The hypothesis has led to models for EEG predicting correctly the band structure and even individual resonance bands and also generalizing the notion of EEG [K16] . Also a generalization of the notion of genetic code emerges resolving the paradoxes related to the standard dogma [K28, K16] . A particularly fascinating implication is the possibility to identify great leaps in evolution as phase transitions in which new higher level of dark matter emerges [K16] .

It seems safe to conclude that the dark matter hierarchy with levels labelled by the values of Planck constants explains the macroscopic and macro-temporal quantum coherence naturally. That this explanation is consistent with the explanation based on spin glass degeneracy is suggested by following observations. First, the argument supporting spin glass degeneracy as an explanation of the macro-temporal quantum coherence does not involve the value of $\hbar$ at all. Secondly, the failure of the perturbation theory assumed to lead to the increase of Planck constant and formation of macroscopic quantum phases could be precisely due to the emergence of a large number of new degrees of freedom due to spin glass degeneracy. Thirdly, the phase transition increasing Planck constant has concrete topological interpretation in terms of many-sheeted space-time consistent with the spin glass degeneracy.

*2. Dark matter hierarchy and the notion of self*

The vision about dark matter hierarchy leads to a more refined view about self hierarchy and hierarchy of moments of consciousness [K15, K16] . The larger the value of Planck constant, the longer the subjectively experienced duration and the average geometric duration $T(k) \propto \hbar$ of the quantum jump.

Quantum jumps form also a hierarchy with respect to p-adic and dark hierarchies and the geometric durations of quantum jumps scale like $\hbar$. Dark matter hierarchy suggests also a slight modification of the notion of self. Each self involves a hierarchy of dark matter levels, and one is led to ask whether the highest level in this hierarchy corresponds to single quantum jump rather than a sequence of quantum jumps. The averaging of conscious experience over quantum jumps would occur only for sub-selves at lower levels of dark matter hierarchy and these mental images would be ordered, and single moment of consciousness would be experienced as a history of events. The quantum parallel dissipation at the lower levels would give rise to the experience of flow of time. For instance, hadron as a macro-temporal quantum system in the characteristic time scale of hadron is a dissipating system at quark and gluon level corresponding to shorter p-adic time scales. One can ask whether even entire life cycle could be regarded as a single quantum jump at the highest level so that consciousness would not be completely lost even during deep sleep. This would allow to understand why we seem to know directly that this biological body of mine existed yesterday.

The fact that we can remember phone numbers with 5 to 9 digits supports the view that self corresponds at the highest dark matter level to single moment of consciousness. Self would experience the average over the sequence of moments of consciousness associated with each sub-self but there would be no averaging over the separate mental images of this kind, be their parallel or serial. These mental images correspond to sub-selves having shorter wake-up periods than self and would be experienced as being time ordered. Hence the digits in the phone number are experienced as separate mental images and ordered with respect to experienced time.

*3. The time span of long term memories as signature for the level of dark matter hierarchy*

The basic question is what time scale can one assign to the geometric duration of quantum jump measured naturally as the size scale of the space-time region about which quantum jump gives conscious information. This scale is naturally the size scale in which the non-determinism of quantum jump is localized. During years I have made several guesses about this time scales but zero energy ontology and the vision about fractal hierarchy of quantum jumps within quantum jumps leads to a unique identification.

Causal diamond as an imbedding space correlate of self defines the time scale $\tau$ for the space-time region about which the consciousness experience is about. The temporal distances between the tips of $CD$ as come as integer multiples of $CP_2$ length scales and for prime multiples correspond to what I have christened as secondary p-adic time scales. A reasonable guess is that secondary p-adic time scales are selected during evolution and the primes near powers of two are especially favored. For electron, which corresponds to Mersenne prime $M_{127} = 2^{127} - 1$ this scale corresponds to .1 seconds defining the fundamental time scale of living matter via 10 Hz biorhythm (alpha rhythm). The unexpected prediction is that all elementary particles correspond to time scales possibly relevant to living matter.

Dark matter hierarchy brings additional finesse. For the higher levels of dark matter hierarchy $\tau$ is scaled up by $\hbar/\hbar_0$. One could understand evolutionary leaps as the emergence of higher levels at the level of individual organism making possible intentionality and memory in the time scale defined $\tau$.

Higher levels of dark matter hierarchy provide a neat quantitative view about self hierarchy and its evolution. Various levels of dark matter hierarchy would naturally correspond to higher levels in the hierarchy of consciousness and the typical duration of life cycle would give an idea about the level in question. The level would determine also the time span of long term memories as discussed in [K16] . The emergence of these levels must have meant evolutionary leap since long term memory is also accompanied by ability to anticipate future in the same time scale. This picture would suggest that the basic difference between us and our cousins is not at the level of genome as it is usually understood but at the level of the hierarchy of magnetic bodies [K28, K16]. In fact, higher levels of dark matter hierarchy motivate the introduction of the notions of super-genome and hyper-genome. The genomes of entire organ can join to form super-genome expressing genes coherently. Hyper-genomes would result from the fusion of genomes of different organisms and collective levels of consciousness would express themselves via hyper-genome and make possible social rules and moral.

## 1.3 Bird's eye of view about the topics of the book

The topics of this book are the purely geometric aspects of the vision about physics as an infinite-dimensional Kähler geometry of the "world of classical worlds", with " classical world" identified either as light-like 3-D surface of the unique Bohr orbit like 4-surface traversing through it. The non-determinism of Kähler action forces to generalize the notion of 3-surface so that unions of space-like surfaces with time like separations must be allowed. Zero energy ontology allows to formulate this picture elegantly in terms of causal diamonds defined as intersections of future and past directed light-cones. Also a a geometric realization of coupling constant evolution and finite measurement resolution emerges.

There are two separate tasks involved.

1. Provide configuration space of 3-surfaces with Kähler geometry which is consistent with 4-dimensional general coordinate invariance so that the metric is $\mathrm{Diff}^4$ degenerate. General coordinate invariance implies that the definition of metric must assign to a given light-like 3-surface $X^3$ a 4-surface as a kind of Bohr orbit $X^4(X^3)$.

2. Provide the configuration space with a spinor structure. The great idea is to identify configuration space gamma matrices in terms of super algebra generators expressible using second quantized fermionic oscillator operators for induced free spinor fields at the space-time surface assignable to a given 3-surface. The isometry generators and contractions of Killing vectors with gamma matrices would thus form a generalization of Super Kac-Moody algebra.

The condition of mathematical existence poses surprisingly strong conditions on configuration space metric and spinor structure.

1. From the experience with loop spaces one can expect that there is no hope about existence of well-defined Riemann connection unless this space is union of infinite-dimensional symmetric spaces with constant curvature metric and simple considerations requires that vacuum Einstein equations are satisfied by each component in the union. The coordinates labeling these symmetric spaces are zero modes having interpretation as genuinely classical variables which do not quantum fluctuate since they do not contribute to the line element of the configuration space.

2. The construction of the Kähler structure involves also the identification of complex structure. Direct construction of Kähler function as action associated with a preferred Bohr orbit like extremal for some physically motivated action action leads to a unique result. Second approach is group theoretical and is based on a direct guess of isometries of the infinite-dimensional symmetric space formed by 3-surfaces with fixed values of zero modes. The group of isometries is generalization of Kac-Moody group obtained by replacing finite-dimensional Lie group with the group of symplectic transformations of $\delta M^4_+ \times CP_2$, where $\delta M^4_+$ is the boundary of 4-dimensional future light-cone. A crucial role is played by the generalized conformal invariance assignable to light-like 3-surfaces and to the boundaries of causal diamond. In particular, a generalization of Equivalence Principle can be formulated in terms of generalized coset construction.

3. Fermionic statistics and quantization of spinor fields can be realized in terms of configuration space spinors structure. Quantum criticality and the idea about space-time surfaces as analogs of Bohr orbits have served as basic guiding lines of Quantum TGD. These notions can be formulated more precisely in terms of the modified Dirac equation for induced spinor fields allowing also realization of super-conformal symmetries and quantum gravitational holography. A rather detailed view about how configuration space Kähler function emerges as Dirac determinant allowing a tentative identification of the preferred extremals of Kähler action as surface for which second variation of Kähler action vanishes for some deformations of the surface. The catastrophe theoretic analog for quantum critical space-time surfaces are the points of space spanned by behavior and control variables at which the determinant defined by the second derivatives of potential function with respect to behavior variables vanishes. Number theoretic vision leads to rather detailed view about preferred extremals of Kähler action. In particular, preferred extremals should be what I have dubbed as hyper-quaternionic surfaces. It it still an open question whether this characterization is equivalent with quantum criticality or not.

The seven online books about TGD [K60, K43, K44, K51, K37, K34, K50] and eight online books about TGD inspired theory of consciousness and quantum biology [K55, K7, K41, K6, K22, K27, K30, K49] are warmly recommended for the reader willing to get overall view about what is involved.

## 1.4 The contents of the book

In the following abstracts of various chapters of the book are given in order to provide overall view.

### 1.4.1 Identification of the Configuration Space Kähler Function

There are two basic approaches to quantum TGD. The first approach, which is discussed in this chapter, is a generalization of Einstein's geometrization program of physics to an infinite-dimensional context. Second approach is based on the identification of physics as a generalized number theory. The first approach relies on the vision of quantum physics as infinite-dimensional Kähler geometry for the "world of classical worlds" (WCW) identified as the space of 3-surfaces in in certain 8-dimensional space. There are three separate approaches to the challenge of constructing WCW Kähler geometry and spinor structure. The first approach relies on direct guess of Kähler function. Second approach relies on the construction of Kähler form and metric utilizing the huge symmetries of the geometry needed to guarantee the mathematical existence of Riemann connection. The third approach relies on the construction of spinor structure based on the hypothesis that complexified WCW gamma matrices are representable as linear combinations of fermionic oscillator operator for second quantized free spinor fields at space-time surface and on the geometrization of super-conformal symmetries in terms of WCW spinor structure.

In this chapter the proposal for Kähler function based on the requirement of 4-dimensional General Coordinate Invariance implying that its definition must assign to a given 3-surface a unique space-time surface. Quantum classical correspondence requires that this surface is a preferred extremal of some some general coordinate invariant action, and so called Kähler action is a unique candidate in this respect. The preferred extremal has intepretation as an analog of Bohr orbit so that classical physics becomes and exact part of WCW geometry and therefore also quantum physics.

The basic challenge is the explicit identification of WCW Kähler function $K$. Two assumptions lead to the identification of $K$ as a sum of Chern-Simons type terms associated with the ends of causal diamond and with the light-like wormhole throats at which the signature of the induced metric changes. The first assumption is the weak form of electric magnetic duality. Second assumption is that the Kähler current for preferred extremals satisfies the condition $j_K \wedge dj_K = 0$ implying that the flow parameter of the flow lines of $j_K$ defines a global space-time coordinate. This would mean that the vision about reduction to almost topological QFT would be realized.

Second challenge is the understanding of the space-time correlates of quantum criticality. Electric-magnetic duality helps considerably here. The realization that the hierarchy of Planck constant realized in terms of coverings of the imbedding space follows from basic quantum TGD leads to a further understanding. The extreme non-linearity of canonical momentum densities as functions of time derivatives of the imbedding space coordinates implies that the correspondence between these two variables is not 1-1 so that it is natural to introduce coverings of $CD \times CP_2$. This leads also to a precise geometric characterization of the criticality of the preferred extremals.

### 1.4.2 Construction of Configuration Space Kähler Geometry from Symmetry Principles

There are three separate approaches to the challenge of constructing WCW Kähler geometry and spinor structure. The first one relies on a direct guess of Kähler function. Second approach relies on the construction of Kähler form and metric utilizing the huge symmetries of the geometry needed to guarantee the mathematical existence of Riemann connection. The third approach relies on the construction of spinor structure assuming that complexified WCW gamma matrices are representable as linear combinations of fermionic oscillator operator for the second quantized free spinor fields at space-time surface and on the geometrization of super-conformal symmetries in terms of spinor structure.

In this chapter the construction of Kähler form and metric based on symmetries is discussed. The basic vision is that WCW can be regarded as the space of generalized Feynman diagrams with lines thickned to light-like 3-surfaces and vertices identified as partonic 2-surfaces. In zero energy ontology the strong form of General Coordinate Invariance (GCI) implies effective 2-dimensionality and the basic objects are pairs partonic 2-surfaces $X^2$ at opposite light-like boundaries of causal diamonds ($CD$s).

The hypothesis is that WCW can be regarded as a union of infinite-dimensional symmetric spaces $G/H$ labeled by zero modes having an interpretation as classical, non-quantum fluctuating variables. A crucial role is played by the metric 2-dimensionality of the light-cone boundary $\delta M^4_+$ and of light-like 3-surfaces implying a generalization of conformal invariance. The group $G$ acting as isometries of WCW is tentatively identified as the symplectic group of $\delta M^4_+ \times CP_2$ localized with respect to $X^2$. $H$ is identified as Kac-Moody type group associated with isometries of $H = M^4 \times CP_2$ acting on light-like 3-surfaces and thus on $X^2$.

An explicit construction for the Hamiltonians of WCW isometry algebra as so called flux Hamiltonians is proposed and also the elements of Kähler form can be constructed in terms of these. Explicit expressions for WCW flux Hamiltonians as functionals of complex coordinates of the Cartesisian product of the infinite-dimensional symmetric spaces having as points the partonic 2-surfaces defining the ends of the the light 3-surface (line of generalized Feynman diagram) are proposed.

### 1.4.3 Configuration space spinor structure

There are three separate approaches to the challenge of constructing WCW Kähler geometry and spinor structure. The first approach relies on a direct guess of Kähler function. Second approach relies on the construction of Kähler form and metric utilizing the huge symmetries of the geometry needed to guarantee the mathematical existence of Riemann connection. The third approach discussed in this chapter relies on the construction of spinor structure based on the hypothesis that complexified WCW gamma matrices are representable as linear combinations of fermionic oscillator operator for the second quantized free spinor fields at space-time surface and on the geometrization of super-conformal symmetries in terms of spinor structure. This implies a geometrization of fermionic statistics.

The basic philosophy is that at fundamental level the construction of WCW geometry reduces to the second quantization of the induced spinor fields using Dirac action. This assumption is parallel with the bosonic emergence stating that all gauge bosons are pairs of fermion and antifermion at opposite throats of wormhole contact. Vacuum function is identified as Dirac determinant and the conjecture is that it reduces to the exponent of Kähler function. In order to achieve internal consistency induced gamma matrices appearing in Dirac operator must be replaced by the modified gamma matrices defined uniquely by Kähler action and one must also assume that extremals of Kähler action are in question so that the classical space-time dynamics reduces to a consistency condition. This implies also super-symmetries and the fermionic oscillator algebra at partonic 2-surfaces has intepretation as $\mathcal{N} = \infty$ generalization of space-time super-symmetry algebra different however from standard SUSY algebra in that Majorana spinors are not needed. This algebra serves as a building brick of various super-conformal algebras involved.

The requirement that there exist deformations giving rise to conserved Noether charges requires that the preferred extremals are critical in the sense that the second variation of the Kähler action vanishes for these deformations. Thus Bohr orbit property could correspond to criticality or at least involve it.

Quantum classical correspondence demands that quantum numbers are coded to the properties of the preferred extremals given by the Dirac determinant and this requires a linear coupling to the conserved quantum charges in Cartan algebra. Effective 2-dimensionality allows a measurement interaction term only in 3-D Chern-Simons Dirac action assignable to the wormhole throats and the ends of the space-time surfaces at the boundaries of $CD$. This allows also to have physical propagators reducing to Dirac propagator not possible without the measurement interaction term. An essential point is that the measurement interaction corresponds formally to a gauge transformation for the induced Kähler gauge potential. If one accepts the weak form of electric-magnetic duality Kähler function reduces to a generalized Chern-Simons term and the effect of measurement interaction term to Kähler function reduces effectively to the same gauge transformation.

The basic vision is that WCW gamma matrices are expressible as super-symplectic charges at the boundaries of $CD$. The basic building brick of WCW is the product of infinite-D symmetric spaces assignable to the ends of the propagator line of the generalized Feynman diagram. WCW Kähler metric has in this case "kinetic" parts associated with the ends and "interaction" part between the ends. General expressions for the super-counterparts of WCW flux Hamiltoniansand for the matrix elements of WCW metric in terms of their anticommutators are proposed on basis of this picture.

### 1.4.4 Does modified Dirac action define the fundamental action principle?

The construction of the spinor structure for the world of classical worlds (WCW) leads to the vision that second quantized modified Dirac equation codes for the entire quantum TGD. Among other things this would mean that Dirac determinant would define the vacuum functional of the theory having interpretation as the exponent of Kähler function of WCW and Kähler function would reduce to Kähler action for a preferred extremal of Kähler action. In this chapter the recent view about the modified Dirac action are explained in more detail.

*1. Identification of the modified Dirac action*

The modified Dirac action action involves several terms. The first one is 4-dimensional assignable to Kähler action. Second term is instanton term reducible to an expression restricted to wormhole throats or any light-like 3-surfaces parallel to them in the slicing of space-time surface by light-like 3-surfaces. The third term is assignable to Chern-Simons term and has interpretation as a measurement interaction term linear in Cartan algebra of the isometry group of the imbedding space in order to obtain stringy propagators and also to realize coupling between the quantum numbers associated with super-conformal representations and space-time geometry required by quantum classical correspondence.

This means that 3-D light-like wormhole throats carry induced spinor field which can be regarded as independent degrees of freedom having the spinor fields at partonic 2-surfaces as sources and acting as 3-D sources for the 4-D induced spinor field. The most general measurement interaction would involve the corresponding coupling also for Kähler action but is not physically motivated. There are good arguments in favor of Chern-Simons Dirac action and corresponding measurement interaction.

1. A correlation between 4-D geometry of space-time sheet and quantum numbers is achieved by the identification of exponent of Kähler function as Dirac determinant making possible the entanglement of classical degrees of freedom in the interior of space-time sheet with quantum numbers.

2. Cartan algebra plays a key role not only at quantum level but also at the level of space-time geometry since quantum critical conserved currents vanish for Cartan algebra of isometries and the measurement interaction terms giving rise to conserved currents are possible only for Cartan algebras. Furthermore, modified Dirac equation makes sense only for eigen states of Cartan algebra generators. The hierarchy of Planck constants realized in terms of the book like structure of the generalized imbedding space assigns to each $CD$ (causal diamond) preferred Cartan algebra: in case of Poincare algebra there are two of them corresponding to linear and cylindrical $M^4$ coordinates.

3. Quantum holography and dimensional reduction hierarchy in which partonic 2-surface defined fermionic sources for 3-D fermionic fields at light-like 3-surfaces $Y_l^3$ in turn defining fermionic sources for 4-D spinors find an elegant realization. Effective 2-dimensionality is achieved if the replacement of light-like wormhole throat $X_l^3$ with light-like 3-surface $Y_l^3$ "parallel" with it in the definition of Dirac determinant corresponds to the $U(1)$ gauge transformation $K \rightarrow K + f + \overline{f}$ for Kähler function of WCW so that WCW Kähler metric is not affected. Here $f$ is holomorphic function of WCW ("world of classical worlds") complex coordinates and arbitrary function of zero mode coordinates.

4. An elegant description of the interaction between super-conformal representations realized at partonic 2-surfaces and dynamics of space-time surfaces is achieved since the values of Cartan charges are feeded to the 3-D Dirac equation which also receives mass term at the same time. Almost topological QFT at wormhole throats results at the limit when four-momenta vanish: this is in accordance with the original vision about TGD as almost topological QFT.

5. A detailed view about the physical role of quantum criticality results. Quantum criticality fixes the values of Kähler coupling strength as the analog of critical temperature. Quantum criticality implies that second variation of Kähler action vanishes for critical deformations and the existence of conserved current except in the case of Cartan algebra of isometries. Quantum criticality allows to fix the values of couplings appearing in the measurement interaction by using the condition $K \rightarrow K + f + \overline{f}$. p-Adic coupling constant evolution can be understood also and corresponds to scale hierarchy for the sizes of causal diamonds ($CD$s).

6. The inclusion of imaginary instanton term to the definition of the modified gamma matrices is not consistent with the conjugation of the induced spinor fields. Measurement interaction can be however assigned to both Kähler action and its instanton term. CP breaking, irreversibility and the space-time description of dissipation are closely related and the CP and T oddness of the instanton part of the measurement interaction term could provide first level description for dissipative effects. It must be however emphasized that the mere addition of instanton term to Kähler function could be enough.

7. A radically new view about matter antimatter asymmetry based on zero energy ontology emerges and one could understand the experimental absence of antimatter as being due to the fact antimatter corresponds to negative energy states. The identification of bosons as wormhole contacts is the only possible option in this framework.

8. Almost stringy propagators and a consistency with the identification of wormhole throats as lines of generalized Feynman diagrams is achieved. The notion of bosonic emergence leads to a long sought general master formula for the $M$-matrix elements. The counterpart for fermionic loop defining bosonic inverse propagator at QFT limit is wormhole contact with fermion and cutoffs in mass squared and hyperbolic angle for loop momenta of fermion and antifermion in the rest system of emitting boson have precise geometric counterpart.

*2. Hyper-quaternionicity and quantum criticality*

The conjecture that quantum critical space-time surfaces are hyper-quaternionic in the sense that the modified gamma matrices span a quaternionic subspace of complexified octonions at each point of the space-time surface is consistent with what is known about preferred extremals. The condition that both the modified gamma matrices and spinors are quaternionic at each point of the space-time surface leads to a precise ansatz for the general solution of the modified Dirac equation making sense also in the real context. The octonionic version of the modified Dirac equation is very simple since $SO(7,1)$ as vielbein group is replaced with $G_2$ acting as automorphisms of octonions so that only the neutral Abelian part of the classical electro-weak gauge fields survives the map.

Octonionic gamma matrices provide also a non-associative representation for the 8-D version of Pauli sigma matrices and encourage the identification of 8-D twistors as pairs of octonionic spinors conjectured to be highly relevant also for quantum TGD. Quaternionicity condition implies that octo-twistors reduce to something closely related to ordinary twistors.

*3. The exponent of Kähler function as Dirac determinant for the modified Dirac action*

Although quantum criticality in principle predicts the possible values of Kähler coupling strength, one might hope that there exists even more fundamental approach involving no coupling constants and predicting even quantum criticality and realizing quantum gravitational holography.

1. The Dirac determinant defined by the product of Dirac determinants associated with the light-like partonic 3-surfaces $X^3_l$ associated with a given space-time sheet $X^4$ is the simplest candidate for vacuum functional identifiable as the exponent of the Kähler function. Individual Dirac determinant is defined as the product of eigenvalues of the dimensionally reduced modified Dirac operator $D_{K,3}$ and there are good arguments suggesting that the number of eigenvalues is finite. p-Adicization requires that the eigenvalues belong to a given algebraic extension of rationals. This restriction would imply a hierarchy of physics corresponding to different extensions and could automatically imply the finiteness and algebraic number property of the Dirac determinants if only finite number of eigenvalues would contribute. The regularization would be performed by physics itself if this were the case.

2. It remains to be proven that the product of eigenvalues gives rise to the exponent of Kähler action for the preferred extremal of Kähler action. At this moment the only justification for the conjecture is that this the only thing that one can imagine.

3. A long-standing conjecture has been that the zeros of Riemann Zeta are somehow relevant for quantum TGD. Rieman zeta is however naturally replaced Dirac zeta defined by the eigenvalues of $D_{K,3}$ and closely related to Riemann Zeta since the spectrum consists essentially for the cyclotron energy spectra for localized solutions region of non-vanishing induced Kähler magnetic field and hence is in good approximation integer valued up to some cutoff integer. In zero energy ontology the Dirac zeta function associated with these eigenvalues defines "square root" of thermodynamics assuming that the energy levels of the system in question are expressible as logarithms of the eigenvalues of the modified Dirac operator defining kind of fundamental constants. Critical points correspond to approximate zeros of Dirac zeta and if Kähler function vanishes at criticality as it indeed should, the thermal energies at critical points are in first order approximation proportional to zeros themselves so that a connection between quantum criticality and approximate zeros of Dirac zeta emerges.

4. The discretization induced by the number theoretic braids reduces the world of classical worlds to effectively finite-dimensional space and configuration space Clifford algebra reduces to a finite-dimensional algebra. The interpretation is in terms of finite measurement resolution represented in terms of Jones inclusion $\mathcal{M} \subset \mathcal{N}$ of HFFs with $\mathcal{M}$ taking the role of complex numbers. The finite-D quantum Clifford algebra spanned by fermionic oscillator operators is identified as a representation for the coset space $\mathcal{N}/\mathcal{M}$ describing physical states modulo measurement resolution. In the sectors of generalized imbedding space corresponding to non-standard values of Planck constant quantum version of Clifford algebra is in question.

### 1.4.5 The recent vision about preferred extremals and solutions of the modified Dirac equation

During years several approaches to what preferred extremals of Kähler action and solutions of the modified Dirac equation could be have been proposed and the challenge is to see whether at least some of these approaches are consistent with each other. It is good to list various approaches first.

1. For preferred extremals generalization of conformal invariance to 4-D situation is very attractive approach and leads to concrete conditions formally similar to those encountered in string model. The approach based on basic heuristics for massless equations, on effective 3-dimensionality, and weak form of electric magnetic duality is also promising. An alternative approach is inspired by number theoretical considerations and identifies space-time surfaces as associative or co-associative sub-manifolds of octonionic imbedding space.

2. There are also several approaches for solving the modified Dirac equation. The most promising approach is assumes that the solutions are restricted on 2-D stringy world sheets and/or partonic 2-surfaces. This strange looking view is a rather natural consequence of number theoretic vision.

The conditions stating that electric charge is conserved for preferred extremals is an alternative very promising approach.

In this chapter the question whether these various approaches are mutually consistent is discussed. It indeed turns out that the approach based on the conservation of electric charge leads under rather general assumptions to the proposal that solutions of the modified Dirac equation are localized on 2-dimensional string world sheets and/or partonic 2-surfaces. Einstein's equations are satisfied for the preferred extremals and this implies that the earlier proposal for the realization of Equivalence Principle is not needed. This leads to a considerable progress in the understanding of super Virasoro representations for super-symplectic and super-Kac-Moody algebra. In particular, the proposal is that super-Kac-Moody currents assignable to string world sheets define duals of gauge potentials and their generalization for gravitons: in the approximation that gauge group is Abelian - motivated by the notion of finite measurement resolution - the exponents for the sum of KM charges would define non-integrable phase factors. One can also identify Yangian as the algebra generated by these charges. The approach allows also to understand the special role of the right handed neutrino in SUSY according to TGD.

### 1.4.6 Knots and TGD

Khovanov homology generalizes the Jones polynomial as knot invariant. The challenge is to find a quantum physical construction of Khovanov homology analous to the topological QFT defined by Chern-Simons action allowing to interpret Jones polynomial as vacuum expectation value of Wilson loop in non-Abelian gauge theory.

Witten's approach to Khovanov homology relies on fivebranes as is natural if one tries to define 2-knot invariants in terms of their cobordisms involving violent un-knottings. Despite the difference in approaches it is very useful to try to find the counterparts of this approach in quantum TGD since this would allow to gain new insights to quantum TGD itself as almost topological QFT identified as symplectic theory for 2-knots, braids and braid cobordisms. This comparison turns out to be extremely useful from TGD point of view.

1. A highly unique identification of string world sheets and therefore also of the braids whose ends carry quantum numbers of many particle states at partonic 2-surfaces emerges if one identifies the string word sheets as singular surfaces in the same manner as is done in Witten's approach. This identification need of course not be correct and later in the article a less ad hoc identification is proposed. Even more, the conjectured slicings of preferred extremals by 3-D surfaces and string world sheets central for quantum TGD can be identified uniquely if the identification is accepted. The slicing by 3-surfaces would be interpreted in gauge theory in terms of Higgs= constant surfaces with radial coordinate of $CP_2$ playing the role of Higgs. The slicing by string world sheets would be induced by different choices of $U(2)$ subgroup of $SU(3)$ leaving Higgs=constant surfaces invariant.

2. Also a physical interpretation of the operators Q, F, and P of Khovanov homology emerges. P would correspond to instanton number and F to the fermion number assignable to right handed neutrinos. The breaking of $M^4$ chiral invariance makes possible to realize Q physically. The finding that the generalizations of Wilson loops can be identified in terms of the gerbe fluxes $\int H_A J$ supports the conjecture that TGD as almost topological QFT corresponds essentially to a symplectic theory for braids and 2-knots.

The basic challenge of quantum TGD is to give a precise content to the notion of generalization Feynman diagram and the reduction to braids of some kind is very attractive possibility inspired by zero energy ontology. The point is that no $n > 2$-vertices at the level of braid strands are needed if bosonic emergence holds true.

1. For this purpose the notion of algebraic knot is introduce and the possibility that it could be applied to generalized Feynman diagrams is discussed. The algebraic structrures kei, quandle, rack, and biquandle and their algebraic modifications as such are not enough. The lines of Feynman graphs are replaced by braids and in vertices braid strands redistribute. This poses several challenges: the crossing associated with braiding and crossing occurring in non-planar

Feynman diagrams should be integrated to a more general notion; braids are replaced with sub-manifold braids; braids of braids ....of braids are possible; the redistribution of braid strands in vertices should be algebraized. In the following I try to abstract the basic operations which should be algebraized in the case of generalized Feynman diagrams.

2. One should be also able to concretely identify braids and 2-braids (string world sheets) as well as partonic 2-surfaces and I have discussed several identifications during last years. Legendrian braids turn out to be very natural candidates for braids and their duals for the partonic 2-surfaces. String world sheets in turn could correspond to the analogs of Lagrangian sub-manifolds or two minimal surfaces of space-time surface satisfying the weak form of electric-magnetic duality. The latter opion turns out to be more plausible. This identification - if correct - would solve quantum TGD explicitly at string world sheet level which corresponds to finite measurement resolution.

3. Also a brief summary of generalized Feynman rules in zero energy ontology is proposed. This requires the identification of vertices, propagators, and prescription for integrating over al 3-surfaces. It turns out that the basic building blocks of generalized Feynman diagrams are well-defined.

4. The notion of generalized Feynman diagram leads to a beautiful duality between the descriptions of hadronic reactions in terms of hadrons and partons analogous to gauge-gravity duality and AdS/CFT duality but requiring no additional assumptions. The model of quark gluon plasma as s strongly interacting phase is proposed. Color magnetic flux tubes are responsible for the long range correlations making the plasma phase more like a very large hadron rather than a gas of partons. One also ends up with a simple estimate for the viscosity/entropy ratio using black-hole analogy.

### 1.4.7 Miscellaneous topics

This chapter contains topics which do not fit naturally under any umbrella, but which I feel might be of some relevance. Basically TGD inspired comments to the work of the people not terribly relevant to quantum TGD itself are in question. For few years ago Witten's approach to 3-D quantum gravitation raised a considerable interest and this inspired the comparison of this approach with quantum TGD in which light-like 3-surfaces are in a key role. Few years later the entropic gravity of Verlinde stimulated a lot of fuss in blogs and it is interesting to point out how the formal thermodynamical structure (or actually its "square root") emerges in the fundamental formulation of TGD. Lisi's $E_8$ theory was a further blog favorite and some comments about its failures and possible manners to cure them are discussed. It is also shown ho how $E_8$ can be seed as being replaced with the Kac-Moody algebra associated standard model symmetry group in TGD framework.

# Chapter 2

# Identification of the Configuration Space Kähler Function

## 2.1 Introduction

The motivation or the construction of configuration space geometry is the postulate that physics reduces to the geometry of classical spinor fields in the the "world of the classical worlds" (WCW) identified as the infinite-dimensional configuration space of 3-surfaces of some subspace of $M^4 \times CP_2$. The first candidates were $M^4_+ \times CP_2$ and $M^4 \times CP_2$, where $M^4$ and $M^4_+$ denote Minkowski space and its light cone respectively. The recent identification of WCW is as the the union of sub-WCWs consisting of light-like 3-surface representing generalized Feynman diagrams in $CD \times CP_2$, where $CD$ is intersection of future and past directed light-cones of $M^4$. The details of this identification will be discussed later.

Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that configuration space possesses Kähler geometry. One of the basic features of the Kähler geometry is that it is solely determined by the so called Kähler function, which defines both the Kähler form $J$ and the components of the Kähler metric $g$ in complex coordinates via the formulas [A69]

$$\begin{aligned} J &= i\partial_k\partial_{\bar{l}} K dz^k \wedge d\bar{z}^l \ , \\ ds^2 &= 2\partial_k\partial_{\bar{l}} K dz^k d\bar{z}^l \ . \end{aligned} \tag{2.1.1}$$

Kähler form is covariantly constant two-form and can be regarded as a representation of imaginary unit in the tangent space of the configuration space

$$J_{mr}J^{rn} = -g_m^{\ n} \ . \tag{2.1.2}$$

As a consequence Kähler form defines also symplectic structure in configuration space.

### 2.1.1 Configuration space Kähler metric from Kähler function

The task of finding Kähler geometry for the configuration space reduces to that of finding the Kähler function. The main constraints on the Kähler function result from the requirement of General Coordinate Invariance (GCI) -or more technically Diff$^4$ symmetry and Diff degeneracy. GCI requires that the definition of the Kähler function assigns to a given 3-surface $X^3$ a unique space-time surface $X^4(X^3)$, the generalized Bohr orbit defining the classical physics associated with $X^3$. The natural guess inspired by quantum classical correspondence is that Kähler function is defined by what might be called Kähler action, which is essentially Maxwell action with Maxwell field expressible in terms of $CP_2$ coordinates and that the space-time surface corresponds to a preferred extremal of Kähler action.

One can end up with the identification of the preferred extremal via several routes. Kähler action contains Kähler coupling strength as a temperature like parameter and this leads to the idea of quantum criticality fixing this parameter. One could go even even further, and require that space-time surfaces are critical in the sense that there exist an infinite number of vanishing second variations of Kähler action defining conserved Noether charges. The approach based on the modified Dirac action indeed leads naturally to this picture [K19] . Kähler coupling strength should be however visible in the solutions of field equations somehow before one can say that these two criticalities have something to do with each other. Since Kähler coupling strength does not appear in field equations it can make its way to field equations only via boundary conditions. This is achieved if one accepts the weak form of self-duality discussed in [K10] which roughly states that for the partonic 2-surfaces the induced Kähler electric field is proportional to the Kähler magnetic field strength. The proportionality constant turns out to be essentially the Kähler coupling strength. The simplest hypothesis is that Kähler coupling strength has single universal value for given value of Planck constant and the weak form of self-duality fixes it.

If Kähler action would define a strictly deterministic variational principle, $\mathrm{Diff}^4$ degeneracy and invariance would be achieved by restricting the consideration to 3-surfaces $Y^3$ at the boundary of $M^4_+$ and by defining Kähler function for 3-surfaces $X^3$ at $X^4(Y^3)$ and diffeo-related to $Y^3$ as $K(X^3) = K(Y^3)$. This reduction might be called quantum gravitational holography. The classical non-determinism of the Kähler action introduces complications which might be overcome in zero energy ontology (ZEO). ZEO and strong from of GCI lead to the effective replacement of $X^3$ with partonic 2-surfaces at the ends of $CD$ plus the 4-D tangent space distribution associated with them as basic geometric objects so that one can speak about effective 2-dimensionality and strong form of gravitational holography.

### 2.1.2 Configuration space metric from symmetries

A complementary approach to the problem of constructing configuration space geometry is based on symmetries. The work of Dan [A44] [A44] has demonstrated that the Kähler geometry of loop spaces is unique from the existence of Riemann connection and fixed completely by the Kac Moody symmetries of the space. In 3-dimensional context one has even better reasons to expect uniqueness. The guess is that configuration space is a union symmetric spaces labeled by zero modes not appearing in the line element as differentials and having interpretations as classical degrees providing a rigorous formulation of quantum measurement theory. The generalized conformal invariance of metrically 2-dimensional light like 3-surfaces acting as causal determinants is the corner stone of the construction. The construction works only for 4-dimensional space-time and imbedding space which is a product of four-dimensional Minkowski space or its future light cone with $CP_2$.

In this sequel I will first consider the basic properties of the configuration space, propose an identification of the Kähler function and discuss various physical and mathematical motivations behind the proposed definition. The key feature of the Kähler action is the failure of classical determinism in its standard form, and various implications of the failure are discussed.

## 2.2 Configuration space

The view about configuration space or world of classical worlds (WCW) has developed considerably during the last two decades. Here only the recent view is summarized in order to not load reader with unessential details.

### 2.2.1 Basic notions

The notions of imbedding space, 3-surface (and 4-surface), and configuration space or "world of classical worlds" (WCW), are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in $H$. The basic idea was that the definition of Kähler metric of WCW assigns to each $X^3$ a unique space-time surface $X^4(X^3)$ allowing in this manner to realize GCI. During years these notions have however evolved considerably.

#### The notion of imbedding space

Two generalizations of the notion of imbedding space were forced by number theoretical vision [K53, K54, K52] .

1. p-Adicization forced to generalize the notion of imbedding space by gluing real and p-adic variants of imbedding space together along rationals and common algebraic numbers. The generalized imbedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book. As matter fact, this gluing idea generalizes to the level of WCW.

2. With the discovery of zero energy ontology [K9, K13] it became clear that the so called causal diamonds ($CD$s) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of $CD$ characterizes the position of $CD$ in $H$. If the temporal distance between upper and lower tip of $CD$ is quantized power of 2 multiples of $CP_2$ length, p-adic length scale hypothesis [K3] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of $CD$ can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that $CD$s can contains $CD$s within $CD$s, and measurement resolution dictates the length scale below which the sub-$CD$s are not visible.

3. The realization of the hierarchy of Planck constants [K18] led to a further generalization of the notion of imbedding space. Generalized imbedding space is obtained by gluing together Cartesian products of singular coverings and possibly also factor spaces of $CD$ and $CP_2$ to form a book like structure. There are good physical and mathematical arguments suggesting that only the singular coverings should be allowed [K52] . The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized imbedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each $CD$ and $CP_2$ is replaced with a union of $CD$s and $CP_2$s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

### The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial and the receont view is an outcome of a long and tedious process involving many hastily done mis-interpretations.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to equivalence implied by GCI. There was a problem related to the realization of GCI since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for $Y^3$ at $X^4(X^3)$ and Diff$^4$ related $X^3$ should satisfy $X^4(Y^3) = X^4(X^3)$ .

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the GCI in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. Light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces. Therefore it seems that one must choose between light-like and space-like 3-surfaces or assume generalized GCI requiring that equivalently either space-like 3-surfaces or light-like 3-surfaces at the ends of $CD$s can be identified as the fundamental geometric objects. General GCI requires that the basic objects correspond to the partonic 2-surfaces identified as intersections of these 3-surfaces plus common 4-D tangent space distribution. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. Since the information about normal space of the 2-surface is needed one has only effective 2-dimensionality. Weak form of self-duality [K10] however implies that the normal data (flux Hamiltonians associated with Kähler electric field) reduces to magnetic flux Hamiltonians. This is essential for conformal symmetries and also simplifies the construction enormously.

3. At some stage came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces

representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-$CD$s. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-$CD$s containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

4. A further but inessential complication relates to the hierarchy of Planck constants forcing to generalize the notion of imbedding space and also to the fact that for non-standard values of Planck constant there is symmetry breaking due to preferred plane $M^2$ preferred homologically trivial geodesic sphere of $CP_2$ having interpretation as geometric correlate for the selection of quantization axis. For given sector of $CH$ this means union over choices of this kind.

The basic vision forced by the generalization of GCI has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action and are thus analogous to Bohr orbits. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

The study of the modified Dirac equation led to the realization that classical field equations for Kähler action can be seen as consistency conditions for the modified Dirac action and led to the identification of preferred extremals in terms of criticality. This identification which follows naturally also from quantum criticality.

1. The detailed construction of the generalized eigen modes of the dimensional reduction of the modified Dirac operator $D_K$ associated with Kähler action [K9] relies on the vision that the generalized eigenvalues of this operator code for information about preferred extremal of Kähler action and that vacuum functional identified as Dirac determinant equals to exponent of Kähler action for a preferred extremal.

2. The next step of progress was the realization that the requirement that the conservation of the Noether currents associated with the modified Dirac equation requires that the second variation of the Kähler action vanishes. In strongest form this condition would be satisfied for all variations and in weak sense only for those defining dynamical symmetries. The interpretation is as a space-time correlate for quantum criticality and the vacuum degeneracy of Kähler action makes the criticality plausible. Weak form of electric-magnetic duality gives a precise formulation for how Kähler coupling strength is visible in the properties of preferred extremals. A generalization of the ideas of the catastrophe theory to infinite-dimensional context results. These conditions make sense also in p-adic context and have a number theoretical universal form.

The notion of number theoretical compactication led to important progress in the understanding of the preferred extremals and the conjectures were consistent with what is known about the known extremals.

1. The conclusion was that one can assign to the 4-D tangent space $T(X^4(X^3_l)) \subset M^8$ a subspace $M^2(x) \subset M^4$ having interpretation as the plane of non-physical polarizations. This in the case that the induced metric has Minkowskian signature. If not, and if co-hyper-quaternionic surface is in question, similar assigned should be possible in normal space. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW. This is as it must be since complexification does not make sense in $M^2$ degrees of freedom.

2. In number theoretical framework $M^2(x)$ has interpretation as a preferred hyper-complex sub-space of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of $M^8$. The condition $M^2(x) \subset T(X^4(X^3_l)))$ in principle fixes the tangent space at $X^3_l$, and one has good hopes that the boundary value problem is well-defined and could fix $X^4(X^3)$ at least partially as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2(x) \subset M^4$ plays also other important roles.

3. At the level of $H$ the counterpart for the choice of $M^2(x)$ seems to be following. Suppose that $X^4(X_l^3)$ has Minkowskian signature. One can assign to each point of the $M^4$ projection $P_{M^4}(X^4(X_l^3))$ a sub-space $M^2(x) \subset M^4$ and its complement $E^2(x)$, and the distributions of these planes are integrable and define what I have called Hamilton-Jacobi coordinates which can be assigned to the known extremals of Kähler with Minkowskian signature. This decomposition allows to slice space-time surfaces by string world sheets and their 2-D partonic duals. Also a slicing to 1-D light-like surfaces and their 3-D light-like duals $Y_l^3$ parallel to $X_l^3$ follows under certain conditions on the induced metric of $X^4(X_l^3)$. This decomposition exists for known extremals and has played key role in the recent developments. Physically it means that 4-surface (3-surface) reduces effectively to 3-D (2-D) surface and thus holography at space-time level. A physically attractive realization of the slicings of space-time surface by 3-surfaces and string world sheets is discussed in [K25] by starting from the observation that TGD could define a natural realization of braids, braid cobordisms, and 2-knots.

4. The weakest form of number theoretic compactification [K54] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic $M^8$ can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of $E^4$ in the decomposition $M^8 = M^4 \times E^4$, where $M^4$ corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in $M^8$ is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

If one takes $M^- H$ duality seriously, one must conclude that one can choose any partonic 2-surface in the slicing of $X^4$ as a representative. This means gauge invariance reflect in the definition of Kähler function as $U(1)$ gauge transformation $K \rightarrow K + f + \overline{f}$ having no effect on Kähler metric and Kähler form.

Although the details of this vision might change it can be defended by its ability to fuse together all great visions about quantum TGD. In the sequel the considerations are restricted to 3-surfaces in $M^4_\pm \times CP_2$. The basic outcome is that Kähler metric is expressible using the data at partonic 2-surfaces $X^2 \subset \delta M^4_+ \times CP_2$. The generalization to the actual physical situation requires the replacement of $X^2 \subset \delta M^4_+ \times CP_2$ with unions of partonic 2-surfaces located at light-like boundaries of $CD$s and sub-$CD$s.

### The notion of configuration space

From the beginning there was a problem related to the precise definition of the configuration space ("world of classical worlds" (WCW)). Should one regard $CH$ as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the basis question is "$M^4_+$ or $M^4$?" and that this question had been settled in favor of $M^4_+$ by the fact that $M^4_+$ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering $M^4$ instead of $M^4_+$.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds ($CD$s) define excellent candidates for the fundamental building blocks of the configuration space or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of $H$ defined the sectors of WCW.

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the imbedding space representing the upper and lower boundaries of $CD$. Second conformal symmetry corresponds to light-like 3-surface $X_l^3$, which can be boundaries of $X^4$ and light-like surfaces separating

space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that configuration space (WCW) is a union of configuration spaces associated with the spaces $CD \times CP_2$. $CD$s can contain $CD$s within $CD$s so that a fractal like hierarchy having interpretation in terms of measurement resolution results. It must be however emphasized that Kähler function depends on partonic 2-surfaces at both ends of space-time surface so that WCW is topologically Cartesian product of corresponding symmetric spaces. WCW metric must therefore have parts corresponding to the partonic 2-surfaces (free part) and also an interaction term depending on the partonic 2-surface at the opposite ends of the light-like 3-surface. The conclusion is that geometrization reduces to that for single like of generalized Feynman diagram containing partonic 2-surfaces at its ends. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case corresponding to a line of generalized Feynman diagram. One can also deduce the free part of the metric by restricting the consideration to partonic 2-surfaces at single end of generalized Feynman diagram.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface $X^2$ - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at $X^2$ define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the configuration space metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at configuration space level and TGD is a genuine theory of gravitation at this level.

2. Configuration space can be divided into slices for which the induced Kähler forms of $CP_2$ and $\delta M^4_\pm$ at the partonic 2-surfaces $X^2$ at the light-like boundaries of $CD$s are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of $CD$s).

3. This leads to the identification of the coset space structure of the sub-configuration space associated with given $CD$ in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras (symmetries respecting light-likeness of light-like 3-surfaces). Configuration space in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with Kac-Moody group. Equivalently, the local coset space $S^2 \times CP_2$ is in question: this was one of the first ideas about configuration space which I gave up as too naive!

4. Generalized coset construction and coset space structure have very deep physical meaning since they realize Equivalence Principle at quantum level: the identical actions of Super Virasoro generators for super-symplectic and super Kac-Moody algebras implies that inertial and gravitational four-momenta are identical.

### 2.2.2 Constraints on the configuration space geometry

The constraints on the WCW result both from the infinite dimension of the configuration space and from physically motivated symmetry requirements. There are three basic physical requirements on the configuration space geometry: namely four-dimensional GCI in strong form, Kähler property and the decomposition of configuration space into a union $\cup_i G/H_i$ of symmetric spaces $G/H_i$, each coset space allowing $G$-invariant metric such that $G$ is subgroup of some 'universal group' having natural action on 3-surfaces. Together with the infinite dimensionality of the configuration space these requirements pose extremely strong constraints on the configuration space geometry. In the following we shall consider these requirements in more detail.

#### Diff$^4$ invariance and Diff$^4$ degeneracy

Diff$^4$ plays fundamental role as the gauge group of General Relativity. In string models $Diff^2$ invariance ($Diff^2$ acts on the orbit of the string) plays central role in making possible the elimination of the time like and longitudinal vibrational degrees of freedom of string. Also in the present case the elimination of the tachyons (time like oscillatory modes of 3-surface) is a physical necessity and Diff$^4$ invariance provides an obvious manner to do the job.

In the standard path l integral formulation the realization of Diff$^4$ invariance is an easy task at the formal level. The problem is however that path integral over four-surfaces is plagued by divergences and doesn't make sense. In the present case the configuration space consists of 3-surfaces and only $Diff^3$ emerges automatically as the group of re-parameterizations of 3-surface. Obviously one should somehow define the action of Diff$^4$ in the space of 3-surfaces. Whatever the action of Diff$^4$ is it must leave the configuration space metric invariant. Furthermore, the elimination of tachyons is expected to be possible only provided the time like deformations of the 3-surface correspond to zero norm vector fields of the configuration space so that 3-surface and its Diff$^4$ image have zero distance. The conclusion is that configuration space metric should be both Diff$^4$ invariant and Diff$^4$ degenerate.

The problem is how to define the action of Diff$^4$ in $C(H)$. Obviously the only manner to achieve Diff$^4$ invariance is to require that the very definition of the configuration space metric somehow associates a unique space time surface to a given 3-surface for Diff$^4$ to act on. The obvious physical interpretation of this space time surface is as "classical space time" so that "Classical Physics" would be contained in configuration space geometry. In fact, this space-time surface is analogous to Bohr orbit so that semiclassical quantization rules become an exact part of the quantum theory. It is this requirement, which has turned out to be decisive concerning the understanding of the WCW geometry.

### Decomposition of the configuration space into a union of symmetric spaces $G/H$

The extremely beautiful theory of finite-dimensional symmetric spaces constructed by Elie Cartan suggests that configuration space should possess decomposition into a union of coset spaces $CH = \cup_i G/H_i$ such that the metric inside each coset space $G/H_i$ is left invariant under the infinite dimensional isometry group $G$. The metric equivalence of surfaces inside each coset space $G/H_i$ does not mean that 3-surfaces inside $G/H_i$ are physically equivalent. The reason is that the vacuum functional is exponent of Kähler action which is not isometry invariant so that the 3-surfaces, which correspond to maxima of Kähler function for a given orbit, are in a preferred position physically. For instance, one can imagine of calculating functional integral around this maximum perturbatively. Symmetric space property actually allows also much more powerful non-perturbative approach based on harmonic analysis [K19] . The sum of over $i$ means actually integration over the zero modes of the metric (zero modes correspond to coordinates not appearing as coordinate differentials in the metric tensor).

The coset space $G/H$ is a symmetric space only under very special Lie-algebraic conditions. Denoting the decomposition of the Lie-algebra $g$ of $G$ to the direct sum of $H$ Lie-algebra $h$ and its complement $t$ by $g = h \oplus t$, one has

$$[h,h] \subset h \quad , \quad [h,t] \subset t \quad , \quad [t,t] \subset h \quad .$$

This decomposition turn out to play crucial role in guaranteing that $G$ indeed acts as isometries and that the metric is Ricci flat.

The four-dimensional $Diff$ invariance indeed suggests to a beautiful solution of the problem of identifying $G$. The point is that any 3-surface $X^3$ is $Diff^4$ equivalent to the intersection of $X^4(X^3)$ with the light cone boundary. This in turn implies that 3-surfaces in the space $\delta H = \delta M^4_+ \times CP_2$ should be all what is needed to construct configuration space geometry. The group $G$ can be identified as some subgroup of diffeomorphisms of $\delta H$ and $H_i$ contains that subgroup of $G$, which acts as diffeomorphisms of the 3-surface $X^3$. Since $G$ preserves topology, configuration space must decompose into union $\cup_i G/H_i$, where $i$ labels 3-topologies and various zero modes of the metric. For instance, the elements of the Lie-algebra of $G$ invariant under configuration space complexification correspond to zero modes.

The reduction to the light cone boundary, identifiable as the moment of big bang, looks perhaps odd at first. In fact, it turns out that the classical non-determinism of Kähler action does not allow the complete reduction to the light cone boundary: physically this is a highly desirable implication but means a considerable mathematical challenge.

### Kähler property

Kähler property implies that the tangent space of the configuration space allows complexification and that there exists a covariantly constant two-form $J_{kl}$, which can be regarded as a representation of the imaginary unit in the tangent space of the configuration space:

$$J_k{}^r J_{rl} = -G_{kl} \ . \tag{2.2.1}$$

There are several physical and mathematical reasons suggesting that configuration space metric should possess Kähler property in some generalized sense.

1. The deepest motivation comes from the need to geometrize hermitian conjugation which is basic mathematical operation of quantum theory.

2. Kähler property turns out to be a necessary prerequisite for defining divergence free configuration space integration. We will leave the demonstration of this fact later although the argument as such is completely general.

3. Kähler property very probably implies an infinite-dimensional isometry group. The study of the loop groups $Map(S^1, G)$ [A44] shows that loop group allows only single Kähler metric with well defined Riemann connection and this metric allows local G as its isometries!

   To see this consider the construction of Riemannian connection for $Map(X^3, H)$. The defining formula for the connection is given by the expression

$$\begin{aligned} 2(\nabla_X Y, Z) &= X(Y,Z) + Y(Z,X) - Z(X,Y) \\ &+ ([X,Y],Z) + ([Z,X],Y) - ([Y,Z],X) \end{aligned} \tag{2.2.2}$$

   $X, Y, Z$ are smooth vector fields in $Map(X^3, G)$. This formula defines $\nabla_X Y$ uniquely provided the tangent space of $Map$ is complete with respect to Riemann metric. In the finite-dimensional case completeness means that the inverse of the covariant metric tensor exists so that one can solve the components of connection from the conditions stating the covariant constancy of the metric. In the case of the loop spaces with Kähler metric this is however not the case.

   Now the symmetry comes into the game: if $X, Y, Z$ are left (local gauge) invariant vector fields defined by the Lie-algebra of local $G$ then the first three terms drop away since the scalar products of left invariant vector fields are constants. The expression for the covariant derivative is given by

$$\nabla_X Y = (Ad_X Y - Ad_X^* Y - Ad_Y^* X)/2 \tag{2.2.3}$$

   where $Ad_X^*$ is the adjoint of $Ad_X$ with respect to the metric of the loop space.

   At this point it is important to realize that Freed's argument does not force the isometry group of the configuration space to be $Map(X^3, M^4 \times SU(3))$! Any symmetry group, whose Lie algebra is complete with respect to the configuration space metric ( in the sense that any tangent space vector is expressible as superposition of isometry generators modulo a zero norm tangent vector) is an acceptable alternative.

   The Kähler property of the metric is quite essential in one-dimensional case in that it leads to the requirement of left invariance as a mathematical consistency condition and we expect that dimension three makes no exception in this respect. In 3-dimensional case the degeneracy of the metric turns out to be even larger than in 1-dimensional case due to the four-dimensional Diff degeneracy. So we expect that the metric ought to possess some infinite-dimensional isometry group and that the above formula generalizes also to the 3-dimensional case and to the case of local coset space. Note that in $M^4$ degrees of freedom $Map(X^3, M^4)$ invariance would imply the flatness of the metric in $M^4$ degrees of freedom.

   The physical implications of the above purely mathematical conjecture should not be underestimated. For example, one natural looking manner to construct physical theory would be based on the idea that configuration space geometry is dynamical and this approach is followed in the attempts to construct string theories [B15] . Various physical considerations (in particular the need to obtain oscillator operator algebra) seem to imply that configuration space geometry is necessarily Kähler. The above result however states that configuration space Kähler geometry cannot be dynamical quantity and is dictated solely by the requirement of internal consistency. This result is extremely nice since it has been already found that the definition of the configuration space metric must somehow associate a unique classical space time and "classical physics" to a given 3-surface: uniqueness of the geometry implies the uniqueness of the "classical physics".

4. The choice of the imbedding space becomes highly unique. In fact, the requirement that configuration space is not only symmetric space but also (contact) Kähler manifold inheriting its (degenerate) Kähler structure from the imbedding space suggests that spaces, which are products of four-dimensional Minkowski space with complex projective spaces $CP_n$, are perhaps the only possible candidates for $H$. The reason for the unique position of the four-dimensional Minkowski space turns out to be that the boundary of the light cone of D-dimensional Minkowski space is metrically a sphere $S^{D-2}$ despite its topological dimension $D-1$: for $D=4$ one obtains two-sphere allowing Kähler structure and infinite parameter group of conformal symmetries!

5. It seems possible to understand the basic mathematical structures appearing in string model in terms of the Kähler geometry rather nicely.

   (a) The projective representations of the infinite-dimensional isometry group (not necessarily Map!) correspond to the ordinary representations of the corresponding centrally extended group [A46]. The representations of Kac Moody group indeed play central role in string models [B40, B22] and configuration space approach would explain their occurrence, not as a result of some quantization procedure, but as a consequence of symmetry of the underlying geometric structure.

   (b) The bosonic oscillator operators of string models would correspond to centrally extended Lie-algebra generators of the isometry group acting on spinor fields of the configuration space.

   (c) The "fermionic" fields ( Ramond fields, [B40, B22] ) should correspond to gamma matrices of the configuration space. Fermionic oscillator operators would correspond simply to contractions of isometry generators $j_A^k$ with complexified gamma matrices of configuration space

$$\begin{aligned} \Gamma_A^{\pm} &= j_A^k \Gamma_k^{\pm} \\ \Gamma_k^{\pm} &= (\Gamma^k \pm J^k_{\ l}\Gamma^l)/\sqrt{2} \end{aligned} \tag{2.2.4}$$

   ($J^k_{\ l}$ is the Kähler form of the configuration space) and would create various spin excitations of the configuration space spinor field. $\Gamma_k^{\pm}$ are the complexified gamma matrices, complexification made possible by the Kähler structure of the configuration space.

This suggests that some generalization of the so called Super Kac Moody algebra of string models [B40, B22] should be regarded as a spectrum generating algebra for the solutions of field equations in configuration space.

Although the Kähler structure seems to be physically well motivated there is a rather heavy counter argument against the whole idea. Kähler structure necessitates complex structure in the tangent space of the configuration space. In $CP_2$ degrees of freedom no obvious problems of principle are expected: configuration space should inherit in some sense the complex structure of $CP_2$.

In Minkowski degrees of freedom the signature of the Minkowski metric seems to pose a serious obstacle for complexification: somehow one should get rid of two degrees of freedom so that only two Euclidian degrees of freedom remain. An analogous difficulty is encountered in quantum field theories: only two of the four possible polarizations of gauge boson correspond to physical degrees of freedom: mathematically the wrong polarizations correspond to zero norm states and transverse states span a complex Hilbert space with Euclidian metric. Also in string model analogous situation occurs: in case of $D$-dimensional Minkowski space only $D-2$ transversal degrees of freedom are physical. The solution to the problem seems therefore obvious: configuration space metric must be degenerate so that each vibrational mode spans effectively a 2-dimensional Euclidian plane allowing complexification.

We shall find that the definition of Kähler function to be proposed indeed provides a solution to this problem and also to the problems listed before.

1. The definition of the metric doesn't differentiate between 1- and N-particle sectors, avoids spin statistics difficulty and has the physically appealing property that one can associate to each 3-surface a unique classical space time: classical physics is described by the geometry of the configuration space and d the geometry of the configuration space is determined uniquely by the requirement of mathematical consistency.

2. Complexification is possible only provided the dimension of the Minkowski space equals to four and is due to the effective 3-dimensionality of light-cone boundary.

3. It is possible to identify a unique candidate for the necessary infinite-dimensional isometry group $G$. $G$ is subgroup of the diffeomorphism group of $\delta M^4_+ \times CP_2$. Essential role is played by the fact that the boundary of the four-dimensional light cone, which, despite being topologically 3-dimensional, is metrically two-dimensional Euclidian sphere, and therefore allows infinite-parameter groups of isometries as well as conformal and symplectic symmetries and also Kähler structure unlike the higher-dimensional light cone boundaries. Therefore configuration space metric is Kähler only in the case of four-dimensional Minkowski space and allows symplectic $U(1)$ central extension without conflict with the no-go theorems about higher dimensional central extensions.

   The study of the vacuum degeneracy of Kähler function defined by Kähler action forces to conclude that the isometry group must consist of the symplectic transformations of $\delta H = \delta M^4_+ \times CP_2$. The corresponding Lie algebra can be regarded as a loop algebra associated with the symplectic group of $S^2 \times CP_2$, where $S^2$ is $r_M = constant$ sphere of light cone boundary. Thus the finite-dimensional group $G$ defining loop group in case of string models extends to an infinite-dimensional group in TGD context. This group has a monstrous size. The radial Virasoro localized with respect to $S^2 \times CP_2$ defines naturally complexification for both $G$ and $H$. The general form of the Kähler metric deduced on basis of this symmetry has same qualitative properties as that deduced from Kähler function identified as preferred extremal of Kähler action. Also the zero modes, among them isometry invariants, can be identified.

4. The construction of the configuration space spinor structure is based on the identification of the configuration space gamma matrices as linear superpositions of the oscillator operators associated with the second quantized induced spinor fields. The extension of the symplectic invariance to super symplectic invariance fixes the anti-commutation relations of the induced spinor fields, and configuration space gamma matrices correspond directly to the super generators. Physics as number theory vision suggests strongly that configuration space geometry exists for 8-dimensional imbedding space only and that the choice $M^4_+ \times CP_2$ for the imbedding space is the only possible one.

## 2.3 Identification of the Kähler function

There are three approaches to the construction of the WCW geometry: a direct physics based guess of the Kähler function, a group theoretic approach based on the hypothesis that $CH$ can be regarded as a union of symmetric spaces, and the approach based on the construction of WCW spinor structure first by second quantization of induced spinor fields. Here the first approach is discussed.

### 2.3.1 Definition of Kähler function

**Kähler metric in terms of Kähler function**

Quite generally, Kähler function $K$ defines Kähler metric in complex coordinates via the following formula

$$J_{k\bar{l}} = ig_{k\bar{l}} = i\partial_k\partial_{\bar{l}}K \quad . \tag{2.3.1}$$

Kähler function is defined only modulo a real part of holomorphic function so that one has the gauge symmetry

$$K \rightarrow K + f + \overline{f} \ . \tag{2.3.2}$$

Let $X^3$ be a given 3-surface and let $X^4$ be any four-surface containing $X^3$ as a sub-manifold: $X^4 \supset X^3$. The 4-surface $X^4$ possesses in general boundary. If the 3-surface $X^3$ has nonempty boundary $\delta X^3$ then the boundary of $X^3$ belongs to the boundary of $X^4$: $\delta X^3 \subset \delta X^4$.

### Induced Kähler form and its physical interpretation

Induced Kähler form defines a Maxwell field and it is important to characterize precisely its relationship to the gauge fields as they are defined in gauge theories. Kähler form $J$ is related to the corresponding Maxwell field $F$ via the formula

$$J = xF \ , \ x = \frac{g_K}{\hbar} \ . \tag{2.3.3}$$

Similar relationship holds true also for the other induced gauge fields. The inverse proportionality of $J$ to $\hbar$ does not matter in the ordinary gauge theory context where one routinely choses units by putting $\hbar = 1$ but becomes very important when one considers a hierachy of Planck constants [K18] .

Unless one has $J = (g_K/\hbar_0)$, where $\hbar_0$ corresponds to the ordinary value of Planck constant, $\alpha_K = g_K^2/4\pi\hbar$ together the large Planck constant means weaker interactions and convergence of the functional integral defined by the exponent of Kähler function and one can argue that the convergence of the functional integral is what forces the hierarchy of Planck constants. This is in accordance with the vision that Mother Nature likes theoreticians and takes care that the perturbation theory works by making a phase transition increasing the value of the Planck constant in the situation when perturbation theory fails. This leads to a replacement of the $M^4$ (or more precisely, causal diamond $CD$) and $CP_2$ factors of the imbedding space ($CD \times CP_2$) with its $r = \hbar/\hbar_0$-fold singular covering (one can consider also singular factor spaces). If the components of the space-time surfaces at the sheets of the covering are identical, one can interpret $r$-fold value of Kähler action as a sum of $r$ identical contributions from the sheets of the covering with ordinary value of Planck constant and forget the presence of the covering. Physical states are however different even in the case that one assumes that sheets carry identical quantum states and anyonic phase could correspond to this kind of phase [K40] .

### Kähler action

One can associate to Kähler form Maxwell action and also Chern-Simons anomaly term proportional to $\int_{X^4} J \wedge J$ in well known manner. Chern Simons term is purely topological term and well defined for orientable 4-manifolds, only. Since there is no deep reason for excluding non-orientable space-time surfaces it seems reasonable to drop Chern Simons term from consideration. Therefore Kähler action $S_K(X^4)$ can be defined as

$$S_K(X^4) = k_1 \int_{X^4;X^3 \subset X^4} J \wedge (*J) \ . \tag{2.3.4}$$

The sign of the square root of the metric determinant, appearing implicitly in the formula, is defined in such a manner that the action density is negative for the Euclidian signature of the induced metric and such that for a Minkowskian signature of the induced metric Kähler electric field gives a negative contribution to the action density.

The notational convention

$$k_1 \equiv \frac{1}{16\pi\alpha_K} \ , \tag{2.3.5}$$

where $\alpha_K$ will be referred as Kähler coupling strength will be used in the sequel. If the preferred extremals minimize/maximize [K54] the absolute value of the action in each region where action density has a definite sign, the value of $\alpha_K$ can depend on space-time sheet.

**Kähler function**

One can define the Kähler function in the following manner. Consider first the case $H = M^4_+ \times CP_2$ and neglect for a moment the non-determinism of Kähler action. Let $X^3$ be a 3-surface at the light-cone boundary $\delta M^4_+ \times CP_2$. Define the value $K(X^3)$ of Kähler function $K$ as the value of the Kähler action for some preferred extremal in the set of four-surfaces containing $X^3$ as a sub-manifold:

$$K(X^3) \; = \; K(X^4_{pref}) \;,\; X^4_{pref} \subset \{X^4|X^3 \subset X^4\} \;. \tag{2.3.6}$$

The most plausible identification of preferred extremals is in terms of quantum criticality in the sense that the preferred extremals allow an infinite number of deformations for which the second variation of Kähler action vanishes. Combined with the weak form of electric-magnetic duality forcing appearence of Kähler coupling strength in the boundary conditions at partonic 2-surfaces this condition might be enough to fix preferred extremals completely.

The precise formulation of Quantum TGD has developed rather slowly. Only quite recently- 33 years after the birth of TGD - I have been forced to reconsider the question whether the precise identification of Kähler function. Should Kähler function actually correspond to the Kähler action for the space-time regions with Euclidian signature having interpretation as generalized Feynman graphs? If so what would be the interpretation for the Minkowskian contribution?

1. If one accepts just the formal definition for the square root of the metric determinant, Minkowskian regions would naturally give an imaginary contribution to the exponent defining the vacuum functional. The presence of the phase factor would give a close connection with the path integral approach of quantum field theories and the exponent of Kähler function would make the functional integral well-defined.

2. The weak form of electric magnetic duality would reduce the contributions to Chern-Simons terms from opposite sides of wormhole throats with degenerate four-metric with a constraint term guaranteeing the duality plus measurement interaction terms describing coupling to isometry charges formally representable as an addition of a gauge term to Chern-Simons-Dirac action. The measurement interaction terms would correspond to couplings to four-momenta at Minkowskian side and color charges at Euclidian side and would give different contributions to the Euclidian and Minkowskian Chern-Simons terms.

The motivation for this reconsideration came from the applications of ideas of Floer homology to TGD framework [K64]: the Minkowskian contribution to Kähler action for preferred extremals would define Morse function providing information about WCW homology. Both Kähler and Morse would find place in TGD based world order.

### 2.3.2 What are the values of the Kähler coupling strength?

Since the vacuum functional of the theory turns out to be essentially the exponent $exp(K)$ of the Kähler function, the dynamics depends on the normalization of the Kähler function. Since the Theory of Everything should be unique it would be highly desirable to find arguments fixing the normalization or equivalently the possible values of the Kähler coupling strength $\alpha_K$. Also a discrete spectrum of values is acceptable.

The quantization of Kähler form could result in the following manner. It will be found that Abelian extension of the isometry group results by coupling spinors of the configuration space to a multiple of Kähler potential. This means that Kähler potential plays role of gauge connection so that Kähler form must be integer valued by Dirac quantization condition for magnetic charge. So, if Kähler form is co-homologically nontrivial it is quantized.

Unfortunately, the exact definition of renormalization group concept is not at all obvious. There is however a much more general but more or less equivalent manner to formulate the condition fixing the value of $\alpha_K$. Vacuum functional $exp(K)$ is analogous to the exponent $exp(-H/T)$ appearing in the definition of the partition function of a statistical system and S-matrix elements and other interesting physical quantities are integrals of type $\langle O\rangle = \int exp(K)O\sqrt{G}dV$ and therefore analogous to the thermal averages of various observables. $\alpha_K$ is completely analogous to temperature. The critical points of a statistical system correspond to critical temperatures $T_c$ for which the partition function is nonanalytic function of $T - T_c$ and according RGE hypothesis critical systems correspond to fixed points of renormalization group evolution. Therefore, a mathematically more precise manner to fix the value of $\alpha_K$ is to require that some integrals of type $\langle O \rangle$ (not necessary S-matrix elements) become nonanalytic at $1/\alpha_K - 1/\alpha_K^c$.

This analogy suggests also a physical motivation for the unique value or value spectrum of $\alpha_K$. Below the critical temperature critical systems suffer something analogous to spontaneous magnetization. At the critical point critical systems are characterized by long range correlations and arbitrarily large volumes of magnetized and non-magnetized phases are present. Spontaneous magnetization might correspond to the generation of Kähler magnetic fields: the most probable 3-surfaces are Kähler magnetized for subcritical values of $\alpha_K$. At the critical values of $\alpha_K$ the most probable 3-surfaces contain regions dominated by either Kähler electric and or Kähler magnetic fields: by the compactness of $CP_2$ these regions have in general outer boundaries.

This suggests that 3-space has hierarchical, fractal like structure: 3-surfaces with all sizes (and with outer boundaries) are possible and they have suffered topological condensation on each other. Therefore the critical value of $\alpha_K$ allows the richest possible topological structure for the most probable 3-space. In fact, this hierarchical structure is in accordance with the basic ideas about renormalization group invariance. This hypothesis has highly nontrivial consequences even at the level of ordinary condensed matter physics.

Renormalization group invariance is closely related with criticality. The self duality of the Kähler form and Weyl tensor of $CP_2$ indeed suggest RG invariance. The point is that in $N = 4$ supersymmetric field theories duality transformation relates the strong coupling limit for ordinary particles with the weak coupling limit for magnetic monopoles and vice versa. If the theory is self-dual these limits must be identical so that action and coupling strength must be RG invariant quantities. This form of self-duality cannot hold true in TGD. The weak form of self-duality discussed in [K10] roughly states that for the partonic 2-surface the induce Kähler electric field is proportional to the Kähler magnetic field strength. The proportionality constant is essentially Kähler coupling strength. The simplest hypothesis is that Kähler coupling strength has single universal valiue and the weak form of self-duality fixes it. The proportionality $\alpha_K = g_K^2/4\pi\hbar$ and the proposed quantization of Planck constant requiring a generalization of the imbedding space imply that Kähler coupling strength varies but is constant at a given page of the "Big Book" defined by the generalized imbedding space [K18] .

### 2.3.3 What preferred extremal property means?

The requirement that the 4-surface having given 3-surface as its sub-manifold is absolute minimum of the Kähler action is the most obvious guess for the principle selecting the preferred extremals and has been taken as a working hypothesis for about one and half decades. Quantum criticality of Quantum TGD should have however led to the idea that preferred extremals are critical in the sense that space-time surface allows deformations for which second variation of Kähler action vanishes so that the corresponding Noether currents are conserved.

Further insights emerged through the realization that Noether currents assignable to the modified Dirac equation are conserved only if the first variation of the modified Dirac operator $D_K$ defined by Kähler action vanishes. This is equivalent with the vanishing of the second variation of Kähler action -at least for the variations corresponding to dynamical symmetries having interpretation as dynamical degrees of freedom which are below measurement resolution and therefore effectively gauge symmetries.

The vanishing of the second variation in interior of $X^4(X_l^3)$ is what corresponds exactly to quantum criticality so that the basic vision about quantum dynamics of quantum TGD would lead directly to a precise identification of the preferred extremals.

The vanishing of second variations of preferred extremals -at least for deformations representing dynamical symmetries, suggests a generalization of catastrophe theory of Thom, where the rank of the matrix defined by the second derivatives of potential function defines a hierarchy of criticalities with the tip of bifurcation set of the catastrophe representing the complete vanishing of this matrix. In the recent case this theory would be generalized to infinite-dimensional context. There are three kind of variables now but quantum classical correspondence (holography) allows to reduce the types of variables to two.

1. The variations of $X^4(X_l^3)$ vanishing at the intersections of $X^4(X_l^3)$ with the light-like boundaries

of causal diamonds $CD$ would represent behavior variables. At least the vacuum extremals of Kähler action would represent extremals for which the second variation vanishes identically (the "tip" of the multi-furcation set).

2. The zero modes of Kähler function would define the control variables interpreted as classical degrees of freedom necessary in quantum measurement theory. By effective 2-dimensionality (or holography or quantum classical correspondence) meaning that the configuration space metric is determined by the data coming from partonic 2-surfaces $X^2$ at intersections of $X^3_l$ with boundaries of $CD$, the interiors of 3-surfaces $X^3$ at the boundaries of $CD$s in rough sense correspond to zero modes so that there is indeed huge number of them. Also the variables characterizing 2-surface, which cannot be complexified and thus cannot contribute to the Kähler metric of configuration space represent zero modes. Fixing the interior of the 3-surface would mean fixing of control variables. Extremum property would fix the 4-surface and behavior variables if boundary conditions are fixed to sufficient degree.

3. The complex variables characterizing $X^2$ would represent third kind of variables identified as quantum fluctuating degrees of freedom contributing to the configuration space metric. Quantum classical correspondence requires 1-1 correspondence between zero modes and these variables. This would be essentially holography stating that the 2-D "causal boundary" $X^2$ of $X^3(X^2)$ codes for the interior. Preferred extremal property identified as criticality condition would realize the holography by fixing the values of zero modes once $X^2$ is known and give rise to the holographic correspondence $X^2 \rightarrow X^3(X^2)$. The values of behavior variables determined by extremization would fix then the space-time surface $X^4(X^3_l)$ as a preferred extremal.

4. Clearly, the presence of zero modes would be absolutely essential element of the picture. Quantum criticality, quantum classical correspondence, holography, and preferred extremal property would all represent more or less the same thing. One must of course be very cautious since the boundary conditions at $X^3_l$ involve normal derivative and might bring in delicacies forcing to modify the simplest heuristic picture.

One must be very cautious with what one means with the preferred extremal property and criticality.

1. Does one assign criticality with the partonic 2-surfaces at the ends of $CD$s? Does one restrict it with the throats for which light-like 3-surface has also degenerate induced 4-metric? Or does one assume stronger form of holography requiring a slicing of space-time surface by partonic 2-surfaces and string world sheets and assign criticality to all partonic 2-surfaces. This kind of slicing is suggested by the study of the extremals [K5] , required by the number theoretic vision ($M^8-H$ duality [K52] ), and also by the purely physical condition that a stringy realization of GCI is possible.

2. What is the exact meaning of the preferred extremal property? The assumption that the variations of Kähler action leaving 3-surfaces at the ends of $CD$s invariant would not be consistent with the effective 2-dimensionality. The assumption that the critical deformations leave invariant only partonic 2-surfaces would imply genuine 2-dimensionality. Should one assume that critical deformations leave invariant partonic 2-surface and 3-D tangent space in the direction of space-like 3-surface or light-like 3-surface but not both. This would be consistent with effective 3-dimensionality and would explain why Kac-Moody symmetries associated with the light-like 3-surfaces act as gauge symmetries. This is also essential for the realization of Poincare invariance since the quantization of the light-cone proper time distance between $CD$s implies that infinitesimal Poincare transformations lead out of $CD$ unless compensated by Kac-Moody type transformations acting like gauge transformations. In the similar manner it would explain why symplectic transformations of $\delta CD$ act like gauge transformations.

3. Could one pose the criticality condition for all partonic 2-surfaces in the slicing or only for the throats of light-like 3-surfaces? This hypothesis looks natural but is not necessary. Light-like throats are very singular objects criticality might apply only to their variations only in the limiting sense and it might be necesary to assume criticality for all partonic 2-surfaces.

### 2.3.4 Why non-local Kähler function?

Kähler function is nonlocal functional of 3-surface. Non-locality of the Kähler function seems to be at odds with basic assumptions of local quantum field theories. Why this rather radical departure from the basic assumptions of local quantum field theory? The answer is shortly given: configuration space integration appears in the definition of the inner product for WCW spinor fields and this inner product must be free from perturbative divergences. Consider now the argument more closely.

In the case of finite-dimensional symmetric space with Kähler structure the representations of the isometry group necessitate the modification of the integration measure defining the inner product so that the integration measure becomes proportional to the exponent $exp(K)$ of the Kähler function [B42]. The generalization to infinite-dimensional case is obvious. Also the requirement of Kac-Moody symmetry leads to the presence of this kind of vacuum functional as will be found later. The exponent is in fact uniquely fixed by finiteness requirement. Configuration space integral is of the following form

$$\int \bar{S}_1 exp(K) S_1 \sqrt{g} dX \quad . \tag{2.3.7}$$

One can develop perturbation theory using local complex coordinates around a given 3-surface in the following manner. The $(1,1)$-part of the second variation of the Kähler function defines the metric and therefore propagagator as contravariant metric and the remaining $(2,0)-$ and $(0,2)$-parts of the second variation are treated perturbatively. The most natural choice for the 3-surface are obviously the 3-surfaces, which correspond to extrema of the Kähler function.

When perturbation theory is developed around the 3-surface one obtains two ill-defined determinants.

1. The Gaussian determinant coming from the exponent, which is just the inverse square root for the matrix defined by the metric defining $(1,1)$-part of the second variation of the Kähler function in local coordinates.

2. The metric determinant. The matrix representing covariant metric is however same as the matrix appearing in Gaussian determinant by the defining property of the Kähler metric: in local complex coordinates the matrix defined by second derivatives is of type $(1,1)$. Therefore these two ill defined determinants (recall the presence of Diff degeneracy) cancel each other exactly for a unique choice of the vacuum functional!

Of course, the cancellation of the determinants is not enough. For an arbitrary local action one encounters the standard perturbative divergences. Since most local actions (Chern-Simons term is perhaps an exception [B24] ) for induced geometric quantities are extremely nonlinear there is no hope of obtaining a finite theory. For nonlocal action the situation is however completely different. There are no local interaction vertices and therefore no products of delta functions in perturbation theory.

A further nice feature of the perturbation theory is that the propagator for small deformations is nothing but the contravariant metric. Also the various vertices of the theory are closely related to the metric of the configuration space since they are determined by the Kähler function so that perturbation theory would have a beautiful geometric interpretation. Furthermore, since four-dimensional Diff degeneracy implies that the propagator doesn't couple to un-physical modes.

It should be noticed that divergence cancellation arguments do not necessarily exclude Chern Simons term from vacuum functional defined as imaginary exponent of $exp(ik_2 \int_{X^4} J \wedge J)$. The term is not well defined for non-orientable space-time surfaces and one must assume that $k_2$ vanishes for these surfaces. The presence of this term might provide first principle explanation for CP breaking. If $k_2$ is integer multiple of $1/(8\pi)$ Chern Simons term gives trivial contribution for closed space-time surfaces since instanton number is in question. By adding a suitable boundary term of form $exp(ik_3 \int_{\delta X^3} J \wedge A)$ it is possible to guarantee that the exponent is integer valued for 4-surfaces with boundary, too.

There are two arguments suggesting that local Chern Simons term would not introduce divergences. First, 3-dimensional Chern Simons term for ordinary Abelian gauge field is known to define a divergence free field theory [B24] . The term doesn't depend at all on the induced metric and therefore contains no dimensional parameters ($CP_2$ radius) and its expansion in terms of $CP_2$ coordinate variables is of the form allowed by renormalizable field theory in the sense that only quartic terms appear. This is seen by noticing that there always exist symplectic coordinates, where the expression of the Kähler potential is of the form

$$A = \sum_k P_k dQ^k \ . \tag{2.3.8}$$

The expression for Chern-Simons term in these coordinates is given by

$$k_2 \int_{X^3} \sum_{k,l} P_l dP_k \wedge dQ^k \wedge dQ^l \ , \tag{2.3.9}$$

and clearly quartic $CP_2$ coordinates. A further nice property of the Chern Simons term is that this term is invariant under symplectic transformations of $CP_2$, which are realized as $U(1)$ gauge transformation for the Kähler potential.

## 2.4 Some properties of Kähler action

In this section some properties of Kähler action and Kähler function are discussed in light of experienced gained during about 15 years after the introduction of the notion.

### 2.4.1 Vacuum degeneracy and some of its implications

The vacuum degeneracy is perhaps the most characteristic feature of the Kähler action. Although it is not associated with the preferred extremals of Kähler action, there are good reasons to expect that it has deep consequences concerning the structure of the theory.

#### Vacuum degeneracy of the Kähler action

The basic reason for choosing Kähler action is its enormous vacuum degeneracy, which makes long range interactions possible (the well known problem of the membrane theories is the absence of massless particles [B38] ). The Kähler form of $CP_2$ defines symplectic structure and any 4-surface for which $CP_2$ projection is so called Lagrangian manifold (at most two dimensional manifold with vanishing induced Kähler form), is vacuum extremal due to the vanishing of the induced Kähler form. More explicitly, in the local coordinates, where the vector potential $A$ associated with the Kähler form reads as $A = \sum_k P_k dQ^k$. Lagrangian manifolds are expressible locally in the following form

$$P_k = \partial_k f(Q^i) \ . \tag{2.4.1}$$

where the function $f$ is arbitrary. Notice that for the general $YM$ action surfaces with one-dimensional $CP_2$ projection are vacuum extremals but for Kähler action one obtains additional degeneracy.

There is also a second kind of vacuum degeneracy, which is relevant to the elementary particle physics. The so called $CP_2$ type vacuum extremals are warped imbeddings $X^4$ of $CP_2$ to $H$ such that Minkowski coordinates are functions of a single $CP_2$ coordinate, and the one-dimensional projection of $X^4$ is random light like curve. These extremals have a non-vanishing action but vanishing Poincare charges. Their small deformations are identified as space-time counterparts of fermions and their super partners. Wormhole throats identified as pieces of these extremals are identified as bosons and their super partners.

The conditions stating light likeness are equivalent with the Virasoro conditions of string models and this actually led to the eventualo realization that conformal invariance is a basic symmetry of TGD and that WCW can be regarded as a union of symmetric spaces with isometry groups having identification as symplectic and Kac-Moody type groups assignable to the partonic 2-surfaces.

### Approximate symplectic invariance

Vacuum extremals have diffeomorphisms of $M^4_+$ and $M^4_+$ local symplectic transformations as symmetries. For non-vacuum extremals these symmetries leave induced Kähler form invariant and only induced metric breaks these symmetries. Symplectic transformations of $CP_2$ act on the Maxwell field defined by the induced Kähler form in the same manner as ordinary $U(1)$ gauge symmetries. They are however not gauge symmetries since gauge invariance is still present. In fact, the construction of the configuration space geometry relies on the assumption that symplectic transformations of $\delta M^4_+ \times CP_2$ which infinitesimally correspond to combinations of $M^4_+$ local $CP_2$ symplectic and $CP_2$-local $M^4_+$ symplectic transformations act as isometries of the configuration space. In zero energy ontology these transformations act simultanoeusly on all partonic 2-surfaces characterizing the space-time sheet representing a generalized Feynman diagram inside $CD$.

The fact that $CP_2$ symplectic transformations do not act as genuine gauge transformations means that $U(1)$ gauge invariance is effectively broken. This has non-trivial implications. The field equations allow purely geometric vacuum 4-currents not possible in Maxwell's electrodynamics [K5] . For the known extremals (massless extremals) they are light-like and a possible interpretation is in terms of Bose-Einstein condensates of collinear massless bosons.

### Spin glass degeneracy

Vacuum degeneracy means that all surfaces belonging to $M^4_+ \times Y^2$, $Y^2$ any Lagrangian sub-manifold of $CP_2$ are vacua irrespective of the topology and that symplectic transformations of $CP_2$ generate new surfaces $Y^2$. If preferred extremals are obtained as small deformations of vacuum extremals (for which the criticality is maximal), one expects therefore enormous ground state degeneracy, which could be seen as 4-dimensional counterpart of the spin glass degeneracy. This degeneracy corresponds to the hypothesis that configuration space is a union of symmetric spaces labeled by zero modes which do not appear at the line-element of the configuration space metric.

Zero modes define what might be called the counterpart of spin glass energy landscape and the maxima Kähler function as a function of zero modes define a discrete set which might be called reduced configuration space. Spin glass degeneracy turns out to be crucial element for understanding how macro-temporal quantum coherence emerges in TGD framework. One of the basic ideas about p-adicization is that the maxima of Kähler function define the TGD counterpart of spin glass energy landscape [K53, K21] . The hierarchy of discretizations of the symmetric spaces corresponding to a hierarchy of measurement resolutions [K19] could allow an identification in terms of a hierarchy spin glass energy landscapes so that the algebraic points of the WCW would correspond to the maxima of Kähler function. The hierarchical structure would be due to the failure of strict non-determinism of Kähler action allowing in zero energy ontology to add endlessly details to the space-time sheets representing zero energy states in shorter scale.

### Generalized quantum gravitational holography

The original naive belief was that the construction of the configuration space geometry reduces to $\delta H = \delta M^4_+ \times CP_2$. An analogous idea in string model context became later known as quantum gravitational holography. The basic implication of the vacuum degeneracy is classical non-determinism, which is expected to reflect itself as the properties of the Kähler function and configuration space geometry. Obviously classical non-determinism challenges the notion of quantum gravitational holography.

The hope was that a generalization of the notion of 3-surface is enough to get rid of the degeneracy and save quantum gravitational holography in its simplest form. This would mean that one just replaces space-like 3-surfaces with "association sequences" consisting of sequences of space-like 3-surfaces with time like separations as causal determinants. This would mean that the absolute minima of Kähler function would become degenerate: same space-like 3-surface at $\delta H$ would correspond to several association sequences with the same value of Kähler function.

The life turned out to be more complex than this. $CP_2$ type extremals have Euclidian signature of the induced metric and therefore $CP_2$ type extremals glued to space-time sheet with Minkowskian signature of the induced metric are surrounded by light like surfaces $X^3_l$, which might be called elementary particle horizons. The non-determinism of the $CP_2$ type extremals suggests strongly that also elementary particle horizons behave non-deterministically and must be regarded as causal determinants having time like projection in $M^4_+$. Pieces of $CP_2$ type extremals are good candidates for the wormhole contacts connecting a space-time sheet to a larger space-time sheet and are also surrounded by an elementary particle horizons and non-determinism is also now present. That this non-determinism would allow the proposed simple description seems highly implausible.

Zero energy ontology realized in terms of a hierarchy of $CD$s seems to provide the most plausible treatment of the non-determinism and has indeed led to a breakthrough in the construction and understanding of quantum TGD. At the level of generalized Feynman diagrams sub-$CD$s containing zero energy states represent a hierarchy of radiative corrections so that the classical determinism is direct correlate for the quantum non-determinism. Determinism makes sense only when one has specified the length scale of measurement resolution. One can always add a $CD$ containing a vacuum extremal to get a new zero energy state and a preferred extremal containing more details.

#### Classical non-determinism saves the notion of time

Although classical non-determinism represents a formidable mathematical challenge it is a must for several reasons. Quantum classical correspondence, which has become a basic guide line in the development of TGD, states that all quantum phenomena have classical space-time correlates. This is not new as far as properties of quantum states are considered. What is new that also quantum jumps and quantum jump sequences which define conscious existence in TGD Universe, should have classical space-time correlates: somewhat like written language is correlate for the contents of consciousness of the writer. Classical non-determinism indeed makes this possible. Classical non-determinism makes also possible the realization of statistical ensembles as ensembles formed by strictly deterministic pieces of the space-time sheet so that even thermodynamics has space-time representations. Space-time surface can thus be seen as symbolic representations for the quantum existence.

In canonically quantized general relativity the loss of time is fundamental problem. If quantum gravitational holography would work in the most strict sense, time would be lost also in TGD since all relevant information about quantum states would be determined by the moment of big bang. More precisely, geometro-temporal localization for the contents of conscious experience would not be possible. Classical non-determinism together with quantum-classical correspondence however suggests that it is possible to have quantum jumps in which non-determinism is concentrated in space-time region so that also conscious experience contains information about this region only.

### 2.4.2 Four-dimensional General Coordinate Invariance

The proposed definition of the Kähler function is consistent with GCI and implies also 4-dimensional Diff degeneracy of the Kähler metric. Zero energy ontology inspires strengthening of the GCI in the sense that space-like 3-surfaces at the boundaries of $CD$ are physically equivalent with the light-like 3-surfaces connecting the ends. This imples that basic geometric objects are partonic 2-surfaces at the boundaries of $CD$s identified as the intersections of these two kinds of surfaces. Besides this the distribution of 4-D tangent planes at partonic 2-surfaces would code for physics so that one would have only effective 2-dimensionality. The failure of the non-determinism of Kähler action in the standard sense of the word affects the situation also and one must allow a fractal hierarchy of $CD$s inside $CD$s having interpretation in terms of radiative corrections.

#### Resolution of tachyon difficulty and absence of Diff anomalies

In TGD as in string models the tachyon difficulty is potentially present: unless the time like vibrational excitations possess zero norm they contribute tachyonic term to the mass squared operator of Super Kac Moody algebra. This difficulty is familiar already from string models [B40, B22] .

The degeneracy of the metric with respect to the time like vibrational excitations guarantees that time like excitations do not contribute to the mass squared operator so that mass spectrum is tachyon free. It also implies the decoupling of the tachyons from physical states: the propagator of the theory corresponds essentially to the inverse of the Kähler metric and therefore decouples from time like vibrational excitations. The experience with string model suggests that if metric is degenerate with respect to diffeomorphisms of $X^4(X^3)$ there are indeed good hopes that time like excitations possess vanishing norm with respect to configuration space metric.

The four-dimensional Diff invariance of the Kähler function implies that Diff invariance is guaranteed in the strong sense since the scalar product of two Diff vector fields given by the matrix associated with $(1, 1)$ part of the second variation of the Kähler action vanishes identically. This property gives hopes of obtaining theory, which is free from Diff anomalies: in fact loop space metric is not Diff degenerate and this might be the underlying reason to the problems encountered in string models [B40, B22] .

### Complexification of the configuration space

Strong form of GCI plays a fundamental role in the complexification of the configuration space. GCI in strong form reduces the basic building brick of WCW to the pairs of partonic 2-surfaces and their 4-D tangent space data associated with ends of light-like 3-surface at light-like boundaries of $CD$. At boths end the imbedding space is effectively reduces to $\delta M^4_+ \times CP_2$ (forgetting the complications due to non-determinism of Kähler action). Light cone boundary in turn is metrically 2-dimensional Euclidian sphere allowing infinite-dimensional group of conformal symmetries and Kähler structure. Therefore one can say that in certain sense configuration space metric inherits the Kähler structure of $S^2 \times CP_2$. This mechanism works in case of four-dimensional Minkowski space only: higher-dimensional spheres do not possess even Kähler structure. In fact, it turns out that the quantum fluctuating degrees of freedom can be regarded in well-defined sense as a local variant of $S^2 \times CP_2$ and thus as an infinite-dimensional analog of symmetric space as the considerations of [K10] demonstrate.

The details of the complexification were understood only after the construction of configuration space geometry and spinor structure in terms of second quantized induced spinor fields [K9] . This also allows to make detailed statements about complexification [K10] .

### Contravariant metric and Diff$^4$ degeneracy

Diff degeneracy implies that the definition of the contravariant metric, which corresponds to the propagator associated to small deformations of minimizing surface is not quite straightforward. We believe that this problem is only technical. Certainly this problem is not new, being encountered in both GRT and gauge theories [B17, B16] . In TGD a solution of the problem is provided by the existence of infinite-dimensional isometry group. If the generators of this group form a complete set in the sense that any vector of the tangent space is expressible as as sum of these generators plus some zero norm vector fields then one can restrict the consideration to this subspace and in this subspace the matrix $g(X, Y)$ defined by the components of the metric tensor indeed indeed possesses well defined inverse $g^{-1}(X, Y)$. This procedure is analogous to gauge fixing conditions in gauge theories and coordinate fixing conditions in General Relativity.

It has turned that the representability of WCW as a union of symmetric spaces makes possible an approach to WCW integration based on harmonic analysis replacing the perturbative approach based on perturbative functional integral. This approach allows also a p-adic variant and leads an effective discretization in terms of discrete variants of WCW for which the points of symmetric space consist of algebraic points. There is an infinite number of these discretizations [K53] and the interpretation is in terms of finite measurement resolution. This gives a connection with the p-adicization program, infinite primes, inclusions of hyper-finite factors as representation of the finite measurement resolution, and the hierarchy of Planck constants [K52] so that various approaches to quantum TGD converge nicely.

### General Coordinate Invariance and WCW spinor fields

GCI applies also at the level of quantum states. WCW spinor fields are Diff$^4$ invariant. This in fact fixes not only classical but also quantum dynamics completely. The point is that the values of the configuration space spinor fields must be essentially same for all Diff$^4$ related 3-surfaces at the orbit $X^4$ associated with a given 3-surface. This would mean that the time development of Diff$^4$ invariant configuration spinor field is completely determined by its initial value at the moment of the big bang!

This is of course a naive over statement. The non-determinism of Kähler action and zero energy ontology force to take the causal diamond ($CD$) defined by the intersection of future and past directed light-cones as the basic structural unit of configuration space, and there is fractal hierarchy of $CD$s within $CD$s so that the above statement makes sense only for giving $CD$ in measurement resolution neglecting the presence of smaller $CD$s. Strong form of GCI also implies factorization of WCW spinor fields into a sum of products associated with various partonic 2-surfaces. In particular, one obtains time-like entanglement between positive and negative energy parts of zero energy states and entanglement coefficients define what can be identified as $M$-matrix expressible as a "complex square root" of density matrix and reducing to a product of positive definite diagonal square root of density matrix and unitary S-matrix. The collection of orthonormal $M$-matrices in turn define unitary $U$-matrix between zero energy states. $M$-matrix is the basic object measured in particle physics laboratory.

### 2.4.3 Configuration space geometry, generalized catastrophe theory, and phase transitions

The definition of configuration space geometry has nice catastrophe theoretic interpretation. To understand the connection consider first the definition of the ordinary catastrophe theory [A74] .

1. In catastrophe theory one considers extrema of the potential function depending on dynamical variables $x$ as function of external parameters $c$. The basic space decomposes locally into cartesian product $E = C \times X$ of control variables $c$, appearing as parameters in potential function $V(c, x)$ and of state variables $x$ appearing as dynamical variables. Equilibrium states of the system correspond to the extrema of the potential $V(x, c)$ with respect to the variables $x$ and in the absence of symmetries they form a sub-manifold of $M$ with dimension equal to that of the parameter space $C$. In some regions of $C$ there are several extrema of potential function and the extremum value of $x$ as a function of $c$ is multi-valued. These regions of $C \times X$ are referred to as catastrophes. The simplest example is cusp catastrophe (see Fig. 2.4.3) with two control parameters and one state variable.

2. In catastrophe regions the actual equilibrium state must be selected by some additional physical requirement. If system obeys flow dynamics defined by first order differential equations the catastrophic jumps take place along the folds of the cusp catastrophe (delay rule). On the other hand, the Maxwell rule obeyed by thermodynamic phase transitions states that the equilibrium state corresponds to the absolute minimum of the potential function and the state of system changes in discontinuous manner along the Maxwell line in the middle between the folds of the cusp (see Fig. 2.4.3).

3. As far as discontinuous behavior is considered fold catastrophe is the basic catastrophe: all catastrophes contain folds as there 'satellites' and one aim of the catastrophe theory is to derive all possible manners for the stable organization of folds into higher catastrophes. The fundamental result of the catastrophe theory is that for dimensions $d$ of $C$ smaller than 5 there are only 7 basic catastrophes and polynomial potential functions provide a canonical representation for the catastrophes: fold catastrophe corresponds to third order polynomial (in fold the two real roots become a pair of complex conjugate roots), cusp to fourth order polynomial, etc.

Consider now the TGD counterpart of this. TGD allows allows two kinds of catastrophe theories.

1. The first one is related to Kähler action as a local functional of 4-surface. The nature of this catastrophe theory depends on what one means with the preferred extremals.

2. Second catastrophe theory corresponds to Kähler function a non-local functional of 3-surface. The maxima of the vacuum functional defined as the exponent of Kähler function define what might called effective space-times, and discontinuous jumps changing the values of the parameters characterizing the maxima are possible.

Consider first the option based on Kähler action.

1. Potential function corresponds to Kähler action restricted to the solutions of Euler Lagrange equations. Catastrophe surface corresponds to the four-surfaces found by extremizing Kähler action with respect to to the variables of $X$ (time derivatives of coordinates of $C$ specifying $X^3$ in $H_a$) keeping the variables of $C$ specifying 3-surface $X^3$ fixed. Preferred extremal property is analogous to the Bohr quantization since canonical momenta cannot be chosen freely as in the ordinary initial value problems of the classical physics. Preferred extremals are by definition at criticality. Behavior variables correspond to the deformations of the 4-surface keeping partonic 2-surfaces and 3-D tangent space data fixed and preserving extremal property. Control variables would correspond to these data.

2. At criticality the rank of the infinite-dimensional matrix defined by the second functional derivatives of the Kähler action is reduced. Catastrophes form a hierarchy characterized by the reduction of the rank of this matrix and Thom's catastrophe theory generalizes to infinite-dimensional context. Criticality in this sense would be one aspect of quantum criticality having also other aspects. No discrete jumps would occur and system would only move along the critical surface becoming more or less critical.

3. There can exist however several critical extremals assignable to a given partonic 2-surface but have nothing to do with the catastrophes as defined in Thom's approach. In presence of degeneracy one should be able to choose one of the critical extremals or replace this kind of regions of WCW by their multiple coverings so that single partonic 2-surface is replaced with its multiple copy. The degeneracy of the preferred extremals could be actually a deeper reason for the hierarchy of Planck constants involving in its most plausible version n-fold singular coverings of $CD$ and $CP_2$. This interpretation is very satisfactory since the generalization of the imbedding space and hierarchy of Planck constants would follow naturally from quantum criticality rather than as separate hypothesis.

4. The existence of the catastrophes is implied by the vacuum degeneracy of the Kähler action. For example, for pieces of Minkowski space in $M^4_+ \times CP_2$ the second variation of the Kähler action vanishes identically and only the fourth variation is non-vanishing: these 4-surfaces are analogous to the tip of the cusp catastrophe. There are also space-time surfaces for which the second variation is non-vanishing but degenerate and a hierarchy of subsets in the space of extremal 4-surfaces with decreasing degeneracy of the second variation defines the boundaries of the projection of the catastrophe surface to the space of 3-surfaces. The space-times for which second variation is degenerate contain as subset the critical and initial value sensitive absolute minimum space-times.

Consider next the catastrophe theory defined by Kähler function.

1. In this case the most obvious identification for the behavior variables would be in terms of the space of all 3-surfaces in $CD \times CP_2$ - and if one believes in holography and zero energy ontology - the 2-surfaces assignable the boundaries of causal diamonds ($CD$s).

2. The natural control variables are zero modes whereas behavior variables would correspond to quantum fluctuating degrees of freedom contributing to the configuration space metric. The induced Kähler form at partonic 2-surface would define infinitude of purely classical control variables. There is also a correlation between zero modes identified as degrees of freedom assignable to the interior of 3-surface and quantum fluctuating degrees of freedom assigned to the partonic 2-surfaces. This is nothing but holography and effective 2-dimensionality justifying the basic assumption of quantum measurement theory about the correspondence between classical and quantum variables. The absence of several maxima implies also the presence of saddle surfaces at which the rank of the matrix defined by the second derivatives is reduced. This could lead to a non-positive definite metric. It seems that it is possible to have maxima of Kähler function without losing positive definiteness of the metric since metric is defined as (1,1)-type derivatives with respect to complex coordinates. In case of $CP_2$ however Kähler function has single degenerate maximum corresponding to the homologically trivial geodesic sphere at $r = \infty$. It might happen that also in the case of infinite-D symmetric space finite maxima are impossible.

3. The criticality of Kähler function would be analogous to thermodynamical criticality and to the criticality in the sense of catastrophe theory. In this case Maxwell's rule is possible and even plausible since quantum jump replaces the dynamics defined by a continuous flow.

Cusp catastrophe provides a simple concretization of the situation for the criticality of Kähler action (as distinguished from that for Kähler function).

1. The set $M$ of the critical 4-surfaces corresponds to the $V$-shaped boundary of the 2-D cusp catastrophe in 3-D space to plane. In general case it forms codimension one set in configuration space. In TGD Universe physical system would reside at this line or its generalization to higher dimensional catastrophes. For the criticality associated with Kähler action the transitions would

be smooth transitions between different criticalities characterized by the rank defined above: in the case of cusp from the tip of cusp to the vertex of cusp or vice versa. Evolution could mean a gradual increase of criticality in this sense. If preferred extremals are not unique, cusp catastrophe does not provide any analogy. The strong form of criticality would mean that the system would be always "at the tip of cusp" in metaphoric sense. Vacuum extremals are maximally critical in trivial sense, and the deformations of vacuum extremals could define the hierarchy of criticalities.

2. For the criticality of Kähler action Maxwell's rule stating that discontinuous jumps occur along the middle line of the cusp is in conflict with catastrophe theory predicting that jumps occurs along at criticality. For the criticality of Kähler function -if allowed at all by symmetric space property- Maxwell's rule can hold true but cannot be regarded as a fundamental law. It is of course known that phase transitions can occur in different manners (super heating and super cooling).


Figure 2.1: Cusp catastrophe

## 2.5 Weak form electric-magnetic duality and its implications

The notion of electric-magnetic duality [B8] was proposed first by Olive and Montonen and is central in $\mathcal{N} = 4$ supersymmetric gauge theories. It states that magnetic monopoles and ordinary particles are two different phases of theory and that the description in terms of monopoles can be applied at the limit when the running gauge coupling constant becomes very large and perturbation theory fails to converge. The notion of electric-magnetic self-duality is more natural since for $CP_2$ geometry Kähler form is self-dual and Kähler magnetic monopoles are also Kähler electric monopoles and Kähler coupling strength is by quantum criticality renormalization group invariant rather than running coupling constant. The notion of electric-magnetic (self-)duality emerged already two decades ago in the attempts to formulate the Kähler geometric of world of classical worlds. Quite recently a considerable step of progress took place in the understanding of this notion [K10] . What seems to be essential is that one adopts a weaker form of the self-duality applying at partonic 2-surfaces. What this means will be discussed in the sequel.

Every new idea must be of course taken with a grain of salt but the good sign is that this concept leads to precise predictions. The point is that elementary particles do not generate monopole fields in macroscopic length scales: at least when one considers visible matter. The first question is whether elementary particles could have vanishing magnetic charges: this turns out to be impossible. The next question is how the screening of the magnetic charges could take place and leads to an identification of the physical particles as string like objects identified as pairs magnetic charged wormhole throats connected by magnetic flux tubes.

1. The first implication is a new view about electro-weak massivation reducing it to weak confinement in TGD framework. The second end of the string contains particle having electroweak isospin neutralizing that of elementary fermion and the size scale of the string is electro-weak scale would be in question. Hence the screening of electro-weak force takes place via weak confinement realized in terms of magnetic confinement.

2. This picture generalizes to the case of color confinement. Also quarks correspond to pairs of magnetic monopoles but the charges need not vanish now. Rather, valence quarks would be connected by flux tubes of length of order hadron size such that magnetic charges sum up to zero. For instance, for baryonic valence quarks these charges could be $(2, -1, -1)$ and could be proportional to color hyper charge.

3. The highly non-trivial prediction making more precise the earlier stringy vision is that elementary particles are string like objects in electro-weak scale: this should become manifest at LHC energies.

4. The weak form electric-magnetic duality together with Beltrami flow property of Kähler leads to the reduction of Kähler action to Chern-Simons action so that TGD reduces to almost topological QFT and that Kähler function is explicitly calculable. This has enormous impact concerning practical calculability of the theory.

5. One ends up also to a general solution ansatz for field equations from the condition that the theory reduces to almost topological QFT. The solution ansatz is inspired by the idea that all isometry currents are proportional to Kähler current which is integrable in the sense that the flow parameter associated with its flow lines defines a global coordinate. The proposed solution ansatz would describe a hydrodynamical flow with the property that isometry charges are conserved along the flow lines (Beltrami flow). A general ansatz satisfying the integrability conditions is found. The solution ansatz applies also to the extremals of Chern-Simons action and and to the conserved currents associated with the modified Dirac equation defined as contractions of the modified gamma matrices between the solutions of the modified Dirac equation. The strongest form of the solution ansatz states that various classical and quantum currents flow along flow lines of the Beltrami flow defined by Kähler current (Kähler magnetic field associated with Chern-Simons action). Intuitively this picture is attractive. A more general ansatz would allow several Beltrami flows meaning multi-hydrodynamics. The integrability conditions boil down to two scalar functions: the first one satisfies massless d'Alembert equation in the induced metric and the the gradients of the scalar functions are orthogonal. The interpretation in terms of momentum and polarization directions is natural.

6. The general solution ansatz works for induced Kähler Dirac equation and Chern-Simons Dirac equation and reduces them to ordinary differential equations along flow lines. The induced spinor fields are simply constant along flow lines of indued spinor field for Dirac equation in suitable gauge. Also the generalized eigen modes of the modified Chern-Simons Dirac operator can be deduced explicitly if the throats and the ends of space-time surface at the boundaries of $CD$ are extremals of Chern-Simons action. Chern-Simons Dirac equation reduces to ordinary differential equations along flow lines and one can deduce the general form of the spectrum and the explicit representation of the Dirac determinant in terms of geometric quantities characterizing the 3-surface (eigenvalues are inversely proportional to the lengths of strands of the flow lines in the effective metric defined by the modified gamma matrices).

### 2.5.1 Could a weak form of electric-magnetic duality hold true?

Holography means that the initial data at the partonic 2-surfaces should fix the configuration space metric. A weak form of this condition allows only the partonic 2-surfaces defined by the wormhole throats at which the signature of the induced metric changes. A stronger condition allows all partonic 2-surfaces in the slicing of space-time sheet to partonic 2-surfaces and string world sheets. Number theoretical vision suggests that hyper-quaternionicity *resp.* co-hyperquaternionicity constraint could be enough to fix the initial values of time derivatives of the imbedding space coordinates in the space-time regions with Minkowskian *resp.* Euclidian signature of the induced metric. This is a condition on modified gamma matrices and hyper-quaternionicity states that they span a hyper-quaternionic sub-space.

#### Definition of the weak form of electric-magnetic duality

One can also consider alternative conditions possibly equivalent with this condition. The argument goes as follows.

1. The expression of the matrix elements of the metric and Kähler form of $WCW$ in terms of the Kähler fluxes weighted by Hamiltonians of $\delta M^4_\pm$ at the partonic 2-surface $X^2$ looks very attractive. These expressions however carry no information about the 4-D tangent space of the partonic 2-surfaces so that the theory would reduce to a genuinely 2-dimensional theory, which cannot hold true. One would like to code to the WCW metric also information about the electric part of the induced Kähler form assignable to the complement of the tangent space of $X^2 \subset X^4$.

2. Electric-magnetic duality of the theory looks a highly attractive symmetry. The trivial manner to get electric magnetic duality at the level of the full theory would be via the identification of the flux Hamiltonians as sums of of the magnetic and electric fluxes. The presence of the induced metric is however troublesome since the presence of the induced metric means that the simple transformation properties of flux Hamiltonians under symplectic transformations -in particular color rotations- are lost.

3. A less trivial formulation of electric-magnetic duality would be as an initial condition which eliminates the induced metric from the electric flux. In the Euclidian version of 4-D YM theory this duality allows to solve field equations exactly in terms of instantons. This approach involves also quaternions. These arguments suggest that the duality in some form might work. The full electric magnetic duality is certainly too strong and implies that space-time surface at the partonic 2-surface corresponds to piece of $CP_2$ type vacuum extremal and can hold only in the deep interior of the region with Euclidian signature. In the region surrounding wormhole throat at both sides the condition must be replaced with a weaker condition.

4. To formulate a weaker form of the condition let us introduce coordinates $(x^0, x^3, x^1, x^2)$ such $(x^1, x^2)$ define coordinates for the partonic 2-surface and $(x^0, x^3)$ define coordinates labeling partonic 2-surfaces in the slicing of the space-time surface by partonic 2-surfaces and string world sheets making sense in the regions of space-time sheet with Minkowskian signature. The assumption about the slicing allows to preserve general coordinate invariance. The weakest condition is that the generalized Kähler electric fluxes are apart from constant proportional to Kähler magnetic fluxes. This requires the condition

$$J^{03}\sqrt{g_4} = KJ_{12} \ . \quad (2.5.1)$$

A more general form of this duality is suggested by the considerations of [K24] reducing the hierarchy of Planck constants to basic quantum TGD and also reducing Kähler function for preferred extremals to Chern-Simons terms [B1] at the boundaries of $CD$ and at light-like wormhole throats. This form is following

$$J^{n\beta}\sqrt{g_4} = K\epsilon \times \epsilon^{n\beta\gamma\delta}J_{\gamma\delta}\sqrt{g_4} \ . \quad (2.5.2)$$

Here the index $n$ refers to a normal coordinate for the space-like 3-surface at either boundary of $CD$ or for light-like wormhole throat. $\epsilon$ is a sign factor which is opposite for the two ends of $CD$. It could be also opposite of opposite at the opposite sides of the wormhole throat. Note that the dependence on induced metric disappears at the right hand side and this condition eliminates the potentials singularity due to the reduction of the rank of the induced metric at wormhole throat.

5. Information about the tangent space of the space-time surface can be coded to the configuration space metric with loosing the nice transformation properties of the magnetic flux Hamiltonians if Kähler electric fluxes or sum of magnetic flux and electric flux satisfying this condition are used and $K$ is symplectic invariant. Using the sum

$$J_e + J_m = (1+K)J \ , \quad (2.5.3)$$

where $J$ can denotes the Kähler magnetic flux, makes it possible to have a non-trivial configuration space metric even for $K = 0$, which could correspond to the ends of a cosmic string like solution carrying only Kähler magnetic fields. This condition suggests that it can depend only on Kähler magnetic flux and other symplectic invariants. Whether local symplectic coordinate invariants are possible at all is far from obvious, If the slicing itself is symplectic invariant then $K$ could be a non-constant function of $X^2$ depending on string world sheet coordinates. The light-like radial coordinate of the light-cone boundary indeed defines a symplectically invariant slicing and this slicing could be shifted along the time axis defined by the tips of $CD$.

**Electric-magnetic duality physically**

What could the weak duality condition mean physically? For instance, what constraints are obtained if one assumes that the quantization of electro-weak charges reduces to this condition at classical level?

1. The first thing to notice is that the flux of $J$ over the partonic 2-surface is analogous to magnetic flux

$$Q_m = \frac{e}{\hbar}\oint BdS = n \ .$$

$n$ is non-vanishing only if the surface is homologically non-trivial and gives the homology charge of the partonic 2-surface.

2. The expressions of classical electromagnetic and $Z^0$ fields in terms of Kähler form [L1] , [L1] read as

$$
\begin{aligned}
\gamma &= \frac{eF_{em}}{\hbar} = 3J - sin^2(\theta_W)R_{03} \ , \\
Z^0 &= \frac{g_Z F_Z}{\hbar} = 2R_{03} \ .
\end{aligned}
\tag{2.5.4}
$$

Here $R_{03}$ is one of the components of the curvature tensor in vielbein representation and $F_{em}$ and $F_Z$ correspond to the standard field tensors. From this expression one can deduce

$$
J = \frac{e}{3\hbar}F_{em} + sin^2(\theta_W)\frac{g_Z}{6\hbar}F_Z \ .
\tag{2.5.5}
$$

3. The weak duality condition when integrated over $X^2$ implies

$$
\begin{aligned}
\frac{e^2}{3\hbar}Q_{em} + \frac{g_Z^2 p}{6}Q_{Z,V} &= K\oint J = Kn \ , \\
Q_{Z,V} &= \frac{I_V^3}{2} - Q_{em} \ , \ p = sin^2(\theta_W) \ .
\end{aligned}
\tag{2.5.6}
$$

Here the vectorial part of the $Z^0$ charge rather than as full $Z^0$ charge $Q_Z = I_L^3 + sin^2(\theta_W)Q_{em}$ appears. The reason is that only the vectorial isospin is same for left and right handed components of fermion which are in general mixed for the massive states.

The coefficients are dimensionless and expressible in terms of the gauge coupling strengths and using $\hbar = r\hbar_0$ one can write

$$
\begin{aligned}
\alpha_{em}Q_{em} &+ p\frac{\alpha_Z}{2}Q_{Z,V} = \frac{3}{4\pi}\times rnK \ , \\
\alpha_{em} &= \frac{e^2}{4\pi\hbar_0} \ , \ \alpha_Z = \frac{g_Z^2}{4\pi\hbar_0} = \frac{\alpha_{em}}{p(1-p)} \ .
\end{aligned}
\tag{2.5.7}
$$

4. There is a great temptation to assume that the values of $Q_{em}$ and $Q_Z$ correspond to their quantized values and therefore depend on the quantum state assigned to the partonic 2-surface. The linear coupling of the modified Dirac operator to conserved charges implies correlation between the geometry of space-time sheet and quantum numbers assigned to the partonic 2-surface. The assumption of standard quantized values for $Q_{em}$ and $Q_Z$ would be also seen as the identification of the fine structure constants $\alpha_{em}$ and $\alpha_Z$. This however requires weak isospin invariance.

### The value of $K$ from classical quantization of Kähler electric charge

The value of $K$ can be deduced by requiring classical quantization of Kähler electric charge.

1. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge $g_K$ would give the condition $K = g_K^2/\hbar$, where $g_K$ is Kähler coupling constant which should invariant under coupling constant evolution by quantum criticality. Within experimental uncertainties one has $\alpha_K = g_K^2/4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where $\alpha_{em}$ is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

2. The quantization of Planck constants makes the condition highly non-trivial. The most general quantization of $r$ is as rationals but there are good arguments favoring the quantization as integers corresponding to the allowance of only singular coverings of $CD$ andn $CP_2$. The point is that in this case a given value of Planck constant corresponds to a finite number pages of the "Big Book". The quantization of the Planck constant implies a further quantization of $K$ and would suggest that $K$ scales as $1/r$ unless the spectrum of values of $Q_{em}$ and $Q_Z$ allowed by the quantization condition scales as $r$. This is quite possible and the interpretation would be that each of the $r$ sheets of the covering carries (possibly same) elementary charge. Kind of discrete variant of a full Fermi sphere would be in question. The interpretation in terms of anyonic phases [K40] supports this interpretation.

3. The identification of $J$ as a counterpart of $eB/\hbar$ means that Kähler action and thus also Kähler function is proportional to $1/\alpha_K$ and therefore to $\hbar$. This implies that for large values of $\hbar$ Kähler coupling strength $g_K^2/4\pi$ becomes very small and large fluctuations are suppressed in the functional integral. The basic motivation for introducing the hierarchy of Planck constants was indeed that the scaling $\alpha \rightarrow \alpha/r$ allows to achieve the convergence of perturbation theory: Nature itself would solve the problems of the theoretician. This of course does not mean that the physical states would remain as such and the replacement of single particles with anyonic states in order to satisfy the condition for $K$ would realize this concretely.

4. The condition $K = g_K^2/\hbar$ implies that the Kähler magnetic charge is always accompanied by Kähler electric charge. A more general condition would read as

$$K = n \times \frac{g_K^2}{\hbar}, n \in Z \ . \tag{2.5.8}$$

   This would apply in the case of cosmic strings and would allow vanishing Kähler charge possible when the partonic 2-surface has opposite fermion and antifermion numbers (for both leptons and quarks) so that Kähler electric charge should vanish. For instance, for neutrinos the vanishing of electric charge strongly suggests $n = 0$ besides the condition that abelian $Z^0$ flux contributing to em charge vanishes.

It took a year to realize that this value of $K$ is natural at the Minkowskian side of the wormhole throat. At the Euclidian side much more natural condition is

$$K = \frac{1}{hbar} \ . \tag{2.5.9}$$

In fact, the self-duality of $CP_2$ Kähler form favours this boundary condition at the Euclidian side of the wormhole throat. Also the fact that one cannot distinguish between electric and magnetic charges in Euclidian region since all charges are magnetic can be used to argue in favor of this form. The same constraint arises from the condition that the action for $CP_2$ type vacuum extremal has the value required by the argument leading to a prediction for gravitational constant in terms of the square of $CP_2$ radius and $\alpha_K$ the effective replacement $g_K^2 \rightarrow 1$ would spoil the argument.

The boundary condition $J_E = J_B$ for the electric and magnetic parts of Kählwer form at the Euclidian side of the wormhole throat inspires the question whether all Euclidian regions could be self-dual so that the density of Kähler action would be just the instanton density. Self-duality follows if the deformation of the metric induced by the deformation of the canonically imbedded $CP_2$ is such that in $CP_2$ coordinates for the Euclidian region the tensor $(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})/\sqrt{g}$ remains invariant. This is certainly the case for $CP_2$ type vacuum extremals since by the light-likeness of $M^4$ projection the metric remains invariant. Also conformal scalings of the induced metric would satisfy this condition. Conformal scaling is not consistent with the degeneracy of the 4-metric at the wormhole throat. Full self-duality is indeed an un-necessarily strong condition.

#### Reduction of the quantization of Kähler electric charge to that of electromagnetic charge

The best manner to learn more is to challenge the form of the weak electric-magnetic duality based on the induced Kähler form.

1. Physically it would seem more sensible to pose the duality on electromagnetic charge rather than Kähler charge. This would replace induced Kähler form with electromagnetic field, which is a linear combination of induced K"ahler field and classical $Z^0$ field

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{03} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \tag{2.5.10}$$

   Here $Z_0 = 2R_{03}$ is the appropriate component of $CP_2$ curvature form [L1]. For a vanishing Weinberg angle the condition reduces to that for Kähler form.

2. For the Euclidian space-time regions having interpretation as lines of generalized Feynman diagrams Weinberg angle should be non-vanishing. In Minkowskian regions Weinberg angle could however vanish. If so, the condition guaranteing that electromagnetic charge of the partonic 2-surfaces equals to the above condition stating that the em charge assignable to the fermion content of the partonic 2-surfaces reduces to the classical Kähler electric flux at the Minkowskian side of the wormhole throat. One can argue that Weinberg angle must increase smoothly from a vanishing value at both sides of wormhole throat to its value in the deep interior of the Euclidian region.

3. The vanishing of the Weinberg angle in Minkowskian regions conforms with the physical intuition. Above elementary particle length scales one sees only the classical electric field reducing to the induced Kähler form and classical $Z^0$ fields and color gauge fields are effectively absent. Only in phases with a large value of Planck constant classical $Z^0$ field and other classical weak fields and color gauge field could make themselves visible. Cell membrane could be one such system [K42]. This conforms with the general picture about color confinement and weak massivation.

The GRT limit of TGD suggests a further reason for why Weinberg angle should vanish in Minkowskian regions.

1. The value of the Kähler coupling strength mut be very near to the value of the fine structure constant in electron length scale and these constants can be assumed to be equal.

2. GRT limit of TGD with space-time surfaces replaced with abstract 4-geometries would naturally correspond to Einstein-Maxwell theory with cosmological constant which is non-vanishing only in Euclidian regions of space-time so that both Reissner-Nordström metric and $CP_2$ are allowed as simplest possible solutions of field equations [K58]. The extremely small value of the observed cosmological constant needed in GRT type cosmology could be equal to the large cosmological constant associated with $CP_2$ metric multiplied with the 3-volume fraction of Euclidian regions.

3. Also at GRT limit quantum theory would reduce to almost topological QFT since Einstein-Maxwell action reduces to 3-D term by field equations implying the vanishing of the Maxwell current and of the curvature scalar in Minkowskian regions and curvature scalar + cosmological constant term in Euclidian regions. The weak form of electric-magnetic duality would guarantee also now the preferred extremal property and prevent the reduction to a mere topological QFT.

4. GRT limit would make sense only for a vanishing Weinberg angle in Minkowskian regions. A non-vanishing Weinberg angle would make sense in the deep interior of the Euclidian regions where the approximation as a small deformation of $CP_2$ makes sense.

The weak form of electric-magnetic duality has surprisingly strong implications for the basic view about quantum TGD as following considerations show.

### 2.5.2 Magnetic confinement, the short range of weak forces, and color confinement

The weak form of electric-magnetic duality has surprisingly strong implications if one combines it with some very general empirical facts such as the non-existence of magnetic monopole fields in macroscopic length scales.

#### How can one avoid macroscopic magnetic monopole fields?

Monopole fields are experimentally absent in length scales above order weak boson length scale and one should have a mechanism neutralizing the monopole charge. How electroweak interactions become short ranged in TGD framework is still a poorly understood problem. What suggests itself is the neutralization of the weak isospin above the intermediate gauge boson Compton length by neutral Higgs bosons. Could the two neutralization mechanisms be combined to single one?

1. In the case of fermions and their super partners the opposite magnetic monopole would be a wormhole throat. If the magnetically charged wormhole contact is electromagnetically neutral but has vectorial weak isospin neutralizing the weak vectorial isospin of the fermion only the electromagnetic charge of the fermion is visible on longer length scales. The distance of this wormhole throat from the fermionic one should be of the order weak boson Compton length. An interpretation as a bound state of fermion and a wormhole throat state with the quantum numbers of a neutral Higgs boson would therefore make sense. The neutralizing throat would have quantum numbers of $X_{-1/2} = \nu_L \overline{\nu}_R$ or $X_{1/2} = \overline{\nu}_L \nu_R$. $\nu_L \overline{\nu}_R$ would not be neutral Higgs boson (which should correspond to a wormhole contact) but a super-partner of left-handed neutrino obtained by adding a right handed neutrino. This mechanism would apply separately to the fermionic and anti-fermionic throats of the gauge bosons and corresponding space-time sheets and leave only electromagnetic interaction as a long ranged interaction.

2. One can of course wonder what is the situation situation for the bosonic wormhole throats feeding gauge fluxes between space-time sheets. It would seem that these wormhole throats must always appear as pairs such that for the second member of the pair monopole charges and $I_V^3$ cancel each other at both space-time sheets involved so that one obtains at both space-time sheets magnetic dipoles of size of weak boson Compton length. The proposed magnetic character of fundamental particles should become visible at TeV energies so that LHC might have surprises in store!

#### Magnetic confinement and color confinement

Magnetic confinement generalizes also to the case of color interactions. One can consider also the situation in which the magnetic charges of quarks (more generally, of color excited leptons and quarks) do not vanish and they form color and magnetic singles in the hadronic length scale. This would mean that magnetic charges of the state $q_{\pm 1/2} - X_{\mp 1/2}$ representing the physical quark would not vanish and magnetic confinement would accompany also color confinement. This would explain why free quarks are not observed. To how degree then quark confinement corresponds to magnetic confinement is an interesting question.

For quark and antiquark of meson the magnetic charges of quark and antiquark would be opposite and meson would correspond to a Kähler magnetic flux so that a stringy view about meson emerges. For valence quarks of baryon the vanishing of the net magnetic charge takes place provided that the magnetic net charges are $(\pm 2, \mp 1, \mp 1)$. This brings in mind the spectrum of color hyper charges coming as $(\pm 2, \mp 1, \mp 1)/3$ and one can indeed ask whether color hyper-charge correlates with the Kähler magnetic charge. The geometric picture would be three strings connected to single vertex. Amusingly, the idea that color hypercharge could be proportional to color hyper charge popped up during the first year of TGD when I had not yet discovered $CP_2$ and believed on $M^4 \times S^2$.

p-Adic length scale hypothesis and hierarchy of Planck constants defining a hierarchy of dark variants of particles suggest the existence of scaled up copies of QCD type physics and weak physics. For p-adically scaled up variants the mass scales would be scaled by a power of $\sqrt{2}$ in the most general case. The dark variants of the particle would have the same mass as the original one. In particular, Mersenne primes $M_k = 2^k - 1$ and Gaussian Mersennes $M_{G,k} = (1+i)^k - 1$ has been proposed to define zoomed copies of these physics. At the level of magnetic confinement this would mean hierarchy of length scales for the magnetic confinement.

One particular proposal is that the Mersenne prime $M_{89}$ should define a scaled up variant of the ordinary hadron physics with mass scaled up roughly by a factor $2^{(107-89)/2} = 512$. The size scale of color confinement for this physics would be same as the weal length scale. It would look more natural that the weak confinement for the quarks of $M_{89}$ physics takes place in some shorter scale and $M_{61}$ is the first Mersenne prime to be considered. The mass scale of $M_{61}$ weak bosons would be by a factor $2^{(89-61)/2} = 2^{14}$ higher and about $1.6 \times 10^4$ TeV. $M_{89}$ quarks would have virtually no weak interactions but would possess color interactions with weak confinement length scale reflecting themselves as new kind of jets at collisions above TeV energies.

In the biologically especially important length scale range 10 nm -2500 nm there are as many as four Gaussian Mersennes corresponding to $M_{G,k}$, $k = 151, 157, 163, 167$. This would suggest that the existence of scaled up scales of magnetic-, weak- and color confinement. An especially interesting possibly testable prediction is the existence of magnetic monopole pairs with the size scale in this range. There are recent claims about experimental evidence for magnetic monopole pairs [D3] .

#### Magnetic confinement and stringy picture in TGD sense

The connection between magnetic confinement and weak confinement is rather natural if one recalls that electric-magnetic duality in super-symmetric quantum field theories means that the descriptions in terms of particles and monopoles are in some sense dual descriptions. Fermions would be replaced by string like objects defined by the magnetic flux tubes and bosons as pairs of wormhole contacts would correspond to pairs of the flux tubes. Therefore the sharp distinction between gravitons and physical particles would disappear.

The reason why gravitons are necessarily stringy objects formed by a pair of wormhole contacts is that one cannot construct spin two objects using only single fermion states at wormhole throats. Of course, also super partners of these states with higher spin obtained by adding fermions and anti-fermions at the wormhole throat but these do not give rise to graviton like states [K20] . The upper and lower wormhole throat pairs would be quantum superpositions of fermion anti-fermion pairs with sum over all fermions. The reason is that otherwise one cannot realize graviton emission in terms of joining of the ends of light-like 3-surfaces together. Also now magnetic monopole charges are necessary but now there is no need to assign the entities $X_{\pm}$ with gravitons.

Graviton string is characterized by some p-adic length scale and one can argue that below this length scale the charges of the fermions become visible. Mersenne hypothesis suggests that some Mersenne prime is in question. One proposal is that gravitonic size scale is given by electronic Mersenne prime $M_{127}$. It is however difficult to test whether graviton has a structure visible below this length scale.

What happens to the generalized Feynman diagrams is an interesting question. It is not at all clear how closely they relate to ordinary Feynman diagrams. All depends on what one is ready to assume about what happens in the vertices. One could of course hope that zero energy ontology could allow some very simple description allowing perhaps to get rid of the problematic aspects of Feynman diagrams.

1. Consider first the recent view about generalized Feynman diagrams which relies zero energy ontology. A highly attractive assumption is that the particles appearing at wormhole throats are on mass shell particles. For incoming and outgoing elementary bosons and their super partners they would be positive it resp. negative energy states with parallel on mass shell momenta. For virtual bosons they the wormhole throats would have opposite sign of energy and the sum of on mass shell states would give virtual net momenta. This would make possible twistor description of virtual particles allowing only massless particles (in 4-D sense usually and in 8-D sense in TGD framework). The notion of virtual fermion makes sense only if one assumes in the interaction region a topological condensation creating another wormhole throat having no fermionic quantum numbers.

2. The addition of the particles $X^{\pm}$ replaces generalized Feynman diagrams with the analogs of stringy diagrams with lines replaced by pairs of lines corresponding to fermion and $X_{\pm 1/2}$. The members of these pairs would correspond to 3-D light-like surfaces glued together at the vertices of generalized Feynman diagrams. The analog of 3-vertex would not be splitting of the string to

form shorter strings but the replication of the entire string to form two strings with same length or fusion of two strings to single string along all their points rather than along ends to form a longer string. It is not clear whether the duality symmetry of stringy diagrams can hold true for the TGD variants of stringy diagrams.

3. How should one describe the bound state formed by the fermion and $X^{\pm}$? Should one describe the state as superposition of non-parallel on mass shell states so that the composite state would be automatically massive? The description as superposition of on mass shell states does not conform with the idea that bound state formation requires binding energy. In TGD framework the notion of negentropic entanglement has been suggested to make possible the analogs of bound states consisting of on mass shell states so that the binding energy is zero [K31] . If this kind of states are in question the description of virtual states in terms of on mass shell states is not lost. Of course, one cannot exclude the possibility that there is infinite number of this kind of states serving as analogs for the excitations of string like object.

4. What happens to the states formed by fermions and $X_{\pm 1/2}$ in the internal lines of the Feynman diagram? Twistor philosophy suggests that only the higher on mass shell excitations are possible. If this picture is correct, the situation would not change in an essential manner from the earlier one.

The highly non-trivial prediction of the magnetic confinement is that elementary particles should have stringy character in electro-weak length scales and could behaving to become manifest at LHC energies. This adds one further item to the list of non-trivial predictions of TGD about physics at LHC energies [K32] .

#### Should $J + J_1$ appear in Kähler action?

The presence of the $S^2$ Kähler form $J_1$ in the weak form of electric-magnetic duality was originally suggested by an erratic argument about the reduction to almost topological QFT to be described in the next subsection. In any case this argument raises the question whether one could replace $J$ with $J + J_1$ in the Kähler action. This would not affect the basic non-vacuum extremals but would modify the vacuum degeneracy of the Kähler action. Canonically imbedded $M^4$ would become a monopole configuration with an infinite magnetic energy and Kähler action due to the monopole singularity at the line connecting tips of the $CD$. Action and energy can be made small by drilling a small hole around origin. This is however not consistent with the weak form of electro-weak duality. Amusingly, the modified Dirac equation reduces to ordinary massless Dirac equation in $M^4$.

This extremal can be transformed to a vacuum extremal by assuming that the solution is also a $CP_2$ magnetic monopole with opposite contribution to the magnetic charge so that $J + J_1 = 0$ holds true. This is achieved if one can regard space-time surface as a map $M^4 \to CP_2$ reducing to a map $(\Theta, \Phi) = (\theta, \pm\phi)$ with the sign chosen by properly projecting the homologically non-trivial $r_M = constant$ spheres of $CD$ to the homologically non-trivial geodesic sphere of $CP_2$. Symplectic transformations of $S^2 \times CP_2$ produce new vacuum extremals of this kind. Using Darboux coordinates in which one has $J = \sum_{k=1,2} P_k dQ^k$ and assuming that $(P_1, Q_1)$ corresponds to the $CP_2$ image of $S^2$, one can take $Q_2$ to be arbitrary function of $P^2$, which in turn is an arbitrary function of $M^4$ coordinates to obtain even more general vacuum extremals with 3-D $CP_2$ projection. Therefore the spectrum of vacuum extremals, which is very relevant for the TGD based description of gravitation in long length scales because it allows to satisfy Einstein's equations as an additional condition, looks much richer than for the original option, and it is natural to ask whether this option might make sense.

An objection is that $J_1$ is a radial monopole field and this breaks Lorentz invariance to $SO(3)$. Lorentz invariance is broken to $SO(3)$ for a given $CD$ also by the presence of the preferred time direction defined by the time-like line connecting the tips of the $CD$ becoming carrying the monopole charge but is compensated since Lorentz boosts of $CD$s are possible. Could one consider similar compensation also now? Certainly the extremely small breaking of Lorentz invariance and the vanishing of the monopole charge for the vacuum extremals is all that is needed at the space-time level. No new gauge fields would be introduced since only the Kähler field part of photon and $Z^0$ boson would receive an additional contribution.

The ultimate fate of the modification depends on whether it is consistent with the general relativistic description of gravitation. Since a breaking of spherical symmetry is involved, it is not at all clear whether one can find vacuum extremals which represent small deformations of the Reissner-Nordström metric and Robertson-Walker metric. The argument below shows that this option does not allow the imbedding of small deformations of physically plausible space-time metrics as vacuum extremals.

The basic vacuum extremal whose deformations should give vacuum extremals allowing interpretation as solutions of Einstein's equations is given by a map $M^4 \rightarrow CP_2$ projecting the $r_M$ constant spheres $S^2$ of $M^2$ to the homologically non-trivial geodesic sphere of $CP_2$. The winding number of this map is $-1$ in order to achieve vanishing of the induced Kähler form $J+J_1$. For instance, the following two canonical forms of the map are possible

$$\begin{aligned} (\Theta, \Psi) &= (\theta_M, -\phi_M) \ , \\ (\Theta, \Psi) &= (\pi - \theta_M, \phi_M) \ . \end{aligned} \tag{2.5.11}$$

Here $(\Theta, \Psi)$ refers to the geodesic sphere of $CP_2$ and $(\theta_M, \phi_M)$ to the sphere of $M^4$.
The resulting space-time surface is not flat and Einstein tensor is non-vanishing. More complex metrics can be constructed from this metric by a deformation making the $CP_2$ projection 3-dimensional.

Using the expression of the $CP_2$ line element in Eguchi-Hanson coordinates [L10]

$$\frac{ds^2}{R^2} = \frac{dr^2}{F^2} + \frac{r^2}{F}(d\Psi + cos\Theta d\Phi)^2 + \frac{r^2}{4F}(d\Theta^2 + fracr^2 4F sin^2\Theta d\Phi^2) \tag{2.5.12}$$

and s the relationship $r = tan(\Theta)$, one obtains following expression for the $CP_2$ metric

$$\frac{ds^2}{R^2} = d\theta_M^2 + sin^2(\theta_M)\left[(d\phi_M + cos(\theta)d\Phi)^2 + \frac{1}{4}(d\theta^2 + sin^2(\theta)d\Phi^2\right] \ . \tag{2.5.13}$$

The resulting metric is obtained from the metric of $S^2$ by replacing $d\phi^2$ which 3-D line element. The factor $sin^2(\theta_M)$ implies that the induced metric becomes singular at North and South poles of $S^2$. In particular, the gravitational potential is proportional to $sin^2(\theta_M)$ so that gravitational force in the radial direction vanishes at equators. It is very difficult to imagine any manner to produce a small deformation of Reissner-Nordstrm metric or Robertson-Walker metric. Hence it seems that the vacuum extremals produce by $J + J_1$ option are not physical.

### 2.5.3 Could Quantum TGD reduce to almost topological QFT?

There seems to be a profound connection with the earlier unrealistic proposal that TGD reduces to almost topological quantum theory in the sense that the counterpart of Chern-Simons action assigned with the wormhole throats somehow dictates the dynamics. This proposal can be formulated also for the modified Dirac action action. I gave up this proposal but the following argument shows that Kähler action with weak form of electric-magnetic duality effectively reduces to Chern-Simons action plus Coulomb term.

1. Kähler action density can be written as a 4-dimensional integral of the Coulomb term $j_K^\alpha A_\alpha$ plus and integral of the boundary term $J^{n\beta}A_\beta\sqrt{g_4}$ over the wormhole throats and of the quantity $J^{0\beta}A_\beta\sqrt{g_4}$ over the ends of the 3-surface.

2. If the self-duality conditions generalize to $J^{n\beta} = 4\pi\alpha_K\epsilon^{n\beta\gamma\delta}J_{\gamma\delta}$ at throats and to $J^{0\beta} = 4\pi\alpha_K\epsilon^{0\beta\gamma\delta}J_{\gamma\delta}$ at the ends, the Kähler function reduces to the counterpart of Chern-Simons action evaluated at the ends and throats. It would have same value for each branch and the replacement $\hbar_0 \rightarrow r\hbar_0$ would effectively describe this. Boundary conditions would however give $1/r$ factor so that $\hbar$ would disappear from the Kähler function! The original attempt to realize quantum TGD as an almost topological QFT was in terms of Chern-Simons action but was

given up. It is somewhat surprising that Kähler action gives Chern-Simons action in the vacuum sector defined as sector for which Kähler current is light-like or vanishes.

Holography encourages to ask whether also the Coulomb interaction terms could vanish. This kind of dimensional reduction would mean an enormous simplification since TGD would reduce to an almost topological QFT. The attribute "almost" would come from the fact that one has non-vanishing classical Noether charges defined by Kähler action and non-trivial quantum dynamics in $M^4$ degrees of freedom. One could also assign to space-time surfaces conserved four-momenta which is not possible in topological QFTs. For this reason the conditions guaranteeing the vanishing of Coulomb interaction term deserve a detailed analysis.

1. For the known extremals $j_K^\alpha$ either vanishes or is light-like ("massless extremals" for which weak self-duality condition does not make sense [K5] ) so that the Coulombic term vanishes identically in the gauge used. The addition of a gradient to $A$ induces terms located at the ends and wormhole throats of the space-time surface but this term must be cancelled by the other boundary terms by gauge invariance of Kähler action. This implies that the $M^4$ part of WCW metric vanishes in this case. Therefore massless extremals as such are not physically realistic: wormhole throats representing particles are needed.

2. The original naive conclusion was that since Chern-Simons action depends on $CP_2$ coordinates only, its variation with respect to Minkowski coordinates must vanish so that the WCW metric would be trivial in $M^4$ degrees of freedom. This conclusion is in conflict with quantum classical correspondence and was indeed too hasty. The point is that the allowed variations of Kähler function must respect the weak electro-magnetic duality which relates Kähler electric field depending on the induced 4-metric at 3-surface to the Kähler magnetic field. Therefore the dependence on $M^4$ coordinates creeps via a Lagrange multiplier term

$$\int \Lambda_\alpha(J^{n\alpha} - K\epsilon^{n\alpha\beta\gamma}J_{\beta\gamma})\sqrt{g_4}d^3x \ . \tag{2.5.14}$$

The (1,1) part of second variation contributing to $M^4$ metric comes from this term.

3. This erratic conclusion about the vanishing of $M^4$ part WCW metric raised the question about how to achieve a non-trivial metric in $M^4$ degrees of freedom. The proposal was a modification of the weak form of electric-magnetic duality. Besides $CP_2$ Kähler form there would be the Kähler form assignable to the light-cone boundary reducing to that for $r_M = constant$ sphere - call it $J^1$. The generalization of the weak form of self-duality would be $J^{n\beta} = \epsilon^{n\beta\gamma\delta}K(J_{\gamma\delta} + \epsilon J^1_{\gamma\delta})$. This form implies that the boundary term gives a non-trivial contribution to the $M^4$ part of the WCW metric even without the constraint from electric-magnetic duality. Kähler charge is not affected unless the partonic 2-surface contains the tip of $CD$ in its interior. In this case the value of Kähler charge is shifted by a topological contribution. Whether this term can survive depends on whether the resulting vacuum extremals are consistent with the basic facts about classical gravitation.

4. The Coulombic interaction term is not invariant under gauge transformations. The good news is that this might allow to find a gauge in which the Coulomb term vanishes. The vanishing condition fixing the gauge transformation $\phi$ is

$$j_K^\alpha\partial_\alpha\phi = -j^\alpha A_\alpha \ . \tag{2.5.15}$$

This differential equation can be reduced to an ordinary differential equation along the flow lines $j_K$ by using $dx^\alpha/dt = j_K^\alpha$. Global solution is obtained only if one can combine the flow parameter $t$ with three other coordinates- say those at the either end of $CD$ to form space-time coordinates. The condition is that the parameter defining the coordinate differential is proportional to the covariant form of Kähler current: $dt = \phi j_K$. This condition in turn implies $d^2t = d(\phi j_K) = d(\phi j_K) = d\phi \wedge j_K + \phi dj_K = 0$ implying $j_K \wedge dj_K = 0$ or more concretely,

$$\epsilon^{\alpha\beta\gamma\delta} j^K_\beta \partial_\gamma j^K_\delta = 0 \ . \tag{2.5.16}$$

$j_K$ is a four-dimensional counterpart of Beltrami field [B29] and could be called generalized Beltrami field.

The integrability conditions follow also from the construction of the extremals of Kähler action [K5] . The conjecture was that for the extremals the 4-dimensional Lorentz force vanishes (no dissipation): this requires $j_K \wedge J = 0$. One manner to guarantee this is the topologization of the Kähler current meaning that it is proportional to the instanton current: $j_K = \phi j_I$, where $j_I = {}^*(J \wedge A)$ is the instanton current, which is not conserved for 4-D $CP_2$ projection. The conservation of $j_K$ implies the condition $j_I^\alpha \partial_\alpha \phi = \partial_\alpha j^\alpha \phi$ and from this $\phi$ can be integrated if the integrability condition $j_I \wedge dj_I = 0$ holds true implying the same condition for $j_K$. By introducing at least 3 or $CP_2$ coordinates as space-time coordinates, one finds that the contravariant form of $j_I$ is purely topological so that the integrability condition fixes the dependence on $M^4$ coordinates and this selection is coded into the scalar function $\phi$. These functions define families of conserved currents $j_K^\alpha \phi$ and $j_I^\alpha \phi$ and could be also interpreted as conserved currents associated with the critical deformations of the space-time surface.

5. There are gauge transformations respecting the vanishing of the Coulomb term. The vanishing condition for the Coulomb term is gauge invariant only under the gauge transformations $A \to A + \nabla\phi$ for which the scalar function the integral $\int j_K^\alpha \partial_\alpha \phi$ reduces to a total divergence a giving an integral over various 3-surfaces at the ends of $CD$ and at throats vanishes. This is satisfied if the allowed gauge transformations define conserved currents

$$D_\alpha (j^\alpha \phi) = 0 \ . \tag{2.5.17}$$

As a consequence Coulomb term reduces to a difference of the conserved charges $Q_\phi^e = \int j^0 \phi \sqrt{g_4} d^3x$ at the ends of the CD vanishing identically. The change of the imons type term is trivial if the total weighted Kähler magnetic flux $Q_\phi^m = \sum \int J \phi dA$ over wormhole throats is conserved. The existence of an infinite number of conserved weighted magnetic fluxes is in accordance with the electric-magnetic duality. How these fluxes relate to the flux Hamiltonians central for WCW geometry is not quite clear.

6. The gauge transformations respecting the reduction to almost topological QFT should have some special physical meaning. The measurement interaction term in the modified Dirac interaction corresponds to a critical deformation of the space-time sheet and is realized as an addition of a gauge part to the Kähler gauge potential of $CP_2$. It would be natural to identify this gauge transformation giving rise to a conserved charge so that the conserved charges would provide a representation for the charges associated with the infinitesimal critical deformations not affecting Kähler action. The gauge transformed Kähler potential couples to the modified Dirac equation and its effect could be visible in the value of Kähler function and therefore also in the properties of the preferred extremal. The effect on WCW metric would however vanish since $K$ would transform only by an addition of a real part of a holomorphic function. Kähler function is identified as a Dirac determinant for Chern-Simons Dirac action and the spectrum of this operator should not be invariant under these gauge transformations if this picture is correct. This is is achieved if the gauge transformation is carried only in the Dirac action corresponding to the Chern-Simons term: this assumption is motivated by the breaking of time reversal invariance induced by quantum measurements. The modification of Kähler action can be guessed to correspond just to the Chern-Simons contribution from the instanton term.

7. A reasonable looking guess for the explicit realization of the quantum classical correspondence between quantum numbers and space-time geometry is that the deformation of the preferred extremal due to the addition of the measurement interaction term is induced by a $U(1)$ gauge transformation induced by a transformation of $\delta CD \times CP_2$ generating the gauge transformation represented by $\phi$. This interpretation makes sense if the fluxes defined by $Q^m_\phi$ and corresponding Hamiltonians affect only zero modes rather than quantum fluctuating degrees of freedom.

To sum up, one could understand the basic properties of WCW metric in this framework. Effective 2-dimensionality would result from the existence of an infinite number of conserved charges in two different time directions (genuine conservation laws plus gauge fixing). The infinite-dimensional symmetric space for given values of zero modes corresponds to the Cartesian product of the WCWs associated with the partonic 2-surfaces at both ends of $CD$ and the generalized Chern-Simons term decomposes into a sum of terms from the ends giving single particle Kähler functions and to the terms from light-like wormhole throats giving interaction term between positive and negative energy parts of the state. Hence Kähler function could be calculated without any knowledge about the interior of the space-time sheets and TGD would reduce to almost topological QFT as speculated earlier. Needless to say this would have immense boost to the program of constructing WCW Kähler geometry.

### 2.5.4 Kähler action for Euclidian regions as Kähler function and Kähler action for Minkowskian regions as Morse function?

One of the nasty questions about the interpretation of Kähler action relates to the square root of the metric determinant. If one proceeds completely straightforwardly, the only reason conclusion is that the square root is imaginary in Minkowskian space-time regions so that Kähler action would be complex. The Euclidian contribution would have a natural interpretation as positive definite Kähler function but how should one interpret the imaginary Minkowskian contribution? Certainly the path integral approach to quantum field theories supports its presence. For some mysterious reason I was able to forget this nasty question and serious consideration of the obvious answer to it. Only when I worked betweeen possibile connections between TGD and Floer homology [K64] I realized that the Minkowskian contribution is an excellent candidate for Morse function whose critical points give information about WCW homology. This would fit nicely with the vision about TGD as almost topological QFT.

Euclidian regions would guarantee the convergence of the functional integral and one would have a mathematically well-defined theory. Minkowskian contribution would give the quantal interference effects and stationary phase approximation. The analog of Floer homology would represent quantum superpositions of critical points identifiable as ground states defined by the extrema of Kähler action for Minkowskian regions. Perturbative approach to quantum TGD would rely on functional integrals around the extrema of Kähler function. One would have maxima also for the Kähler function but only in the zero modes not contributing to the WCW metric.

There is a further question related to almost topological QFT character of TGD. Should one assume that the reduction to Chern-Simons terms occurs for the preferred extremals in *both* Minkowskian and Euclidian regions or only in Minkowskian regions?

1. All arguments for this have been represented for Minkowskian regions [K19] involve local light-like momentum direction which does not make sense in the Euclidian regions. This does not however kill the argument: one can have non-trivial solutions of Laplacian equation in the region of $CP_2$ bounded by wormhole throats: for $CP_2$ itself only covariantly constant right-handed neutrino represents this kind of solution and at the same time supersymmetry. In the general case solutions of Laplacian represent broken super-symmetries and should be in one-one correspondences with the solutions of the modified Dirac equation. The interpretation for the counterparts of momentum and polarization would be in terms of classical representation of color quantum numbers.

2. If the reduction occurs in Euclidian regions, it gives in the case of $CP_2$ two 3-D terms corresponding to two 3-D gluing regions for three coordinate patches needed to define coordinates and spinor connection for $CP_2$ so that one would have two Chern-Simons terms. I have earlier claimed that without any other contributions the first term would be identical with that from Minkowskian region apart from imaginary unit and different coefficient. This statement is

wrong since the space-like parts of the corresponding 3-surfaces are discjoint for Euclidian and Minkowskian regions.

3. There is also another very delicate issue involved. Quantum classical correspondence requires that the quantum numbers of partonic states must be coded to the space-time geometry, and this is achieved by adding to the action a measurement interaction term which reduces to what is almost a gauge term present only in Chern-Simons-Dirac equation but not at space-time interior [K19]. This term would represent a coupling to Poincare quantum numbers at the Minkowskian side and to color and electro-weak quantum numbers at $CP_2$ side. Therefore the net Chern-Simons contributions would be different.

4. There is also a very beautiful argument stating that Dirac determinant for Chern-Simons-Dirac action equals to Kähler function, which would be lost if Euclidian regions would not obey holography. The argument obviously generalizes and applies to both Morse and Kähler function which are definitely not proportional to each other.

The Minkowskian contribution of Kähler action is imaginary due to the negative of the metric determinant and gives a phase factor to vacuum functional reducing to Chern-Simons terms at wormhole throats. Ground state degeneracy due to the possibility of having both signs for Minkowskian contribution to the exponent of vacuum functional provides a general view about the description of CP breaking in TGD framework.

1. In TGD framework path integral is replaced by inner product involving integral over WCV. The vacuum functional and its conjugate are associated with the states in the inner product so that the phases of vacuum functionals cancel if only one sign for the phase is allowed. Minkowskian contribution would have no physical significance. This of course cannot be the case. The ground state is actually degenerate corresponding to the phase factor and its complex conjugate since $\sqrt{g}$ can have two signs in Minkowskian regions. Therefore the inner products between states associated with the two ground states define $2 \times 2$ matrix and non-diagonal elements contain interference terms due to the presence of the phase factor. At the limit of full $CP_2$ type vacuum extremal the two ground states would reduce to each other and the determinant of the matrix would vanish.

2. A small mixing of the two ground states would give rise to CP breaking and the first principle description of CP breaking in systems like $K - \overline{K}$ and of CKM matrix should reduce to this mixing. $K^0$ mesons would be CP even and odd states in the first approximation and correspond to the sum and difference of the ground states. Small mixing would be present having exponential sensitivity to the actions of $CP_2$ type extremals representing wormhole throats. This might allow to understand qualitatively why the mixing is about 50 times larger than expected for $B^0$ mesons.

3. There is a strong temptation to assign the two ground states with two possible arrows of geometric time. At the level of M-matrix the two arrows would correspond to state preparation at either upper or lower boundary of CD. Do long- and shortlived neutral K mesons correspond to almost fifty-fifty orthogonal superpositions for the two arrow of geometric time or almost completely to a fixed arrow of time induced by environment? Is the dominant part of the arrow same for both or is it opposite for long and short-lived neutral measons? Different lifetimes would suggest that the arrow must be the same and apart from small leakage that induced by environment. CP breaking would be induced by the fact that CP is performed only $K^0$ but not for the environment in the construction of states. One can probably imagine also alternative interpretations.

### 2.5.5 A general solution ansatz based on almost topological QFT property

The basic vision behind the ansatz is the reduction of quantum TGD to almost topological field theory. This requires that the flow parameters associated with the flow lines of isometry currents and Kähler current extend to global coordinates. This leads to integrability conditions implying generalized Beltrami flow and Kähler action for the preferred extremals reduces to Chern-Simons action when weak electro-weak duality is applied as boundary conditions. The strongest form of the hydrodynamical interpretation requires that all conserved currents are parallel to Kähler current. In the more general case one would have several hydrodynamic flows. Also the braidings (several of them for the most general ansatz) assigned with the light-like 3-surfaces are naturally defined by the flow lines of conserved currents. The independent behavior of particles at different flow lines can be seen as a realization of the complete integrability of the theory. In free quantum field theories on mass shell Fourier components are in a similar role but the geometric interpretation in terms of flow is of course lacking. This picture should generalize also to the solution of the modified Dirac equation.

### Basic field equations

Consider first the equations at general level.

1. The breaking of the Poincare symmetry due to the presence of monopole field occurs and leads to the isometry group $T\times SO(3)\times SU(3)$ corresponding to time translations, rotations, and color group. The Cartan algebra is four-dimensional and field equations reduce to the conservation laws of energy $E$, angular momentum $J$, color isospin $I_3$, and color hypercharge $Y$.

2. Quite generally, one can write the field equations as conservation laws for $I, J, I_3$, and $Y$.

$$D_\alpha\left[D_\beta(J^{\alpha\beta}H_A) - j^\alpha_K H^A + T^{\alpha\beta}j^l_A h_{kl}\partial_\beta h^l\right] = 0 \ . \tag{2.5.18}$$

   The first term gives a contraction of the symmetric Ricci tensor with antisymmetric Kähler form and vanishes so that one has

$$D_\alpha\left[j^\alpha_K H^A - T^{\alpha\beta}j^k_A h_{kl}\partial_\beta h^l\right] = 0 \ . \tag{2.5.19}$$

   For energy one has $H_A = 1$ and energy current associated with the flow lines is proportional to the Kähler current. Its divergence vanishes identically.

3. One can express the divergence of the term involving energy momentum tensor as as sum of terms involving $j^\alpha_K J_{\alpha\beta}$ and contraction of second fundamental form with energy momentum tensor so that one obtains

$$j^\alpha_K D_\alpha H^A = j^\alpha_K J_\alpha{}^\beta j^A_\beta + T^{\alpha\beta}H^k_{\alpha\beta}j^A_k \ . \tag{2.5.20}$$

### Hydrodynamical solution ansatz

The characteristic feature of the solution ansatz would be the reduction of the dynamics to hydrodynamics analogous to that for a continuous distribution of particles iniatially at the end of $X^3$ of the light-like 3-surface moving along flow lines defined by currents $j_A$ satisfying the integrability condition $j_A\wedge dj_A = 0$. Field theory would reduce effectively to particle mechanics along flow lines with conserved charges defined by various isometry currents. The strongest condition is that all isometry currents $j_A$ and also Kähler current $j_K$ are proportional to the same current $j$. The more general option corresponds to multi-hydrodynamics.

1. *Solution ansatz*

Conserved currents are analogous to hydrodynamical currents in the sense that the flow parameter along flow lines extends to a global space-time coordinate. The conserved current is proportional to the gradient $\nabla\Phi$ of the coordinate varying along the flow lines: $J = \Psi\nabla\Phi$ and by a proper choice of $\Psi$ one can allow to have conservation. The initial values of $\Psi$ and $\Phi$ can be selected freely along the flow lines beginning from either the end of the space-time surface or from wormhole throats.

If one requires hydrodynamics also for Chern-Simons action (effective 2-dimensionality is required for preferred extremals), the initial values of scalar functions can be chosen freely only at the partonic 2-surfaces. The freedom to chose the intial values of the charges conserved along flow lines at the partonic 2-surfaces means the existence of an infinite number of conserved charges so that the theory would be integrable and even in two different coordinate directions. The basic difference as compared to ordinary conservation laws is that the conserved currents are parallel and their flow parameter extends to a global coordinate.

1. The most general assumption is that the conserved isometry currents

$$J_A^\alpha = j_K^\alpha H^A - T^{\alpha\beta} j_A^k h_{kl} \partial_\beta h^l \tag{2.5.21}$$

   and Kähler current as well as instanto current are integrable in the sense that $J_A \wedge J_A = 0$ and $j_K \wedge j_K = 0$ hold true. One could imagine the possibility that the currents are not parallel. If instanton current and Kähler current are proportional to each other, Coulomb interaction term in the Kähler action vanishes and almost topological QFT property is achieved.

2. The integrability condition $dJ_A \wedge J_A = 0$ is satisfied if one one has

$$J_A = \Psi_A d\Phi_A \ . \tag{2.5.22}$$

   The ansatz allows a gauge transformation induced by a symplectic transformation of $S^2.\Phi_A$ is same for Kähler current and instanton current.

3. The conservation of $J_A$ gives

$$d * (\Psi_A d\Phi_A) = 0 \ . \tag{2.5.23}$$

   This would mean separate hydrodynamics for each of the currents involved. In principle there is not need to assume any further conditions and one can imagine infinite basis of scalar function pairs $(\Psi_A, \Phi_A)$ since criticality implies infinite number deformations implying conserved Noether currents.

4. The conservation condition reduces to d'Alembert equation in the induced metric if one assumes that $\nabla\Psi_A$ is orthogonal with every $d\Phi_A$.

$$d * d\Phi_A = 0 \ , \ d\Psi_A \cdot d\Phi_A = 0 \ . \tag{2.5.24}$$

   Taking $x = \Phi_A$ as a coordinate the orthogonality condition states $g^{xj}\partial_j \Psi_A = 0$ and in the general case one cannot solve the condition by simply assuming that $\Psi_A$ depends on the coordinates transversal to $\Phi_A$ only. These conditions bring in mind $p \cdot p = 0$ and $p \cdot e$ condition for massless modes of Maxwell field having fixed momentum and polarization. $d\Phi_A$ would correspond to $p$ and $d\Psi_A$ to polarization. The condition that each isometry current corresponds its own pair $(\Psi_A, \Phi_A)$ would mean that each isometry current corresponds to independent light-like momentum and polarization. Ordinary free quantum field theory would support this view whereas hydrodynamics and QFT limit of TGD would support single flow.

These are the most general hydrodynamical conditions that one can assume. One can consider also more restricted scenarios.

1. The strongest ansatz is inspired by the hydrodynamical picture in which all conserved isometry charges flow along same flow lines so that one would have

$$J_A = \Psi_A d\Phi \ . \tag{2.5.25}$$

In this case same $\Phi$ would satisfy simultaneously the d'Alembert type equations.

$$d * d\Phi = 0 \ , \ d\Psi_A \cdot d\Phi = 0. \tag{2.5.26}$$

This would mean that the massless modes associated with isometry currents move in parallel manner but can have different polarizations. The spinor modes associated with light-light like 3-surfaces carry parallel four-momenta, which suggest that this option is correct. This allows a very general family of solutions and one can have a complete 3-dimensional basis of functions $\Psi_A$ with gradient orthogonal to $d\Phi$.

2. Isometry invariance under $T \times SO(3) \times SU(3)$ allows to consider the possibility that one has

$$J_A = k_A \Psi_A d\Phi_{G(A)} \ , \ d * (d\Phi_G(A)) = 0 \ , \ d\Psi_A \cdot d\Phi_G(A)) = 0 \ . \tag{2.5.27}$$

where $G(A)$ is $T$ for energy current, $SO(3)$ for angular momentum currents and $SU(3)$ for color currents. Energy would thus flow along its own flux lines, angular momentum along its own flow lines, and color quantum numbers along their own flow lines. For instance, color currents would differ from each other only by a numerical constant. The replacement of $\Psi_A$ with $\Psi_{G(A)}$ would be too strong a condition since Killing vector fields are not related by a constant factor.

To sum up, the most general option is that each conserved current $J_A$ defines its own integrable flow lines defined by the scalar function pair $(\Psi_A, \Phi_A)$. A complete basis of scalar functions satisfying the d'Alembert type equation guaranteing current conservation could be imagined with restrictions coming from the effective 2-dimensionality reducing the scalar function basis effectively to the partonic 2-surface. The diametrically opposite option corresponds to the basis obtained by assuming that only single $\Phi$ is involved. The ansatz does not distinguish between $J$ and $J + J_1$ options.

The proposed solution ansatz can be compared to the earlier ansatz [K24] stating that Kähler current is topologized in the sense that for $D(CP_2) = 3$ it is proportional to the identically conserved instanton current (so that 4-D Lorentz force vanishes) and vanishes for $D(CP_2) = 4$ (Maxwell phase). This hypothesis requires that instanton current is Beltrami field for $D(CP_2) = 3$. In the recent case the assumption that also instanton current satisfies the Beltrami hypothesis in strong sense (single function $\Phi$) generalizes the topologization hypothesis for $D(CP_2) = 3$ and guarantees that Coulomb term in Kähler action vanishes identically. A weaker form is obtained by replacing Kähler potential by its gauge transform in which case one also obtains a boundary term. As a matter fact, the topologization hypothesis applies to isometry currents also for $D(CP_2) = 4$ although instanton current is not conserved anymore. One can consider variants of instanton current since both $(A_1, J_1)$ and $(A, J)$ are available.

### Can one require the extremal property in the case of Chern-Simons action?

Effective 2-dimensionality is achieved if the ends and wormhole throats are extremals of Chern-Simons action. The strongest condition would be that space-time surfaces allow orthogonal slicings by 3-surfaces which are extremals of Chern-Simons action.

Also in this case one can require that the flow parameter associated with the flow lines of the isometry currents extends to a global coordinate. Kähler magnetic field $B = *J$ defines a conserved current so that all conserved currents would flow along the field lines of $B$ and one would have 3-D

Beltrami flow. Note that in magnetohydrodynamics the standard assumption is that currents flow along the field lines of the magnetic field.

For wormhole throats light-likeness causes some complications since the induced metric is degenerate and the contravariant metric must be restricted to the complement of the light-like direction. This means that d'Alembert equation reduces to 2-dimensional Laplace equation. For space-like 3-surfaces one obtains the counterpart of Laplace equation with partonic 2-surfaces serving as sources. The interpretation in terms of analogs of Coulomb potentials created by 2-D charge distributions would be natural.

If $J + J1$ appears in Kähler action the extremals need not have 2-dimensional $CP_2$ projection as they must have for $J$ option, and one can hope of obtaining large enough solution family consistent with effective 2-dimensionality. The field equations can be reduced to conservation conditions for the isometry currents for $SO(3) \times SU(3)$ along flow lines.

### 2.5.6 Holomorphic factorization of Kähler function

One can guess the general form of the core part of the Kähler function as function of complex coordinates assignable to the partonic surfaces at positive and negative energy ends of $CD$. It its convenient to restrict the consideration to the simplest possible non-trivial case which is represented by single propagator line connecting the ends of $CD$.

1. The propagator line corresponds to a symmetric space defined as a coset space $G/H$ of the symplectic group and Kac-Moody group. This coset space is as a manifold Cartesian product $(G/H) \times (G/H)$ of symmetric spaces $G/H$ associated with ends of the line. Kähler metric contains also an interaction term between the factors of the Cartesian product so that Kähler function can be said to reduce to a sum of "kinetic" terms and interaction term.

2. The exponent of Kähler function depends on both ends of the line and this means that the geometries at the ends are correlated in the sense that that Kähler form contains interaction terms between the line ends. It is however not quite clear whether it contains separate "kinetic" or self interaction terms assignable to the line ends. For Kähler function the kinetic and interaction terms should have the following general expressions as functions of complex WCW coordinates:

$$\begin{aligned} K_{kin,i} &= \sum_n f_{i,n}(Z_i)\overline{f_{i,n}(Z_i)} + c.c \ , \\ K_{int} &= \sum_n g_{1,n}(Z_1)\overline{g_{2,n}(Z_2)} + c.c \ , i = 1,2 \ . \end{aligned} \tag{2.5.28}$$

Here $K_{kin,i}$ define "kinetic" terms and $K_{int}$ defines interaction term. One would have what might be called holomorphic factorization suggesting a connection with conformal field theories. $K_{kin}$ would correspond to the Chern-Simons term assignable to the ends of the line and $K_{int}$ to the Chern-Simons terms assignable to the wormhole throats.

### 2.5.7 Could the dynamics of Kähler action predict the hierarchy of Planck constants?

The original justification for the hierarchy of Planck constants came from the indications that Planck constant could have large values in both astrophysical systems involving dark matter and also in biology. The realization of the hierarchy in terms of the singular coverings and possibly also factor spaces of $CD$ and $CP_2$ emerged from consistency conditions. The formula for the Planck constant involves heuristic guess work and physical plausibility arguments. There are good arguments in favor of the hypothesis that only coverings are possible. Only a finite number of pages of the Big Book correspond to a given value of Planck constant, biological evolution corresponds to a gradual dispersion to the pages of the Big Book with larger Planck constant, and a connection with the hierarchy of infinite primes and p-adicization program based on the mathematical realization of finite measurement resolution emerges.

One can however ask whether this hierarchy could emerge directly from the basic quantum TGD rather than as a separate hypothesis. The following arguments suggest that this might be possible. One finds also a precise geometric interpretation of preferred extremal property interpreted as criticality in zero energy ontology.

**1-1 correspondence between canonical momentum densities and time derivatives fails for Kähler action**

The basic motivation for the geometrization program was the observation that canonical quantization for TGD fails. To see what is involved let us try to perform a canonical quantization in zero energy ontology at the 3-D surfaces located at the light-like boundaries of $CD \times CP_2$.

1. In canonical quantization canonical momentum densities $\pi_k^0 \equiv \pi_k = \partial L_K/\partial(\partial_0 h^k)$, where $\partial_0 h^k$ denotes the time derivative of imbedding space coordinate, are the physically natural quantities in terms of which to fix the initial values: once their value distribution is fixed also conserved charges are fixed. Also the weak form of electric-magnetic duality given by $J^{03}\sqrt{g_4} = 4\pi\alpha_K J_{12}$ and a mild generalization of this condition to be discussed below can be interpreted as a manner to fix the values of conserved gauge charges (not Noether charges) to their quantized values since Kähler magnetic flux equals to the integer giving the homology class of the (wormhole) throat. This condition alone need not characterize criticality, which requires an infinite number of deformations of $X^4$ for which the second variation of the Kähler action vanishes and implies infinite number conserved charges. This in fact gives hopes of replacing $\pi_k$ with these conserved Noether charges.

2. Canonical quantization requires that $\partial_0 h^k$ in the energy is expressed in terms of $\pi_k$. The equation defining $\pi_k$ in terms of $\partial_0 h^k$ is however highly non-linear although algebraic. By taking squares the equations reduces to equations for rational functions of $\partial_0 h^k$. $\partial_0 h^k$ appears in contravariant and covariant metric at most quadratically and in the induced Kähler electric field linearly and by multplying the equations by $det(g_4)^3$ one can transform the equations to a polynomial form so that in principle $\partial_0 h^k$ can obtained as a solution of polynomial equations.

3. One can always eliminate one half of the coordinates by choosing 4 imbedding space coordinates as the coordinates of the spacetime surface so that the initial value conditions reduce to those for the canonical momentum densities associated with the remaining four coordinates. For instance, for space-time surfaces representable as map $M^4 \to CP_2$ $M^4$ coordinates are natural and the time derivatives $\partial_0 s^k$ of $CP_2$ coordinates are multivalued. One would obtain four polynomial equations with $\partial_0 s^k$ as unknowns. In regions where $CP_2$ projection is 4-dimensional -in particular for the deformations of $CP_2$ vacuum extremals the natural coordinates are $CP_2$ coordinates and one can regard $\partial_0 m^k$ as unknows. For the deformations of cosmic strings, which are of form $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$, one can use coordinates of $M^2 \times S^2$, where $S^2$ is geodesic sphere as natural coordinates and regard as unknowns $E^2$ coordinates and remaining $CP_2$ coordinates.

4. One can imagine solving one of the four polynomials equations for time derivaties in terms of other obtaining $N$ roots. Then one would substitute these roots to the remaining 3 conditions to obtain algebraic equations from which one solves then second variable. Obviously situation is very complex without additional symmetries. The criticality of the preferred extremals might however give additional conditions allowing simplifications. The reasons for giving up the canonical quantization program was following. For the vacuum extremals of Kähler action $\pi_k$ are however identically vanishing and this means that there is an infinite number of value distributions for $\partial_0 h^k$. For small deformations of vacuum extremals one might however hope a finite number of solutions to the conditions and thus finite number of space-time surfaces carrying same conserved charges.

If one assumes that physics is characterized by the values of the conserved charges one must treat the the many-valuedness of $\partial_0 h^k$. The most obvious guess is that one should replace the space of space-like 4-surfaces corresponding to different roots $\partial_0 h^k = F^k(\pi_l)$ with four-surfaces in the covering space of $CD \times CP_2$ corresponding to different branches of the many-valued function $\partial_0 h^k = F(\pi_l)$ co-inciding at the ends of $CD$.

**Do the coverings forces by the many-valuedness of $\partial_0 h^k$ correspond to the coverings associated with the hierarchy of Planck constants?**

The obvious question is whether this covering space actually corresponds to the covering spaces associated with the hierarchy of Planck constants. This would conform with quantum classical correspondence. The hierarchy of Planck constants and hierarchy of covering spaces was introduced to cure the failure of the perturbation theory at quantum level. At classical level the multivaluedness of $\partial_0 h^k$ means a failure of perturbative canonical quantization and forces the introduction of the covering spaces. The interpretation would be that when the density of matter becomes critical the space-time surface splits to several branches so that the density at each branches is sub-critical. It is of course not at all obvious whether the proposed structure of the Big Book is really consistent with this hypothesis and one also consider modifications of this structure if necessary. The manner to proceed is by making questions.

1. The proposed picture would give only single integer characterizing the covering. Two integers assignable to $CD$ and $CP_2$ degrees of freedom are however needed. How these two coverings could emerge?

   (a) One should fix also the values of $\pi_k^n = \partial L_K/\partial h_n^k$, where $n$ refers to space-like normal coordinate at the wormhole throats. If one requires that charges do not flow between regions with different signatures of the metric the natural condition is $\pi_k^n = 0$ and allows also multi-valued solution. Since wormhole throats carry magnetic charge and since weak form of electric-magnetic duality is assumed, one can assume that $CP_2$ projection is four-dimensional so that one can use $CP_2$ coordinates and regard $\partial_0 m^k$ as un-knows. The basic idea about topological condensation in turn suggests that $M^4$ projection can be assumed to be 4-D inside space-like 3-surfaces so that here $\partial_0 s^k$ are the unknowns. At partonic 2-surfaces one would have conditions for both $\pi_k^0$ and $\pi_k^n$. One might hope that the numbers of solutions are finite for preferred extremals because of their symmetries and given by $n_a$ for $\partial_0 m^k$ and by $n_b$ for $\partial_0 s^k$. The optimistic guess is that $n_a$ and $n_b$ corresponds to the numbers of sheets for singular coverings of $CD$ and $CP_2$. The covering could be visualized as replacement of space-time surfaces with space-time surfaces which have $n_a n_b$ branches. $n_b$ branches would degenerate to single branch at the ends of diagrams of the generaled Feynman graph and $n_a$ branches would degenerate to single one at wormhole throats.

   (b) This picture is not quite correct yet. The fixing of $\pi_k^0$ and $\pi_k^n$ should relate closely to the effective 2-dimensionality as an additional condition perhaps crucial for criticality. One could argue that both $\pi_k^0$ and $\pi_k^n$ must be fixed at $X^3$ and $X_l^3$ in order to effectively bring in dynamics in two directions so that $X^3$ could be interpreted as a an orbit of partonic 2-surface in space-like direction and $X_l^3$ as its orbit in light-like direction. The additional conditions could be seen as gauge conditions made possible by symplectic and Kac-Moody type conformal symmetries. The conditions for $\pi_0^k$ would give $n_b$ branches in $CP_2$ degrees of freedom and the conditions for $\pi_k^n$ would split each of these branches to $n_a$ branches.

   (c) The existence of these two kinds of conserved charges (possibly vanishing for $\pi_k^n$) could relate also very closely to the slicing of the space-time sheets by string world sheets and partonic 2-surfaces.

2. Should one then treat these branches as separate space-time surfaces or as a single space-time surface? The treatment as a single surface seems to be the correct thing to do. Classically the conserved changes would be $n_a n_b$ times larger than for single branch. Kähler action need not (but could!) be same for different branches but the total action is $n_a n_b$ times the average action and this effectively corresponds to the replacement of the $\hbar_0/g_K^2$ factor of the action with $\hbar/g_K^2$, $r \equiv \hbar/\hbar_0 = n_a n_b$. Since the conserved quantum charges are proportional to $\hbar$ one could argue that $r = n_a n_b$ tells only that the charge conserved charge is $n_a n_b$ times larger than without multi-valuedness. $\hbar$ would be only effectively $n_a n_b$ fold. This is of course poor man's argument but might catch something essential about the situation.

3. How could one interpret the condition $J^{03}\sqrt{g_4} = 4\pi\alpha_K J_{12}$ and its generalization to be discussed below in this framework? The first observation is that the total Kähler electric charge is by $\alpha_K \propto 1/(n_a n_b)$ same always. The interpretation would be in terms of charge fractionization

meaning that each branch would carry Kähler electric charge $Q_K = ng_K/n_an_b$. I have indeed suggested explanation of charge fractionization and quantum Hall effect based on this picture.

4. The vision about the hierarchy of Planck constants involves also assumptions about imbedding space metric. The assumption that the $M^4$ covariant metric is proportional to $\hbar^2$ follows from the physical idea about $\hbar$ scaling of quantum lengths as what Compton length is. One can always introduce scaled $M^4$ coordinates bringing $M^4$ metric into the standard form by scaling up the $M^4$ size of $CD$. It is not clear whether the scaling up of $CD$ size follows automatically from the proposed scenario. The basic question is why the $M^4$ size scale of the critical extremals must scale like $n_an_b$? This should somehow relate to the weak self-duality conditions implying that Kähler field at each branch is reduced by a factor $1/r$ at each branch. Field equations should posses a dynamical symmetry involving the scaling of $CD$ by integer $k$ and $J^{0\beta}\sqrt{g_4}$ and $J^{n\beta}\sqrt{g_4}$ by $1/k$. The scaling of $CD$ should be due to the scaling up of the $M^4$ time interval during which the branched light-like 3-surface returns back to a non-branched one.

5. The proposed view about hierarchy of Planck constants is that the singular coverings reduce to single-sheeted coverings at $M^2 \subset M^4$ for $CD$ and to $S^2 \subset CP_2$ for $CP_2$. Here $S^2$ is any homologically trivial geodesic sphere of $CP_2$ and has vanishing Kähler form. Weak self-duality condition is indeed consistent with any value of $\hbar$ and impies that the vacuum property for the partonic 2-surface implies vacuum property for the entire space-time sheet as holography indeed requires. This condition however generalizes. In weak self-duality conditions the value of $\hbar$ is free for any 2-D Lagrangian sub-manifold of $CP_2$.

   The branching along $M^2$ would mean that the branches of preferred extremals always collapse to single branch when their $M^4$ projection belongs to $M^2$. Magnetically charged light-light-like throats cannot have $M^4$ projection in $M^2$ so that self-duality conditions for different values of $\hbar$ do not lead to inconsistencies. For spacelike 3-surfaces at the boundaries of $CD$ the condition would mean that the $M^4$ projection becomes light-like geodesic. Straight cosmic strings would have $M^2$ as $M^4$ projection. Also $CP_2$ type vacuum extremals for which the random light-like projection in $M^4$ belongs to $M^2$ would represent this of situation. One can ask whether the degeneration of branches actually takes place along any string like object $X^2 \times Y^2$, where $X^2$ defines a minimal surface in $M^4$. For these the weak self-duality condition would imply $\hbar = \infty$ at the ends of the string. It is very plausible that string like objects feed their magnetic fluxes to larger space-times sheets through wormhole contacts so that these conditions are not encountered.

**Connection with the criticality of preferred extremals**

Also a connection with quantum criticality and the criticality of the preferred extremals suggests itself. Criticality for the preferred extremals must be a property of space-like 3-surfaces and light-like 3-surfaces with degenerate 4-metric and the degeneration of the $n_an_b$ branches of the space-time surface at the its ends and at wormhole throats is exactly what happens at criticality. For instance, in catastrophe theory roots of the polynomial equation giving extrema of a potential as function of control parameters co-incide at criticality. If this picture is correct the hierarchy of Planck constants would be an outcome of criticality and of preferred extremal property and preferred extremals would be just those multi-branched space-time surfaces for which branches co-incide at the the boundaries of $CD \times CP_2$ and at the throats.

## 2.6 Does the exponent of Chern-Simons action reduce to the exponent of the area of minimal surfaces?

As I scanned of hep-th I found an interesting article by Giordano, Peschanski, and Seki [B32] based on AdS/CFT correspondence. What is studied is the high energy behavior of the gluon-gluon and quark-quark scattering amplitudes of $\mathcal{N} = 4$ SUSY.

1. The proposal made earlier by Aldaya and Maldacena [B28] is that gluon-gluon scattering amplitudes are proportional to the imaginary exponent of the area of a minimal surface in $AdS_5$

whose boundary is identified as *momentum space*. The boundary of the minimal surface would be polygon with light-like edges: this polygon and its dual are familiar from twistor approach.

2. Giordano, Peschanski, and Seki claim that quark-quark scattering amplitude for heavy quarks corresponds to the exponent of the area for a minimal surface in the *Euclidian* version of $AdS_5$ which is hyperbolic space (space with a constant negative curvature): it is interpreted as a counterpart of configuration space rather than momentum space and amplitudes are obtained by analytic continuation. For instance, a universal Regge behavior is obtained. For general amplitudes the exponent of the area alone is not enough since it does not depend on gluon quantum numbers and vertex operators at the edges of the boundary polygon are needed.

In the following my intention is to consider the formulation of this conjecture in quantum TGD framework. I hasten to inform that I am not a specialist in AdS/CFT and can make only general comments inspired by analogies with TGD.

### 2.6.1 Why Chern-Simons action should reduce to area for minimal surfaces?

The minimal surface conjectures are highly interesting from TGD point of view. The weak form of electric magnetic duality implies the reduction of Kähler action to 3-D Chern-Simons terms. Effective 2-dimensionality implied by the strong form of General Coordinate Invariance suggests a further reduction of Chern-Simons terms to 2-D terms and the areas of string world sheet and of partonic 2-surface are the only non-topological options that one can imagine. Skeptic could of course argue that the exponent of the minimal surface area results as a characterizer of the quantum state rather than vacuum functional. In the following I defend the minimal interpretation as Chern-Simons terms.

Let us look this conjecture in more detail.

1. In zero energy ontology twistor approach is very natural since all physical states are bound states of massless particles. Also virtual particles are composites of massless states. The possibility to have both signs of energy makes possible space-like momenta for wormhole contacts. Mass shell conditions at internal lines imply extremely strong constraints on the virtual momenta and both UV and IR finiteness are expected to hold true.

2. The weak form of electric magnetic duality [K19] implies that the exponent of Kähler action reduces to the exponent of Chern-Simons term for 3-D space-like surfaces at the ends of space-time surface inside $CD$ and for light-like 3-surfaces. The coefficient of this term is complex since the contribution of Minkowskian regions of the space-time surface is imaginary ($\sqrt{g_4}$ is imaginary) and that of Euclidian regions (generalized Feynman diagrams) real. The Chern-Simons term from Minkowskian regions is like Morse function and that from Euclidian regions defines Kähler function and stationary phase approximation makes sense. The two contributions are different since the space-like 3-surfaces contributing to Kähler function and Morse function are different.

3. Electric magnetic duality [K19] leads also to the conclusion that wormhole throats carrying elementary particle quantum numbers are Kähler magnetic monopoles. This forces to identify elementary particles as string like objects with ends having opposite monopole charges. Also more complex configurations are possible.

   It is not quite clear what the scale of the stringyness is. The natural first guess inspired by quantum classical correspondence is that it corresponds to the p-adic length scale of the particle characterizing its Compton length. Second possibility is that it corresponds to electroweak scale. For leptons stringyness in Compton length scale might not have any fatal implications since the second end of string contains only neutrinos neutralizing the weak isospin of the state. This kind of monopole pairs could appear even in condensed matter scales: in particular if the proposed hierarchy of Planck constants [K18] is realized.

4. Strong form of General Coordinate Invariance requires effective 2-dimensionality. In given UV and IR resolutions either partonic 2-surfaces or string world sheets form a finite hierarchy of $CD$s inside $CD$s with given $CD$ characterized by a discrete scale coming as an integer multiple of a

fundamental scale (essentially $CP_2$ size). The string world sheets have boundaries consisting of either light-like curves in induced metric at light-like wormhole throats and space-like curves at the ends of $CD$ whose $M^4$ projections are light-like. These braids intersect partonic 2-surfaces at discrete points carrying fermionic quantum numbers.

This implies a rather concrete analogy with $AdS_5 \times S_5$ duality, which describes gluons as open strings. In zero energy ontology (ZEO) string world sheets are indeed a fundamental notion and the natural conjecture is that these surfaces are minimal surfaces whose area by quantum classical correspondence depends on the quantum numbers of the external particles. String tension in turn should depend on gauge couplings -perhaps only Kähler coupling strength- and geometric parameters like the size scale of $CD$ and the p-adic length scale of the particle.

5. Are the minimal surfaces in question minimal surfaces of the imbedding space $M^4 \times CP_2$ or of the space-time surface $X^4$? All possible 2-surfaces at the boundary of $CD$ must be allowed so that they cannot correspond to minimal surfaces in $M^4 \times CP_2$ unless one assumes that they emerge in stationary phase approximation only. The boundary conditions at the ends of $CD$ could however be such that *any* partonic 2-surface correspond to a minimal surfaces in $X^4$. Same applies to string world sheets. One might even hope that these conditions combined with the weak form of electric magnetic duality fixes completely the boundary conditions at wormhole throats and space-like ends of space-time surface.

   The trace of the second fundamental form orthogonal to the string world sheet/partonic 2-surface as sub-manifold of space-time surface would vanish: this is nothing but a generalization of the geodesic motion obtained by replacing word line with a 2-D surface. It does not imply the vanishing of the trace of the second fundamental form in $M^4 \times CP_2$ having interpretation as a generalization of particle acceleration [K58]. Effective 2-dimensionality would be realized if Chern-Simons terms reduce to a sum of the areas of these minimal surfaces.

These arguments suggest that scattering amplitudes are proportional to the product of exponents of 2-dimensional actions which can be either imaginary or real. Imaginary exponent would be proportional to the total area of string world sheets and the imaginary unit would come naturally from $\sqrt{g_2}$. Teal exponent proportional to the total area of partonic 2-surfaces. The coefficient of these areas would not in general be same.

The equality of the Minkowskian and Euclidian Chern-Simons terms is suggestive but not necessarily true since there could be also other Chern-Simons contributions than those assignable to wormhole throats and the ends of space-time. The equality would imply that the total area of string world sheets equals to the total area of partonic 2-surfaces suggesting strongly a duality meaning that either Euclidian or Minkowskian regions carry the needed information.

### 2.6.2 IR cutoff and connection with p-adic physics

In twistor approach the IR cutoff is necessary to get rid of IR divergences. Also in the $AdS_5$ approach the condition that the minimal surface area is finite requires an IR cutoff. The problem is that there is no natural IR cutoff. In TGD framework zero energy ontology brings in a natural IR cutoff via the finite and quantized size scale of $CD$ guaranteeing that the minimal surfaces involved have a finite area. This implies that also particles usually regarded as massless have a small mass characterized by the size of $CD$. The size scale of $CD$ would correspond to the scale parameter $R$ assigned with the metric of $AdS_5$.

1. String tension relates in $AdS_5$ approach to the gauge coupling $g_{YM}$ and to the number $N_c$ of colors by the formula

$$\lambda = g_{YM}^2 N_c = \frac{R^2}{\alpha'} \ . \tag{2.6.1}$$

   $1/N_c$-expansion is in terms of $1/\sqrt{\lambda}$. The formula has an alternative form as an expression for the string tension

$$\alpha' = \frac{R^2}{\sqrt{g_{YM}^2 N_c}} \ . \tag{2.6.2}$$

The analog this formula in TGD framework suggests an connection with p-adic length scale hypothesis.

1. As already noticed, the natural counterpart for the scale $R$ could be the discrete value of the size scale of $CD$. Since the symplectic group assignable to $\delta M^4_\pm \times CP_2$ (or the upper or lower boundary of $CD$) is the natural generalization of the gauge group, it would seem that $N_c = \infty$ holds true in the absence of cutoff. At the limit $N_c = \infty$ only planar diagrams would contribute to YM scattering amplitudes. Finite measurement resolution must make the effective value of $N_c$ finite so that also $\lambda$ would be finite. String tension would depend on both the size of $CD$ and the effective number of symplectic colors.

2. If $\alpha'$ is characterized by the square of the Compton length of the particle, $\lambda$ would be essentially the square of the ratio of $CD$ size scale given by secondary p-adic lengths and of the primary p-adic length scale associated with the particle: $\lambda = g_{YM}^2\sqrt{p}$, where $p$ is the p-adic prime characterizing the particle. Favored values of the p-adic prime correspond to primes near powers of two. The effective number of symplectic colors would be $N_c = \sqrt{p}/g_{YM}^2$ and the expansion would come in powers of $g_{YM}^2/\sqrt{p}$. For electron one would have $p = M_{127} = 2^{127-1}$ so that the expansion would converge extremely fast. Together with the amazing success of the p-adic mass calculations based on p-adic thermodynamics for the scaling generator $L_0$ [K34] this suggests a deep connection with p-adic physics and number theoretic universality.

### 2.6.3 Could Kähler action reduce to Kähler magnetic flux over string world sheets and partonic 2-surfaces?

Can one consider alternative identifications of Kähler action for preferred extremals? The only alternative identification of Kähler function that I can imagine is that Kähler action proportional to the Kähler magnetic flux $\int_{Y^2} J$ or Kähler electric flux $\int_{Y^2} *J$ for string world sheets and possibly also partonic 2-surfaces. These fluxes are dimensionless numbers. If the weak form of electric-magnetic duality holds true also at string world sheets, the two options are equivalent apart from a proportionality constant.

1. For Kähler magnetic flux there would be no explicit dependence on the induced metric. This is in accordance with the almost topological QFT property.

2. Unless the weak form of electric-magnetic duality holds true, the Kähler electric flux has an explicit dependence on the induced metric but in a scaling invariant manner. The most obvious objection relates to the sign factor of the dual flux which depends on the orientation of the string world sheet and thus changes sign when the orientation of space-time sheet is changed by changing that of the string world sheet. This is in conflict with the independence of Kähler action on orientation. One can however argue that the orientation makes itself actually physically visible via the weak form of electric-magnetic duality and that the change of the orientation as a symmetry is dynamically broken. This breaking would be anagous to parity breaking at the level of imbedding space.

3. In [K25] it is proposed that braids defined by the boundaries of string world sheets could correspond to Legendrian sub-manifolds, whereas partonic 2-surfaces could the duals of Legendrian manifolds, so that braiding would take place dynamically. The identification of the Kähler action as Kähler magnetic flux associated with string world sheets and possibly also partonic 2-surfaces is consistent with the assumption that the extremal of Kähler action in question. Indeed, the Legendrian property says that the projection of the Kähler gauge potential on braid strand vanishes and this expresses the extremality of the Kähler magnetic flux.

The assumption that Kähler action is proportional to Kähler magnetic flux seems to be consistent with the minimal surface property. The weak form of electric-magnetic duality gives a constraint on the normal derivatives of imbedding space coordinates at the string world sheet and minimal surface property strengthens these constraints. One could perhaps say that space-time surface chooses its shape in such a manner that the string world sheet has a minimal area.

The open questions are following.

1. Does Kähler action for the preferred exremals reduce to the area of the string world sheet or to Kähler flux, or are the representations equivalent so that the induced Kähler form would effectively define area form? If the Kähler form form associated with the induced metric on string world sheet is proportional to the induced Kähler form the Kähler magnetic flux is proportional to the area and Kähler action reduces to genuine area. This condition looks like a natural additional constraint on string world sheets besides minimal surface property.

2. The proportionality of the induced Kähler form and Kähler form of the induced 2-metric implies as such only the extremal property against the symplectic variations so that one cannot have minimal surface property at imbedding space level. Minimality at space-time level is however possible since space-time surface itself can arrange the situation so that general variations deforming the string world sheet along space-time surface reduce to symlectic variations at the level of the imbedding space.

3. Does the situation depend on whether the string world sheet is in Minkowskian or Euclidian space-time region? The problem is that in Euclidian regions the value of Kähler action is positive definite and it is not obvious why the Kähler magnetic flux for Euclidian string world sheets should have a fixed sign. Could weak form of electric-magnetic duality fix the sign?

Irrespective whether the Kähler action is proportional to the total area or the Kähler electric flux over string world sheets, the theory would be exactly solvable at string world sheet level (finite measurement resolution).

### 2.6.4 What is the interpretation of Yangian duality in TGD framework?

Minimal surfaces in both configuration space and momentum space are used in the above mentioned two articles [B28, B32]. The possibility of these two descriptions must reflect the Yangian symmetry unifying the conformal symmetries of Minkowski space and momentum space in twistorial approach.

The minimal surfaces in $X^4 \subset M^4 \times CP_2$ are natural in TGD framework. Could also the minimal surfaces in momentum space have some interpretation in TGD framework? Ore more generally, what could be the interpretation of the dual descriptions provided by twistor diagrams with light-like edges and dual twistor diagrams with light-like vertices? One can imagine many interpretations but zero energy ontology suggests an especially attractive and natural interpretation of this duality as the exchange of the roles of wormhole throats carrying always on mass shell massless momenta and wormhole contacts carrying in general off-mass shell momenta and massive momenta in incoming lines.

1. For configuration space twistor diagrams vertices correspond to incoming and outgoing light-like momenta. The light-like momenta associated with the wormhole throats of the incoming and outgoing lines of generalized Feynman diagram could correspond to the light-like momenta associated with the vertices of the polygon. The internal lines defined by wormhole contacts carrying virtual off mass shell momenta would naturally correspond to to edges of the twistor diagram.

2. What about dual twistor diagrams in which light-like momenta correspond to lines? Zero energy ontology implies that virtual wormhole throats carry on mass shell massless momenta whereas incoming wormhole contacts in general carry massive particles: this guarantees the absence of IR divergences. Could one identify the momenta of internal wormhole throats as light-like momenta associated with the lines dual twistor diagrams and the incoming net momenta assignable to wormhole contacts as incoming and outgoing momenta.

Also the transition from Minkowskian to Euclidian signature by Wick rotation could have interpretation in TGD framework. Space-time surfaces decompose into Minkowskian and Euclidian regions.

The latter ones represent generalized Feynman diagrams. This suggests a generalization of Wick rotation. The string world sheets in Euclidian regions would define the analogs of the minimal surfaces in Euclidian $AdS_5$ and the string world sheets in Minkowskian regions the analogs of Minkowskian $AdS_5$. The magnitudes of the areas would be identical so that they might be seen as analytical continuations of each other in some sense. Note that partonic 2-surfaces would belong to the intersection of Euclidian and Minkowskian space-time regions. This argument tells nothing about possible momentum space analog of $M^4 \times CP_2$.

# Chapter 3

# Construction of Configuration Space Kähler Geometry from Symmetry Principles

## 3.1 Introduction

The most general expectation is that configuration space can be regarded as a union of coset spaces which are infinite-dimensional symmetric spaces with Kähler structure: $C(H) = \cup_i G/H(i)$. Index $i$ labels 3-topology and zero modes. The group $G$, which can depend on 3-surface, can be identified as a subgroup of diffeomorphisms of $\delta M^4_+ \times CP_2$ and $H$ must contain as its subgroup a group, whose action reduces to $Diff(X^3)$ so that these transformations leave 3-surface invariant.

The task is to identify plausible candidate for $G$ and $H$ and to show that the tangent space of the configuration space allows Kähler structure, in other words that the Lie-algebras of $G$ and $H(i)$ allow complexification. One must also identify the zero modes and construct integration measure for the functional integral in these degrees of freedom. Besides this one must deduce information about the explicit form of configuration space metric from symmetry considerations combined with the hypothesis that Kähler function is Kähler action for a preferred extremal of Kähler action. One must of course understand what "preferred" means.

### 3.1.1 General Coordinate Invariance and generalized quantum gravitational holography

The basic motivation for the construction of configuration space geometry is the vision that physics reduces to the geometry of classical spinor fields in the infinite-dimensional configuration space of 3-surfaces of $M^4_+ \times CP_2$ or of $M^4 \times CP_2$. Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that configuration space possesses Kähler geometry. Kähler geometry is coded into Kähler function.

The original belief was that the four-dimensional general coordinate invariance of Kähler function reduces the construction of the geometry to that for the boundary of configuration space consisting of 3-surfaces on $\delta M^4_+ \times CP_2$, the moment of big bang. The proposal was that Kähler function $K(Y^3)$ could be defined as a preferred extremal of so called Kähler action for the unique space-time surface $X^4(Y^3)$ going through given 3-surface $Y^3$ at $\delta M^4_+ \times CP_2$. For Diff$^4$ transforms of $Y^3$ at $X^4(Y^3)$ Kähler function would have the same value so that Diff$^4$ invariance and degeneracy would be the outcome. The proposal was that the preferred extremals are absolute minima of Kähler action.

This picture turned out to be too simple.

1. I have already described the recent view about light-like 3-surfaces as generalized Feynman diagrams and space-time surfaces as preferred extremals of Kähler action and will not repeat what has been said.

2. It has also become obvious that the gigantic symmetries associated with $\delta M^4_\pm \times CP_2 \subset CD \times CP_2$ manifest themselves as the properties of propagators and vertices. Cosmological considerations,

Poincare invariance, and the new view about energy favor the decomposition of the configuration space to a union of configuration spaces assignable to causal diamonds $CD$s defined as intersections of future and past directed light-cones. The minimum assumption is that $CD$s label the sectors of $CH$: the nice feature of this option is that the considerations of this chapter restricted to $\delta M^4_+ \times CP_2$ generalize almost trivially. This option is beautiful because the center of mass degrees of freedom associated with the different sectors of $CH$ would correspond to $M^4$ itself and its Cartesian powers.

The definition of the Kähler function requires that the many-to-one correspondence $X^3 \rightarrow X^4(X^3)$ must be replaced by a bijective correspondence in the sense that $X^3_l$ as light-like 3-surface is unique among all its Diff$^4$ translates. This also allows physically preferred "gauge fixing" allowing to get rid of the mathematical complications due to Diff4 degeneracy. The internal geometry of the space-time sheet must define the preferred 3-surface $X^3_l$.

The realization of this vision means a considerable mathematical challenge. The effective metric 2-dimensionality of 3-dimensional light-like surfaces $X^3_l$ of $M^4$ implies generalized conformal and symplectic symmetries allowing to generalize quantum gravitational holography from light like boundary so that the complexities due to the non-determinism can be taken into account properly.

### 3.1.2 Light like 3-D causal determinants and effective 2-dimensionality

The light like 3-surfaces $X^3_l$ of space-time surface appear as 3-D causal determinants. Basic examples are boundaries and elementary particle horizons at which Minkowskian signature of the induced metric transforms to Euclidian one. This brings in a second conformal symmetry related to the metric 2-dimensionality of the 3-D light-like 3-surface. This symmetry is identifiable as TGD counterpart of the Kac Moody symmetry of string models. The challenge is to understand the relationship of this symmetry to configuration space geometry and the interaction between the two conformal symmetries.

1. Field-particle duality is realized. Light-like 3-surfaces $X^3_l$ -generalized Feynman diagrams - correspond to the particle aspect of field-particle duality whereas the physics in the interior of space-time surface $X^4(X^3_l)$ would correspond to the field aspect. Generalized Feynman diagrams in 4-D sense could be identified as regions of space-time surface having Euclidian signature.

2. One could also say that light-like 3-surfaces $X^3_l$ and the space-like 3-surfaces $X^3$ in the intersections of $X^4(X^3_l) \cap CD \times CP_2$ where the causal diamond $CD$ is defined as the intersections of future and past directed light-cones provide dual descriptions.

3. Generalized coset construction implies that the differences of super-symplectic and Super Kac-Moody type Super Virasoro generators annihilated physical states. This implies Equivalence Principle. This construction in turn led to the realization that configuration space for fixed values of zero modes - in particular the values of the induced Kähler form of $\delta M^4_\pm \times CP_2$ - allows identification as a coset space obtained by dividing the symplectic group of $\delta M^4_\pm \times CP_2$ with Kac-Moody group, whose generators vanish at $X^2 = X^3_l \times \delta M^4_\pm \times CP_2$. One can say that quantum fluctuating degrees of freedom in a very concrete sense correspond to the local variant of $S^2 \times CP_2$.

The analog of conformal invariance in the light-like direction of $X^3_l$ and in the light-like radial direction of $\delta M^4_\pm$ implies that the data at either $X^3$ or $X^3_l$ should be enough to determine configuration space geometry. This implies that the relevant data is contained to their intersection $X^2$ at least for finite regions of $X^3$. This is the case if the deformations of $X^3_l$ not affecting $X^2$ and preserving light likeness corresponding to zero modes or gauge degrees of freedom and induce deformations of $X^3$ also acting as zero modes. The outcome is effective 2-dimensionality. One must be however cautious in order to not make over-statements. The reduction to 2-D theory in global sense would trivialize the theory and the reduction to 2-D theory must takes places for finite region of $X^3$ only so one has in well defined sense three-dimensionality in discrete sense. A more precise formulation of this vision is in terms of hierarchy of $CD$s containing $CD$s containing.... The introduction of sub-$CD$:s brings in improved measurement resolution and means also that effective 2-dimensionality is realized in the scale of sub-$CD$ only.

One cannot over-emphasize the importance of the effective 2-dimensionality. It indeed simplifies dramatically the earlier formulas for configuration space metric involving 3-dimensional integrals over

$X^3 \subset M^4_+ \times CP_2$ reducing now to 2-dimensional integrals. Note that $X^3$ is determined by preferred extremal property of $X^4(X^3_l)$ once $X^3_l$ is fixed and one can hope that this mapping is one-to-one.

### 3.1.3 Magic properties of light cone boundary and isometries of configuration space

The special conformal, metric and symplectic properties of the light cone of four-dimensional Minkowski space: $\delta M^4_+$, the boundary of four-dimensional light cone is metrically 2-dimensional(!) sphere allowing infinite-dimensional group of conformal transformations and isometries(!) as well as Kähler structure. Kähler structure is not unique: possible Kähler structures of light cone boundary are parameterized by Lobatchevski space $SO(3,1)/SO(3)$. The requirement that the isotropy group $SO(3)$ of $S^2$ corresponds to the isotropy group of the unique classical 3-momentum assigned to $X^4(Y^3)$ defined as a preferred extremum of Kähler action, fixes the choice of the complex structure uniquely. Therefore group theoretical approach and the approach based on Kähler action complement each other.

1. The allowance of an infinite-dimensional group of isometries isomorphic to the group of conformal transformations of 2-sphere is completely unique feature of the 4-dimensional light cone boundary. Even more, in case of $\delta M^4_+ \times CP_2$ the isometry group of $\delta M^4_+$ becomes localized with respect to $CP_2$! Furthermore, the Kähler structure of $\delta M^4_+$ defines also symplectic structure.

   Hence any function of $\delta M^4_+ \times CP_2$ would serve as a Hamiltonian transformation acting in both $CP_2$ and $\delta M^4_+$ degrees of freedom. These transformations obviously differ from ordinary local gauge transformations. This group leaves the symplectic form of $\delta M^4_+ \times CP_2$, defined as the sum of light cone and $CP_2$ symplectic forms, invariant. The group of symplectic transformations of $\delta M^4_+ \times CP_2$ is a good candidate for the isometry group of the configuration space.

2. The approximate symplectic invariance of Kähler action is broken only by gravitational effects and is exact for vacuum extremals. If Kähler function were exactly invariant under the symplectic transformations of $CP_2$, $CP_2$ symplectic transformations wiykd correspond to zero modes having zero norm in the Kähler metric of configuration space. This does not make sense since symplectic transformations of $\delta M^4 \times CP_2$ actually parameterize the quantum fluctuation degrees of freedom.

3. The groups $G$ and $H$, and thus configuration space itself, should inherit the complex structure of the light cone boundary. The diffeomorphims of $M^4$ act as dynamical symmetries of vacuum extremals. The radial Virasoro localized with respect to $S^2 \times CP_2$ could in turn act in zero modes perhaps inducing conformal transformations: note that these transformations lead out from the symmetric space associated with given values of zero modes.

### 3.1.4 Symplectic transformations of $\delta M^4_+ \times CP_2$ as isometries of configuration space

The symplectic transformations of $\delta M^4_+ \times CP_2$ are excellent candidates for inducing symplectic transformations of the configuration space acting as isometries. There are however deep differences with respect to the Kac Moody algebras.

1. The conformal algebra of the configuration space is gigantic when compared with the Virasoro + Kac Moody algebras of string models as is clear from the fact that the Lie-algebra generator of a symplectic transformation of $\delta M^4_+ \times CP_2$ corresponding to a Hamiltonian which is product of functions defined in $\delta M^4_+$ and $CP_2$ is sum of generator of $\delta M^4_+$-local symplectic transformation of $CP_2$ and $CP_2$-local symplectic transformations of $\delta M^4_+$. This means also that the notion of local gauge transformation generalizes.

2. The physical interpretation is also quite different: the relevant quantum numbers label the unitary representations of Lorentz group and color group, and the four-momentum labeling the states of Kac Moody representations is not present. Physical states carrying no energy and momentum at quantum level are predicted. The appearance of a new kind of angular momentum not assignable to elementary particles might shed some light to the longstanding problem of baryonic spin (quarks are not responsible for the entire spin of proton). The possibility of a new kind of color might have implications even in macroscopic length scales.

3. The central extension induced from the natural central extension associated with $\delta M^4_+ \times CP_2$ Poisson brackets is anti-symmetric with respect to the generators of the symplectic algebra rather than symmetric as in the case of Kac Moody algebras associated with loop spaces. At first this seems to mean a dramatic difference. For instance, in the case of $CP_2$ symplectic transformations localized with respect to $\delta M^4_+$ the central extension would vanish for Cartan algebra, which means a profound physical difference. For $\delta M^4_+ \times CP_2$ symplectic algebra a generalization of the Kac Moody type structure however emerges naturally.

   The point is that $\delta M^4_+$-local $CP_2$ symplectic transformations are accompanied by $CP_2$ local $\delta M^4_+$ symplectic transformations. Therefore the Poisson bracket of two $\delta M^4_+$ local $CP_2$ Hamiltonians involves a term analogous to a central extension term symmetric with respect to $CP_2$ Hamiltonians, and resulting from the $\delta M^4_+$ bracket of functions multiplying the Hamiltonians. This additional term could give the entire bracket of the configuration space Hamiltonians at the maximum of the Kähler function where one expects that $CP_2$ Hamiltonians vanish and have a form essentially identical with Kac Moody central extension because it is indeed symmetric with respect to indices of the symplectic group.

### 3.1.5 Does the symmetric space property reduce to coset construction for Super Virasoro algebras?

The idea about symmetric space is extremely beautiful but it took a long time and several false alarms before the time was ripe for identifying the precise form of the Cartan decomposition $g = t + h$ satisfying the defining conditions

$$g = t + h \ , \quad [t,t] \subset h \ , \quad [h,t] \subset t \ . \tag{3.1.1}$$

The ultimate solution of the puzzle turned out to be amazingly simple and came only after quantum TGD was understood well enough.

Configuration space geometry allows two super-conformal symmetries. The first one corresponds to super-symplectic transformations acting at the level of imbedding space. The second one corresponds to super Kac-Moody symmetry acting as deformations of light-like 3-surfaces respecting their light-likeness. Super Kac-Moody algebra can be regarded as sub-algebra of super-symplectic algebra, and quantum states correspond to the coset representations for these two algebras so that the differences of the corresponding super-Virasoro generators annihilate physical states. This obviously generalizes Goddard-Olive-Kent construction [A68] . The physical interpretation is in terms of Equivalence Principle. After having realized this it took still some time to realize that this coset representation and therefore also Equivalence Principle also corresponds to the coset structure of the configuration space!

The conclusion would be that $t$ corresponds to super-symplectic algebra made also local with respect to $X^3$ and $h$ corresponds to super Kac-Moody algebra. The experience with finite-dimensional coset spaces would suggest that super Kac-Moody generators interpreted in terms of $h$ leave the points of configuration space analogous to the origin of say $CP_2$ invariant and in fact vanish at this point. Therefore super Kac-Moody generators should vanish for those 3-surfaces $X^3_l$ which correspond to the origin of coset space. The maxima of Kähler function could correspond to this kind of points and could play also an essential role in the integration over configuration space by generalizing the Gaussian integration of free quantum field theories.

### 3.1.6 What effective 2-dimensionality and holography really mean?

Concerning the interpretation of Kac-Moody algebra there are some poorly understood points, which directly relate to what one means with holography.

1. The strongest view about effective 2-dimensionality (holography) is that for preferred extremals the partonic 2-surfaces $X^2$ at the ends of $CD$ act as causal determinants fixing $X^3_l$ in the resolution defined by $CD$. A weaker view about holography is that light-like 3-surfaces with fixed ends give rise to same configuration space metric and the deformations of these surfaces by Kac-Moody algebra correspond to zero modes just like the interior degrees of freedom for space-like 3-surface do. Which of these options is the correct one? The same question can be posed in the case of space-like 3-surfaces.

2. The non-trivial action of Kac-Moody algebra in the interior of $X_l^3$ together with effective 2-dimensionality and holography would encourage the interpretation of Kac-Moody symmetries acting trivially at $X^2$ as gauge symmetries. Light-like 3-surfaces having fixed partonic 2-surfaces at their ends would be equivalent physically and effective 2-dimensionality and holography would be realized modulo gauge transformations.

3. There are also Kac-Moody generators which do not vanish at the ends of the $X_l^3$, and these would act as physical symmetries and their action would reduce at $X^2$ to symplectic action. This Kac-Moody algebra should appear in p-adic mass calculations. This seems to be in conflict with the idea that coset construction corresponds to coset space construction. Perhaps strict correspondence is too naive an assumption. Why couldn't one use the larger Kac-Moody algebra in coset construction and smaller Kac-Moody algebra in coset space construction?

4. Gauge symmetry property means that the Kähler metric of the configuration space is same for all gauge equivalent choices of $X_l^3$ and Kac-Moody deformations correspond to zero modes. Kähler function could differ by a real part of a holomorphic function of configuration space coordinates representing now Kac-Moody transforms of $X_l^3$. If Dirac determinant gives the exponent of Kähler function, the eigenvalues of the modified Dirac action can differ only by scalings with are products of holomorphic function of configuration space coordinates and its conjugates labeling different Kac-Moody transforms of $X_l^3$. This condition makes sense if one restricts the consideration to the finite number of eigenvalues $\lambda_k$ assigned to $D_K$. The introduction of instanton term transforming the eigenvalues to $\lambda_k + \sqrt{n}$ would not allow his scaling.

   Either one must assume more general spectrum of form $\lambda_k + \sqrt{n}x_k$ with $\lambda_k$ and $x_k$ scaling in identical manner or that $n = 0$ modes are enough to define Kähler function. The latter option might be correct since the preferred extremal realizes effective 2-dimensionality at space-time level and conformal excitations break it so that they should not contribute to Kähler function. Also number theoretic universality favors this option. One cannot however exclude the first option. It must be admitted that the situation is not completely understood.

### 3.1.7 About the relationship between super-symplectic and super Kac-Moody algebras

The relationship between Kac-Moody and symplectic algebras is now relatively well understood but the physical interpretation of Kac-Moody algebra deserves attention. There are two Kac-Moody algebras: the smaller one leaves partonic 2-surfaces invariant and second one affects also them. Both of them are in dual relation to the symplectic algebra and these relations correspond to coset space construction and coset construction.

TGD inspired quantum measurement theory suggests that the super-symplectic algebra and smaller Kac-Moody algebra correspond to each other like classical and quantal degrees of freedom. Hence smaller Kac-Moody algebra would act in the zero modes of the configuration space metric. In the proposed construction this indeed is the case for Kac Moody algebra elements leaving partonic 2-surface invariant and appearing in the *coset space construction* but not for those Kac-Moody algebra elements affecting partonic 2-surface and allowing interpretation as sub-algebra of symplectic algebra and appearing in *coset construction*. This interpretation conforms also with the fact that Kac-Moody algebra generates massive excitations in p-adic thermodynamics.

In TGD inspired quantum measurement theory zero modes correspond to classical non-quantum fluctuating dynamical variables in 1-1 correspondence with quantum fluctuating degrees of freedom like the positions of the pointer of the measurement apparatus with the directions of spin of electron. Hence Kac-Moody algebra would define configuration space coordinates in terms of the map induced by correlation between classical and quantal degrees of freedom induced by entanglement. The choice of gauge selecting one particular light-like 3-surface $X_l^3$ could have thus interpretation as a map mapping quantum degrees of freedom to classical ones. This choice of gauge could be achieved by the addition of phase factor depending on quantum numbers assigned with the braid strands so that stationary phase approximation would select the preferred 3-surface with fluctuations around them allowed.

The dual relation between super symplectic algebra and bigger Kac-Moody algebra is realized in terms of coset construction. The idea inspired by Olive-Goddard-Kent coset construction is that the generators of Super Virasoro algebra corresponds to the differences of those associated with Super Kac-Moody and super-symplectic algebras. The justification comes from the miraculous geometry of the light cone boundary implying that Super Kac-Moody conformal symmetries of $X^2$ can be compensated by super-symplectic local radial scalings so that the differences of corresponding Super Virasoro generators annihilate physical states. If the central extension parameters are same, the resulting central extension is trivial. What is done is to construct first a state with a non-positive conformal weight using super-symplectic generators, and then to apply Super-Kac Moody generators to compensate this conformal weight to get a state with vanishing conformal weight. Mass squared would however correspond to either Super-Kac Moody or super-symplectic mass. The identity of these masses gives rise to Equivalence Principle as a one manifestation of the coset representation.

### 3.1.8 Attempts to identify configuration space Hamiltonians

I have made several attempts to identify configuration space Hamiltonians. The first two candidates referred to as magnetic and electric Hamiltonians, emerged in a relatively early stage. The third candidate identifies Hamiltonians as Noether charges and is motivated by the QFT analogy. Magnetic option is the simplest one and the only one consistent with the interpretation of Kac-Moody symmetries leaving the ends of $X_l^3$ invariant.

#### Magnetic Hamiltonians

Assuming that the elements of the radial Virasoro algebra of $\delta M_+^4$ have zero norm, one ends up with an explicit identification of the symplectic structures of the configuration space. There is almost unique identification for the symplectic structure. Configuration space counterparts of $\delta M^4 \times CP_2$ Hamiltonians are defined by the generalized signed and and unsigned Kähler magnetic fluxes

$$\begin{aligned} Q_m(H_A, X^2) &= Z \int_{X^2} H_A J \sqrt{g_2} d^2x \ , \\ Q_m^+(H_A, r_M) &= Z \int_{X^2} H_A |J| \sqrt{g_2} d^2x \ , \\ J &\equiv \epsilon^{\alpha\beta} J_{\alpha\beta} \ . \end{aligned}$$

$H_A$ is $CP_2$ Hamiltonian multiplied by a function of coordinates of light cone boundary belonging to a unitary representation of the Lorentz group. $Z$ is a conformal factor depending on symplectic invariants. The symplectic structure is induced by the symplectic structure of $CP_2$.

The most general flux is superposition of signed and unsigned fluxes $Q_m$ and $Q_m^+$.

$$Q_m^{\alpha,\beta}(H_A, X^2) = \alpha Q_m(H_A, X^2) + \beta Q_m^+(H_A, X^2) \ .$$

Thus it seems that symmetry arguments fix the form of the configuration space metric apart from the presence of a conformal factor $Z$ multiplying the magnetic flux and the degeneracy related to the signed and unsigned fluxes.

Holography requires that the relevant data about configuration space geometry is contained by 2-D surfaces $X^2$ at the intersections of light-like 3-surfaces $\delta M_\pm^4 + \times CP_2$ defining the boundaries of causal diamonds. In this case the entire Hamiltonian could be defined as the sum of magnetic fluxes over surfaces $X_i^2 \subset X^3$.

The key feature of these Hamiltonians is that they depend on $X^2$ only. This conforms with the interpretation of Kac-Moody transformations leaving $X^2$ invariant as gauge symmetries deforming light-like 3-surfaces and leaving configuration space metric as such. By the identify $g_{k\bar{l}} = iJ_{k\bar{l}}$ the half brackets $j^{Ak} J_{k\bar{l}} j^{B\bar{l}} = \partial_k H_A J^{k\bar{l}} \partial_{\bar{l}} H^B$ would define the matrix elements of both Kähler metric and Kähler form: this means a tight constraint if Kähler action defines the metric and magnetic Hamiltonians are the correct choice.

#### Electric Hamiltonians and electric-magnetic duality

Preferred extremal property allows to consider the possibility that one can identify configuration space Hamiltonians as classical charges $Q_e(H_A)$ associated with the Hamiltonians of the symplectic transformations of the light cone boundary, that is as variational derivatives of the Kähler action with respect to the infinitesimal deformations induced by $\delta M_+^4 \times CP_2$ Hamiltonians.

Alternatively, one might simply replace Kähler magnetic field $J$ with Kähler electric field defined by space-time dual $*J$ in the formulas of previous section. These Hamiltonians are analogous to Kähler electric charge and the hypothesis motivated by the experience with the instantons of the Euclidian Yang Mills theories and 'Yin-Yang' principle, as well as by the duality of $CP_2$ geometry, is that for the preferred extremals of the Kähler action these Hamiltonians are affinely related:

$$Q_e(H_A) = Z\left[Q_m(H_A) + q_e(H_A)\right] \quad .$$

Here $Z$ and $q_e$ are constants depending on symplectic invariants only. Thus the equivalence of the two approaches to the construction of configuration space geometry boils down to the hypothesis of a physically well motivated electric-magnetic duality.

The crucial technical idea is to regard configuration space metric as a quadratic form in the entire Lie-algebra of the isometry group $G$ such that the matrix elements of the metric vanish in the subalgebra $H$ of $G$ acting as $Diff^3(X^3)$. The Lie-algebra of $G$ with degenerate metric in the sense that $H$ vector fields possess zero norm, can be regarded as a tangent space basis for the configuration space at point $X^3$ at which $H$ acts as an isotropy group: at other points of the configuration space $H$ is different. For given values of zero modes the maximum of Kähler function is the best candidate for $X^3$. This picture applies also in symplectic degrees of freedom.

There are objections against electric representation.

1. Without additional assumptions the Hamiltonians obtained by replacing induced Kähler form with its dual brings in the dependence on the induced metric of space-time surface at $X^2$ so that configuration space Hamiltonians do not transform nicely under symplectic transformations. Only if the contravariant Kähler electric field defines a symplectic invariant - maybe the preferred extremal property could guarantee this- electric representation of the Hamiltonians looks attractive. Electric-magnetic duality would follow trivially if the self duality of the induced Kähler form of $CP_2$ is preserved in the induction procedure at $X^2$.

2. Kac-Moody transformations vanishing at $X^2$ are not expected to leave the Hamiltonians invariant since they affect the induced metric. This is however highly desirable if effective 2-dimensionality holds true as gauge invariance.

### 3.1.9 For the reader

Few words about the representation of ideas are in order. For a long time the books about TGD served as kind of lab note books - a bottom-up representation providing kind of a ladder making clear the evolution of ideas. This led gradually to a rather chaotic situation in which it was difficult for me to control the internal consistency and for the possible reader to distinguish between the big ideas and ad hoc guesses, most of them related to the detailed realization of big visions. Therefore I decided to clean up a lot of the ad hoc stuff. I have also changed the representation so that it is more top-down and tries to achieve over-all views.

There are several visions about what TGD is and I have worked hardly to achieve a fusion of this visions. Hence simple linear representation in which reader climbs to a tree of wisdom is impossible. I must summarize overall view from the beginning and refer to the results deduced in chapters towards the end of the book and also to ideas discussed in other books. For instance, the construction of configuration space spinor structure discussed in the last chapter [K9] provides the understanding necessary to make the construction of configuration space geometry more detailed. Also number theoretical vision discussed in another book [K51] is necessary. Somehow it seems that a graphic representation emphasizing visually the big picture should be needed to make the representation more comprehensible.

## 3.2 How to generalize the construction of configuration space geometry to take into account the classical non-determinism?

If the imbedding space were $H_+ = M^4_+ \times CP_2$ and if Kähler action were deterministic, the construction of configuration space geometry reduces to $\delta M^4_+ \times CP_2$. Thus in this limit quantum holography principle [B19, B33] would be satisfied also in TGD framework and actually reduce to the general coordinate invariance. The classical non-determinism of Kähler action however means that this construction is not quite enough and the challenge is to generalize the construction.

### 3.2.1 Quantum holography in the sense of quantum gravity theories

In string theory context quantum holography is more or less synonymous with Maldacena conjecture [B33] which (very roughly) states that string theory in Anti-de-Sitter space AdS is equivalent with a conformal field theory at the boundary of AdS. In purely quantum gravitational context [B19] , quantum holography principle states that quantum gravitational interactions at high energy limit in AdS can be described using a topological field theory reducing to a conformal (and non-gravitational) field theory defined at the *time like* boundary of the AdS. Thus the time like boundary plays the role of a dynamical hologram containing all information about correlation functions of $d+1$ dimensional theory. This reduction also conforms with the fact that black hole entropy is proportional to the horizon area rather than the volume inside horizon.

Holography principle reduces to general coordinate invariance in TGD. If the action principle assigning space-time surface to a given 3-surface $X^3$ at light cone boundary were completely deterministic, four-dimensional general coordinate invariance would reduce the construction of the configuration geometry for the space of 3-surfaces in $M^4_+ \times CP_2$ to the construction of the geometry at the boundary of the configuration space consisting of 3-surfaces in $\delta M^4_+ \times CP_2$ (moment of big bang). Also the quantum theory would reduce to the boundary of the future light cone.

The classical non-determinism of Kähler action however implies that quantum holography in this strong form fails. This is very desirable from the point of view of both physics and consciousness theory. Classical determinism would also mean that time would be lost in TGD as it is lost in GRT. Classical non-determinism is also absolutely essential for quantum consciousness and makes possible conscious experiences with contents localized into finite time interval despite the fact that quantum jumps occur between configuration space spinor fields defining what I have used to call quantum histories. Classical non-determinism makes it also possible to generalize quantum-classical correspondence in the sense that classical non-determinism at the space-time level provides correlate for quantum non-determinism. The failure of classical determinism is a difficult challenge for the construction of the configuration space geometry. One might however hope that the notion of quantum holography generalizes.

### 3.2.2 How the classical determinism fails in TGD?

One might hope that determinism in a generalized sense might be achieved by generalizing the notion of 3-surface by allowing unions of space-like 3-surfaces with time like separations with very strong but not complete correlations between the space-like 3-surfaces. In this case the non-determinism would mean that the 3-surfaces $Y^3$ at light cone boundary correspond to at most enumerable number of preferred extremals $X^4(Y^3)$ of Kähler action so that one would get finite or at most enumerably infinite number of replicas of a given configuration space region and the construction would still reduce to the light cone boundary.

1. This is probably quite too simplistic view. Any 4-surface which has $CP_2$ projection which belongs to so called Lagrange manifold of $CP_2$ having by definition vanishing induced Kähler form is vacuum extremal. Thus there is an infinite variety of 6-dimensional sub-manifolds of $H$ for which all extremals of Kähler action are vacua.

2. $CP_2$ type vacuum extremals are different since they possess non-vanishing Kähler form and Kähler action. They are identifiable as classical counterparts of elementary particles have $M^4_+$ projection which is a random light like curve (this in fact gives rise to conformal invariance identifiable as counterpart of quaternion conformal invariance). Thus there are good reasons to suspect that classical non-determinism might destroy the dream about complete reduction to the light cone boundary.

3. The wormhole contacts connecting different space-time sheets together can be seen as pieces of $CP_2$ type extremals and one expects that the non-determinism is still there and that the metrically 2-dimensional elementary particle horizons (light like 3-surfaces of $H$ surrounding wormhole contacts and having time-like $M^4_+$ projection) might be a crucial element in the understanding of quantum TGD. The non-determinism of $CP_2$ type extremals is absolutely crucial

for the ordinary elementary particle physics. It seems that the conformal symmetries responsible for the ordinary elementary particle quantum numbers acting in these degrees of freedom do not contribute to the configuration space metric line element.

4. The possibility of space-time sheets with a negative time orientation with ensuing negative sign of classical energy is a further blow against $\delta M^4_+$ reductionism. Space-time sheets can be created as pairs of positive and negative energy space-time sheet from vacuum and this forces to modify radically the ontology of physics. Crossing symmetry allows to interpret particle reactions as a creation of zero energy states from vacuum, and the identification of the gravitational energy as the difference between positive and negative energies of matter supports the view that the net inertial (conserved Poincare-) energy of the universe vanishes both in quantal and classical sense. This option resolves unpleasant questions about net conserved quantum numbers of Universe, and provides an elegant interpretation of the vacuum extremals as correlates for systems with vanishing Poincare energy. This option is the only possible alternative from the point of view of TGD inspired cosmology where Robertson-Walker metrics are vacuum extremals with respect to inertial energy. In particular, super-symplectic invariance transforms to a fundamental symmetry of elementary particle physics besides the conformal symmetry associated with 3-D light like causal determinants which means a dramatic departure from string models unless it is somehow equivalent with the super-symplectic symmetry.

The treatment of the non-determinism in a framework in which the prediction of time evolution is seen as initial value problem, seems to be difficult. Also the notion of configuration space becomes a messy concept. Zero energy ontology changes the situation completely. Light-like 3-surfaces become representations of generalized Feynman diagrams and brings in the notion of finite time resolution. One obtains adirect connection with the concepts of quantum field theory with path integral with cutoff replaced with a sum over various preferred extremals with cutoff in time resolution.

### 3.2.3 The notions of imbedding space, 3-surface, and configuration space

The notions of imbedding space, 3-surface (and 4-surface), and configuration space (world of classical worlds (WCW)) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in $H$. The basic idea was that the definition of Kähler metric of WCW assigns to each $X^3$ a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably. Therefore it seems better to begin directly from the recent picture.

#### The notion of imbedding space

Two generalizations of the notion of imbedding space were forced by number theoretical vision [K53, K54, K52] .

1. p-Adicization forced to generalize the notion of imbedding space by gluing real and p-adic variants of imbedding space together along rationals and common algebraic numbers. The generalized imbedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of zero energy ontology [K9, K13] it became clear that the so called causal diamonds ($CD$s) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of $CD$ characterizes the position of $CD$ in $H$. If the temporal distance between upper and lower tip of $CD$ is quantized power of 2 multiples of $CP_2$ length, p-adic length scale hypothesis [K36] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of $CD$ can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that $CD$s can contains $CD$s within $CD$s, and measurement resolution dictates the length scale below which the sub-$CD$s are not visible.

3. The realization of the hierarchy of Planck constants [K18] led to a further generalization of the notion of imbedding space. Generalized imbedding space is obtained by gluing together Cartesian products of singular coverings and factor spaces of $CD$ and $CP_2$ to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized imbedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each $CD$ and $CP_2$ is replaced with a union of $CD$s and $CP_2$s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

4. The construction of quantum theory at partonic level brings in very important delicacies related to the Kähler gauge potential of $CP_2$. Kähler gauge potential must have what one might call pure gauge parts in $M^4$ in order that the theory does not reduce to mere topological quantum field theory. Hence the strict Cartesian product structure $M^4 \times CP_2$ breaks down in a delicate manner. These additional gauge components -present also in $CP_2$- play key role in the model of anyons, charge fractionization, and quantum Hall effect [K40] .

#### The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to Equivalence implied by General Coordinate Invariance. There was a problem related to the realization of General Coordinate Invariance since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for $Y^3$ at $X^4(X^3)$ and Diff$^4$ related $X^3$ should satisfy $X^4(Y^3) = X^4(X^3)$ .

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. It is however essential that information about normal space of the 2-surface is needed.

3. At some stage came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-$CD$s. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-$CD$s containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

4. A further complication relates to the hierarchy of Planck constants forcing to generalize the notion of imbedding space and also to the fact that for non-standard values of Planck constant there is symmetry breaking due to preferred plane $M^2$ preferred homologically trivial geodesic sphere of $CP_2$ having interpretation as geometric correlate for the selection of quantization axis. For given sector of $CH$ this means union over choices of this kind.

The basic vision forced by the generalization of General Coordinate Invariance has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action and are thus analogous to Bohr orbits. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

The obvious but rather ad hoc guess motivated by physical intuition was that preferred extremals correspond to the absolute minima of Kähler action for space-time surfaces containing $X^3$. This choice has some nice implications. For instance, one can develop an argument for the existence of an infinite number of conserved charges. If $X^3$ is light-like surface- either light-like boundary of $X^4$ or light-like 3-surface assignable to a wormhole throat at which the induced metric of $X^4$ changes its signature- this identification circumvents the obvious objections. This option however failed to have a direct analog in the p-adic sectors of the world of classical worlds (WCW). The reason is that minimization does not make sense for the p-adic valued counterpart of Kähler action since it is not even well-defined although the field equations make sense p-adically. Therefore, if absolute minimization makes sense it must have expression as purely algebraic conditions.

Much later number theoretical compactication led to important progress in the understanding of the preferred extremals and the conjectures were consistent with what is known about the known extremals.

1. The conclusion was that one can assign to the 4-D tangent space $T(X^4(X^3_l)) \subset M^8$ a subspace $M^2(x) \subset M^4$ having interpretation as the plane of non-physical polarizations. This in the case that the induced metric has Minkowskian signature. If not, and if co-hyper-quaternionic surface is in question, similar assigned should be possible in normal space. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW. This is as it must be since complexification does not make sense in $M^2$ degrees of freedom.

2. In number theoretical framework $M^2(x)$ has interpretation as a preferred hyper-complex sub-space of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of $M^8$. The condition $M^2(x) \subset T(X^4(X^3_l)))$ in principle fixes the tangent space at $X^3_l$, and one has good hopes that the boundary value problem is well-defined and could fix $X^4(X^3)$ at least partially as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2(x) \subset M^4$ plays also other important roles.

3. At the level of $H$ the counterpart for the choice of $M^2(x)$ seems to be following. Suppose that $X^4(X^3_l)$ has Minkowskian signature. One can assign to each point of the $M^4$ projection $P_{M^4}(X^4(X^3_l))$ a sub-space $M^2(x) \subset M^4$ and its complement $E^2(x)$, and the distributions of these planes are integrable and define what I have called Hamilton-Jacobi coordinates which can be assigned to the known extremals of Kähler with Minkowskian signature. This decomposition allows to slice space-time surfaces by string world sheets and their 2-D partonic duals. Also a slicing to 1-D light-like surfaces and their 3-D light-like duals $Y^3_l$ parallel to $X^3_l$ follows under certain conditions on the induced metric of $X^4(X^3_l)$. This decomposition exists for known extremals and has played key role in the recent developments. Physically it means that 4-surface (3-surface) reduces effectively to 3-D (2-D) surface and thus holography at space-time level.

4. The weakest form of number theoretic compactification [K54] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic $M^8$ can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of $E^4$ in the decomposition $M^8 = M^4 \times E^4$, where $M^4$ corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in $M^8$ is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

The study of the modified Dirac equation meant further steps of progress and lead to a rather detailed view about what preferred extremals are.

1. The detailed construction of the generalized eigen modes of the modified Dirac operator $D_K$ associated with Kähler action [K9] relies on the vision that the generalized eigenvalues of this operator code for information about preferred extremal of Kähler action. The view about TGD as almost topological QFT is realized if the eigenmodes correspond to the solutions of $D_K$, which are effectively 3-dimensional. Otherwise almost topological QFT property would require Chern-Simons action alone and this choice is definitely un-physical. The first guess was that the

eigenmodes are restricted to $X_l^3$ and therefore analogous to spinorial shock waves. As I realized that number theoretical compactification requires the slicing of $X^4(X_l^3)$ by light-like 3-surfaces $Y_l^3$ parallel to $X_l^3$, it became clear that super-conformal gauge invariance with respect to the coordinate labeling the slices is a more natural manner to realized effective 3-dimensionality and guarantees that $Y_l^3$ is gauge equivalent with $X_l^3$ (General Coordinate Invariance).

2. The eigen modes of the modified Dirac operator $D_K$ have the defining property that they are localized in regions of $X_l^3$, where the induced Kähler gauge field is non-vanishing. This guarantees that the number of generalized eigen modes is finite so that Dirac determinant is also finite and algebraic number if eigenvalues are algebraic numbers, and therefore makes sense also in p-adic context although Kähler action itself does not make sense p-adically.

3. The construction of the configuration space geometry in terms of modified Dirac action strengthens also the boundary conditions to the condition that there exists space-time coordinates in which the induced $CP_2$ Kähler form and induced metric satisfy the conditions $J_{ni} = 0$, $g_{ni} = 0$ hold at $X_l^3$. One could say that at $X_l^3$ situation is static both metrically and for the Maxwell field defined by the induced Kähler form.

4. The final step in the rapid evolution of ideas that too place during three months - at least I hope so since I do not want to continue this updating endlessly - was the realization that the introduction of imaginary CP breaking instanton part to the Kähler action is possible and also necessary if one wants a stringy variant of Feynman rules. Imaginary part does not contribute to the configuration space metric. This enriches the spectrum of the modified Dirac operator with an infinite number of conformal excitations breaking the effective 2-dimensionality of 3-surfaces and exact holography. Conformal excitations make possible stringy Feynman diagrammatics [K12] . A breaking of effective 3-dimensionality of space-time surface comes through the non-determinism of Kähler action which indeed is the mechanism breaking the effective 2-dimensionality. Dirac determinant can be defined in terms of zeta function regularization using Riemann Zeta. Finite measurement resolution realized in terms of braids defined on basis of purely physical criteria however forces a cutoff in conformal weight and finiteness so that number theoretical universality is not lost.

5. This picture relying crucially on the the slicing of $X^4(X^3)$ did not yet fix the definition of preferred extremals analytically at the level of field equations. The next step of progress was the realization that the requirement that the conservation of the Noether currents associated with the modified Dirac equation requires that the second variation of the Kähler action vanishes. In strongest form this condition would be satisfied for all variations and in weak sense only for those defining dynamical symmetries. The interpretation is as space-time correlate for quantum criticality and the vacuum degeneracy of Kähler action makes the criticality plausible. A generalization of the ideas of the catastrophe theory to infinite-dimensional context results [K24] . These conditions make sense also in p-adic context and have a number theoretical universal form.

Although the details of this vision might change it can be defended by its ability to fuse together all great visions about quantum TGD. In the sequel the considerations are restricted to 3-surfaces in $M_+^4 \times CP_2$. The basic outcome is that Kähler metric is expressible using the data at partonic 2-surfaces $X^2 \subset \delta M_+^4 \times CP_2$. The generalization to the actual physical situation requires the replacement of $X^2 \subset \delta M_+^4 \times CP_2$ with unions of partonic 2-surfaces located at light-like boundaries of $CD$s and sub-$CD$s.

#### The notion of configuration space

From the beginning there was a problem related to the precise definition of the configuration space ("world of classical worlds" (WCW)). Should one regard $CH$ as the space of 3-surfaces of $M^4 \times CP_2$ or $M_+^4 \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "$M_+^4$ or $M^4$?" had been settled in favor of $M_+^4$ by the fact that $M_+^4$ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M_+^4 \times CP_2$ were interpreted as cosmological rather than laboratory

symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering $M^4$ instead of $M^4_+$.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds ($CD$s) define excellent candidates for the fundamental building blocks of the configuration space or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of $H$ defined the sectors of WCW.

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the imbedding space representing the upper and lower boundaries of $CD$. Second conformal symmetry corresponds to light-like 3-surface $X^3_l$, which can be boundaries of $X^4$ and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that configuration space (WCW) is a union of configuration spaces associated with the spaces $CD \times CP_2$. $CD$s can contain $CD$s within $CD$s so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface $X^2$ - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta} J_{\alpha\beta}$ at $X^2$ define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the configuration space metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at configuration space level and TGD is a genuine theory of gravitation at this level.

2. Configuration space can be divided into slices for which the induced Kähler forms of $CP_2$ and $\delta M^4_\pm$ at the partonic 2-surfaces $X^2$ at the light-like boundaries of $CD$s are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of $CD$s).

3. This leads to the identification of the coset space structure of the sub-configuration space associated with given $CD$ in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras (symmetries respecting light-likeness of light-like 3-surfaces). Configuration space in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with Kac-Moody group. Equivalently, the local coset space $S^2 \times CP_2$ is in question: this was one of the first ideas about configuration space which I gave up as too naive!

4. Generalized coset construction and coset space structure have very deep physical meaning since they realize Equivalence Principle at quantum level: the identical actions of Super Virasoro generators for super-symplectic and super Kac-Moody algebras implies that inertial and gravitational four-momenta are identical.

### 3.2.4 The treatment of non-determinism of Kähler action in zero energy ontology

The non-determinism of Kähler action means that the reduction of the construction of the configuration space geometry to the light cone boundary fails. Besides degeneracy of the preferred extrema of Kähler action, the non-determinism should manifest itself as a presence of causal determinants also other than light cone boundary.

One can imagine two kinds of causal determinants.

1. Elementary particle horizons and light-like boundaries $X_l^3 \subset X^4$ of 4-surfaces representing wormhole throats act as causal determinants for the space-time dynamics defined by Kähler action. The boundary values of this dynamics have been already considered.

2. At imbedding space level causal determinants correspond to light like $CD$ forming a fractal hierarchy of $CD$s within $CD$s. These causal determinants determine the dynamics of zero energy states having interpretation as pairs of initial and final states in standard quantum theory.

The manner to treat the classical non-determinism would be roughly following.

1. The replacement of space-like 3-surface $X^3$ with $X_l^3$ transforms initial value problem for $X^3$ to a boundary value problem for $X_l^3$. In principle one can also use the surfaces $X^3 \subset \delta CD \times CP_2$ if $X_l^3$ fixes $X^4(X_l^3)$ and thus $X^3$ uniquely. For years an important question was whether both $X^3$ and $X_l^3$ contribute separately to the configuration space geometry or whether they provide descriptions, which are in some sense dual. This lead to the notion of 7-3 duality and I even considered the possibility that $\delta M_+^4 \times CP_2$ could be replaced with a more general surface $X_l^3 \times CP_2$ allowing also generalized symplectic and conformal symmetries. 7-3 duality is not a good term since the actual duality actually relates descriptions based on space-like 3-surfaces $X^3$ and light-like 3-surfaces $X_l^3$. Hence it seems that the proper place for 7-3 duality is in paper basked.

2. Only Super-Kac-Moody type conformal algebra makes sense in the interior of $X_l^3$. In the 2-D intersections of $X_l^3$ with the boundary of causal diamond ($CD$) defined as intersection of future and past directed light-cones super-symplectic algebra makes sense. This implies effective two-dimensionality which is broken by the non-determinism represented using the hierarchy of $CD$s meaning that the data from these 2-D surfaces and their normal spaces at boundaries of $CD$s in various scales determine the configuration space metric.

3. An important question has been whether Kac-Moody and super-symplectic algebras provide descriptions which are dual in some sense. At the level of Super-Virasoro algebras duality seems to be satisfied in the sense of generalized coset construction meaning that the differences of Super Virasoro generators of super-symplectic and super Kac-Moody algebras annihilate physical states. Among other things this means that four-momenta assignable to the two Super Virasoro representations are identical. T he interpretation is in terms of a generalization of Equivalence Principle [K9, K13] . This gives also a justification for p-adic thermodynamics applying only to Super Kac-Moody algebra.

4. Light-like 3-surfaces can be regarded also as generalized Feynman diagrams. The finite length resolution mean means also a cutoff in the number of generalized Feynman diagrams and this number remains always finite if the light-like 3-surfaces identifiable as maxima of Kähler function correspond to the diagrams. The finiteness of this number is also essential for number theoretic universality since it guarantees that the elements of $M$-matrix are algebraic numbers if momenta and other quantum numbers have this property. The introduction of new sub-$CD$s means also introduction of zero energy states in corresponding time scale.

5. The notion of finite measurement resolution expressed in terms of hierarchy of $CD$s within $CD$s is important for the treatment of classical non-determinism. In a given resolution the non-determinism of Kähler action remains invisible below the time scale assigned to the smallest $CD$s. One could also say that complete non-determinism characterized in terms path integral with cutoff is replaced in TGD framework with the partial failure of classical non-determinism leading to generalized Feynman diagrams. This gives rise to to discrete coupling constant evolution and avoids the mathematical ill-definedness and infinities plaguing path integral formalism since the functional integral over 3-surfaces is well defined.

6. Dirac determinant defining vacuum functional is assumed to correspond to exponent of Kähler action for its preferred extremal. Dirac determinant is defined as a product of finite number of eigenvalues of the transverse part $D_K(X^2)$ of the modified Dirac operator $D_K$ assumed to have decomposition $D_K = D_K(X^2) + D_K(Y^2)$ reflecting the dual slicings of $X^4$ to string world sheets $Y^2$ and partonic 2-surfaces $X^2$. The existence of the slicing is supported by the properties of known extremals of Kähler action and strongly suggested by number theoretical

compactification, and it implies among other things dimensional reduction to Minkowskian string model like theory and its Euclidian equivalent allowing to understand how Equivalence Principle is realized at space-time level. Finite number for the eigenvalues raises even hope that in a given resolution the functional integral reduces to Gaussian integral over a finite-dimensional space of logarithms of eigenvalues.

7. One can ask why Kähler action and playing with all these delicacies related to the failure of complete determinism. After all, one could formally replace Kähler action with 4-volume as in brane models. Space-time surfaces would be minimal surfaces and Dirac operator would be standard Dirac operator for the induced metric. Dirac determinant would however become infinite since the modes would not be anymore analogs of cyclotron states necessarily localized to a finite region of $X^3_l$. Recall that for Kähler action $X^3_l$ indeed decomposes into patches inside with induced Kähler form is non-vanishing and Dirac determinant defining the exponent of Kähler function is well-defined and finite without any regularization procedure. Hence Kähler action is completely unique.

### 3.2.5 Category theory and configuration space geometry

Due the effects caused by the classical non-determinism even classical TGD universes are very far from simple Cartesian clockworks, and the understanding of the general structure of the configuration space is a formidable challenge. Category theory is a branch of mathematics which is basically a theory about universal aspects of mathematical structures. Thus category theoretical thinking might help in disentangling the complexities of the configuration space geometry and the basic ideas of category theory are discussed in this spirit and as an innocent layman. It indeed turns out that the approach makes highly non-trivial predictions.

In zero energy ontology the effects of non-determinism are taken into account in terms of causal diamonds forming a hierarchical fractal structure. One must allow also the unions of $CD$s, $CD$s within $CD$s, and probably also overlapping of $CD$s, and there are good reasons to expert that $CD$s and corresponding algebraic structures could define categories. If one does not allow overlapping $CD$s then set theoretic inclusion map defines a natural arrow. If one allows both unions and intersections then $CD$s would form a structure analogous to the set of open sets used in set theoretic topology. One could indeed see $CD$s (or rather their Cartesian products with $CP_2$) as analogs of open sets in Minkowskian signature.

So called ribbon categories seem to be tailor made for the formulation of quantum TGD and allow to build bridge to topological and conformal field theories. This discussion based on standard ontology. In [K8] rather detailed category theoretical constructions are discussed. Important role is played by the notion of operad [A20, A54] : this structure can be assigned with both generalized Feynman diagrams and with the hierarchy of symplectic fusion algebras realizing symplectic analogs of the fusion rules of conformal field theories.

## 3.3 Identification of the symmetries and coset space structure of the configuration space

In this section the identification of the isometry group of the configuration space will be discussed at general level.

### 3.3.1 Reduction to the light cone boundary

The reduction to the light cone boundary would occur exactly if Kähler action were strictly deterministic. This is not the case but it is possible to generalize the construction at light cone boundary to the general case if causal diamonds define the basic structural units of the configuration space.

#### Old argument

The identification of the configuration space follows as a consequence of 4-dimensional Diff invariance. The right question to ask is the following one. How could one coordinatize the physical(!) vibrational degrees of freedom for 3-surfaces in Diff$^4$ invariant manner: coordinates should have same values for all Diff$^4$ related 3-surfaces belonging to the orbit of $X^3$? The answer is following:

1. Fix some 3-surface (call it $Y^3$) on the orbit of $X^3$ in Diff$^4$ invariant manner.

2. Use as configuration space coordinates of $X^3$ and all its diffeomorphs the coordinates parameterizing small deformations of $Y^3$. This kind of replacement is physically acceptable since metrically the configuration space is equivalent with $Map/Diff^4$.

3. Require that the fixing procedure is Lorentz invariant, where Lorentz transformations in question leave light $M^4_+$ invariant and thus act as isometries.

The simplest choice of $Y^3$ is the intersection of the orbit of 3-surface ($X^4$) with the set $\delta M^4_+ \times CP_2$ , where $\delta M^4_+$ denotes the boundary of the light cone (moment of big bang):

$$Y^3 = X^4 \cap \delta M^4_+ \times CP_2 \tag{3.3.1}$$

Lorentz invariance allows also the choice $X \times CP_2$, where $X$ corresponds to the hyperboloid $a = \sqrt{(m^0)^2 - r_M^2} = constant$ but only the proposed choice ($a = 0$) leads to a natural complexification in $M^4$ degrees of freedom. This choice is also cosmologically very natural and completely analogous to the quantum gravitational holography of string theories.

Configuration space has a fiber space structure. Base space consists of 3-surfaces $Y^3 \subset \delta M^4_+ \times CP_2$ and fiber consists of 3-surfaces on the orbit of $Y^3$ , which are Diff$^4$ equivalent with $Y^3$. The distance between the surfaces in the fiber is vanishing in configuration space metric. An elegant manner to avoid difficulties caused by Diff$^4$ degeneracy in configuration space integration is to *define* integration measure as integral over the reduced configuration space consisting of 3-surfaces $Y^3$ at the light cone boundary.

Situation is however quite not so simple. The vacuum degeneracy of Kähler action suggests strongly classical non-determinism so that there are several, possibly, infinite number of preferred extremals $X^4(Y^3)$ associated with given $Y^3$ on light cone boundary. This implies additional degeneracy.

One might hope that the reduced configuration space could be replaced by its covering space so that given $Y^3$ corresponds to several points of the covering space and configuration space has many-sheeted structure. Obviously the copies of $Y^3$ have identical geometric properties. Configuration space integral would decompose into a sum of integrals over different sheets of the reduced configuration space. Note that configuration space spinor fields are in general different on different sheets of the reduced configuration space.

Even this is probably not enough: it is quite possible that all light like surfaces of $M^4$ possessing Hamilton Jacobi structure (and thus interpretable as light fronts) are involved with the construction of the configuration space geometry. Because of their metric two-dimensionality the proposed construction should generalize. This would mean that configuration space geometry has also local laboratory scale aspects and that the general ideas might allow testing.

#### New version of the argument

This is was the argument for two decades ago. A more elegant formulation would in terms of light-like 3-surfaces connecting the boundaries of causal diamond taken as basic geometric objects and identified as generalized Feynman diagrams so that they are singular as manifolds at the vertices.

If both formulations are required to be correct, the only conclusion is that effective 2-dimensionality must hold true in the scale of given $CD$. In other words, the intersection $X^2 = X^3_l \cap X^3$ at the boundary of $CD$ is effectively the basic dynamical unit. The failure of strict non-determinism however forces to introduce entire hierarchy of $CD$s responsible also for coupling constant evolution defined in terms of the measurement resolution identified as the size of the smallest $CD$ present.

### 3.3.2 Configuration space as a union of symmetric spaces

In finite-dimensional context globally symmetric spaces are of form $G/H$ and connection and curvature are independent of the metric, provided it is left invariant under $G$. The hope is that same holds true in infinite-dimensional context. The most one can hope of obtaining is the decomposition $C(H) =$

$\cup_i G/H_i$ over orbits of $G$. One could allow also symmetry breaking in the sense that $G$ and $H$ depend on the orbit: $C(H) = \cup_i G_i/H_i$ but it seems that $G$ can be chosen to be same for all orbits. What is essential is that these groups are infinite-dimensional. The basic properties of the coset space decomposition give very strong constraints on the group $H$, which certainly contains the subgroup of $G$, whose action reduces to diffeomorphisms of $X^3$.

#### Consequences of the decomposition

If the decomposition to a union of coset spaces indeed occurs, the consequences for the calculability of the theory are enormous since it suffices to find metric and curvature tensor for single representative 3-surface on a given orbit (contravariant form of metric gives propagator in perturbative calculation of matrix elements as functional integrals over the configuration space). The representative surface can be chosen to correspond to the maximum of Kähler function on a given orbit and one obtains perturbation theory around this maximum (Kähler function is not isometry invariant).

The task is to identify the infinite-dimensional groups $G$ and $H$ and to understand the zero mode structure of the configuration space. Almost twenty (seven according to long held belief!) years after the discovery of the candidate for the Kähler function defining the metric, it became finally clear that these identifications follow quite nicely from $Diff^4$ invariance and $Diff^4$ degeneracy as well as special properties of the Kähler action.

The guess (not the first one!) would be following. $G$ corresponds to the symplectic transformations of $\delta M^4_\pm \times CP_2$ leaving the induced Kähler form invariant. If $G$ acts as isometries the values of Kähler form at partonic 2-surfaces (remember effective 2-dimensionality) are zero modes and configuration space allows slicing to symplectic orbits of the partonic 2-surface with fixed induced Kähler form. Quantum fluctuating degrees of freedom would correspond to symplectic group and to the fluctuations of the induced metric. The group $H$ dividing $G$ would in turn correspond to the Kac-Moody symmetries respecting light-likeness of $X^3_l$ and acting in $X^3_l$ but trivially at the partonic 2-surface $X^2$. This coset structure was originally discovered via coset construction for super Virasoro algebras of super-symplectic and super Kac-Moody algebras and realizes Equivalence Principle at quantum level.

#### Configuration space isometries as a subgroup of $Diff(\delta M^4_+ \times CP_2)$

The reduction to light cone boundary leads to the identification of the isometry group as some subgroup of for the group $G$ for the diffeomorphisms of $\delta M^4_+ \times CP_2$. These diffeomorphisms indeed act in a natural manner in $\delta CH$, the the space of 3-surfaces in $\delta M^4_+ \times CP_2$. Configuration space is expected to decompose to a union of the coset spaces $G/H_i$, where $H_i$ corresponds to some subgroup of $G$ containing the transformations of $G$ acting as diffeomorphisms for given $X^3$. Geometrically the vector fields acting as diffeomorphisms of $X^3$ are tangential to the 3-surface. $H_i$ could depend on the topology of $X^3$ and since $G$ does not change the topology of 3-surface each 3-topology defines separate orbit of $G$. Therefore, the union involves sum over all topologies of $X^3$ plus possibly other 'zero modes'. Different topologies are naturally glued together since singular 3-surfaces intermediate between two 3-topologies correspond to points common to the two sectors with different topologies.

### 3.3.3 Isometries of configuration space geometry as symplectic transformations of $\delta M^4_+ \times CP_2$

During last decade I have considered several candidates for the group $G$ of isometries of the configuration space as the sub-algebra of the subalgebra of $Diff(\delta M^4_+ \times CP_2)$. To begin with let us write the general decomposition of $diff(\delta M^4_+ \times CP_2)$:

$$diff(\delta M^4_+ \times CP_2) = S(CP_2) \times diff(\delta M^4_+) \oplus S(\delta M^4_+) \times diff(CP_2) \ . \tag{3.3.2}$$

Here $S(X)$ denotes the scalar function basis of space $X$. This Lie-algebra is the direct sum of light cone diffeomorphisms made local with respect to $CP_2$ and $CP_2$ diffeomorphisms made local with respect to light cone boundary.

The idea that entire diffeomorphism group would act as isometries looks unrealistic since the theory should be more or less equivalent with topological field theory in this case. Consider now the various candidates for $G$.

1. The fact that symplectic transformations of $CP_2$ and $M^4_+$ diffeomorphisms are dynamical symmetries of the vacuum extremals suggests the possibility that the diffeomorphisms of the light cone boundary and symplectic transformations of $CP_2$ could leave Kähler function invariant and thus correspond to zero modes. The symplectic transformations of $CP_2$ localized with respect to light cone boundary acting as symplectic transformations of $CP_2$ have interpretation as local color transformations and are a good candidate for the isometries. The fact that local color transformations are not even approximate symmetries of Kähler action is not a problem: if they were exact symmetries, Kähler function would be invariant and zero modes would be in question.

2. $CP_2$ local conformal transformations of the light cone boundary act as isometries of $\delta M^4_+$. Besides this there is a huge group of the symplectic symmetries of $\delta M^4_+ \times CP_2$ if light cone boundary is provided with the symplectic structure. Both groups must be considered as candidates for groups of isometries. $\delta M^4_+ \times CP_2$ option exploits fully the special properties of $\delta M^4_+ \times CP_2$, and one can develop simple argument demonstrating that $\delta M^4_+ \times CP_2$ symplectic invariance is the correct option. Also the construction of configuration space gamma matrices as super-symplectic charges supports $\delta M^4_+ \times CP_2$ option.

This picture remained same for a long time. The discovery that Kac-Moody algebra consisting of $X^2$ local symmetries generated by Hamiltonians of isometry sub-algebra of symplectic algebra forced to challenge this picture and ask whether also $X^2$-local transformations of symplectic group could be involved.

1. The basic condition is that the $X^2$ local transformation acts leaves induced Kähler form invariant apart from diffeomorphism. Denote the infinitesimal generator of $X^2$ local symplecto morphism by $\Phi_A(x)j^{Ak}$, where $A$ labels Hamiltonians in the sum and by $j^\alpha$ the generator of $X^2$ diffeomorphism.

2. The invariance of $J = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$ modulo diffeomorphism under the infinitesimal symplectic transformation gives

$$\{H^A, \Phi_A\} \equiv \partial_\alpha H^A \epsilon^{\alpha\beta}\partial_\beta \Phi_A = \partial_\alpha J j^\alpha \quad . \tag{3.3.3}$$

3. Note that here the Poisson bracket is not defined by $J^\alpha\beta$ but $\epsilon^{\alpha\beta}$ defined by the induced metric. Left hand side reflects the failure of symplectomorphism property due to the dependence of $\Phi_A(x)$ on $X^2$ coordinate which and comes from the gradients of $\delta M^4 \times CP_2$ coordinates in the expression of the induced Kähler form. Right hand side corresponds to the action of infinitesimal diffeomorphism.

4. Let us assume that one can restrict the consideration to single Hamiltonian so that the transformation is generated by $\Phi(x)H_A$ and that to each $\Phi(x)$ there corresponds a diffeomorphism of $X^2$, which is a symplectic transformation of $X^2$ with respect to symplectic form $\epsilon^{\alpha\beta}$ and generated by Hamiltonian $\Psi(x)$. This transforms the invariance condition to

$$\{H^A, \Phi\} \equiv \partial_\alpha H^A \epsilon^{\alpha\beta}\partial_\beta \Phi = \partial_\alpha J \epsilon^{\alpha\beta}\partial_\beta \Psi_A = \{J, \Psi_A\} \quad . \tag{3.3.4}$$

This condition can be solved identically by assuming that $\Phi_A$ and $\Psi$ are proportional to arbitrary smooth function of $J$:

$$\Phi = f(J) \ , \quad \Psi_A = -f(J)H_A \quad . \tag{3.3.5}$$

Therefore the $X^2$ local symplectomorphisms of $H$ reduce to symplectic transformations of $X^2$ with Hamiltonians depending on single coordinate $J$ of $X^2$. The analogy with conformal invariance for which transformations depend on single coordinate $z$ is obvious. As far as the anti-commutation relations for induced spinor fields are considered this means that $J = consant$ curves behave as points points. For extrema of $J$ appearing as candidates for points of number theoretic braids $J = constant$ curves reduce to points.

5. From the structure of the conditions it is easy to see that the transformations generate a Lie-algebra. For the transformations $\Phi^1_A H^A$ $\Phi^2_A H^A$ the commutator is

$$\Phi_A^{[1,2]} = f_A^{\ BC}\Phi_B\Phi_C \ , \tag{3.3.6}$$

where $f_A^{\ BC}$ are the structure constants for the symplectic algebra of $\delta M^4_\pm \times CP_2$. From this form it is easy to check that Jacobi identifies are satisfied. The commutator has same form as the commutator of gauge algebra generators. BRST gauge symmetry is perhaps the nearest analog of this symmetry. In the case of isometries these transforms realized local color gauge symmetry in TGD sense.

6. If space-time surface allows a slicing to light-like 3-surfaces $Y^3_l$ parallel to $X^3_l$, these conditions make sense also for the partonic 2-surfaces defined by the intersections of $Y^3_l$ with $\delta M^4_\pm \times CP_2$ and "parallel" to $X^2$. The local symplectic transformations also generalize to their local variants in $X^3_l$. Light-likeness of $X^3_l$ means effective metric 2-dimensionality so that 2-D Kähler metric and symplectic form as well as the invariant $J = \epsilon^{\alpha\beta}J_{\alpha\beta}$ exist. A straightforward calculation shows that the the notion of local symplectic transformation makes sense also now and formulas are exactly the same as above.

### 3.3.4 Identification of Kac-Moody symmetries

The Kac-Moody algebra of symmetries acting as symmetries respecting the light-likeness of 3-surfaces plays a crucial role in the identification of quantum fluctuating configuration space degrees of freedom contributing to the metric.

#### Identification of Kac-Moody algebra

The generators of bosonic super Kac-Moody algebra leave the light-likeness condition $\sqrt{g_3} = 0$ invariant. This gives the condition

$$\delta g_{\alpha\beta} Cof(g^{\alpha\beta}) = 0 \ , \tag{3.3.7}$$

Here $Cof$ refers to matrix cofactor of $g_{\alpha\beta}$ and summation over indices is understood. The conditions can be satisfied if the symmetries act as combinations of infinitesimal diffeomorphisms $x^\mu \rightarrow x^\mu + \xi^\mu$ of $X^3$ and of infinitesimal conformal symmetries of the induced metric

$$\delta g_{\alpha\beta} = \lambda(x) g_{\alpha\beta} + \partial_\mu g_{\alpha\beta}\xi^\mu + g_{\mu\beta}\partial_\alpha\xi^\mu + g_{\alpha\mu}\partial_\beta\xi^\mu \ . \tag{3.3.8}$$

#### Ansatz as an $X^3$-local conformal transformation of imbedding space

Write $\delta h^k$ as a super-position of $X^3$-local infinitesimal diffeomorphisms of the imbedding space generated by vector fields $J^A = j^{A,k}\partial_k$:

$$\delta h^k = c_A(x) j^{A,k} \ . \tag{3.3.9}$$

This gives

$$\begin{aligned} & c_A(x) \left[D_k j^A_l + D_l j^A_k\right]\partial_\alpha h^k \partial_\beta h^l + 2\partial_\alpha c_A h_{kl} j^{A,k}\partial_\beta h^l \\ = \ & \lambda(x) g_{\alpha\beta} + \partial_\mu g_{\alpha\beta}\xi^\mu + g_{\mu\beta}\partial_\alpha\xi^\mu + g_{\alpha\mu}\partial_\beta\xi^\mu \ . \end{aligned} \tag{3.3.10}$$

If an $X^3$-local variant of a conformal transformation of the imbedding space is in question, the first term is proportional to the metric since one has

$$D_k j_l^A + D_l j_k^A = 2h_{kl} \ . \tag{3.3.11}$$

The transformations in question includes conformal transformations of $H_\pm$ and isometries of the imbedding space $H$.

The contribution of the second term must correspond to an infinitesimal diffeomorphism of $X^3$ reducible to infinitesimal conformal transformation $\psi^\mu$:

$$2\partial_\alpha c_A h_{kl} j^{A,k} \partial_\beta h^l = \xi^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\beta}\partial_\alpha \xi^\mu + g_{\alpha\mu}\partial_\beta \xi^\mu \ . \tag{3.3.12}$$

**A rough analysis of the conditions**

One could consider a strategy of fixing $c_A$ and solving solving $\xi^\mu$ from the differential equations. In order to simplify the situation one could assume that $g_{ir} = g_{rr} = 0$. The possibility to cast the metric in this form is plausible since generic 3-manifold allows coordinates in which the metric is diagonal.

1. The equation for $g_{rr}$ gives

$$\partial_r c_A h_{kl} j^{Ak} \partial_r h^k = 0 \ . \tag{3.3.13}$$

The radial derivative of the transformation is orthogonal to $X^3$. No condition on $\xi^\alpha$ results. If $c_A$ has common multiplicative dependence on $c_A = f(r) d_A$ by a one obtains

$$d_A h_{kl} j^{Ak} \partial_r h^k = 0 \ . \tag{3.3.14}$$

so that $J^A$ is orthogonal to the light-like tangent vector $\partial_r h^k$ $X^3$ which is the counterpart for the condition that Kac-Moody algebra acts in the transversal degrees of freedom only. The condition also states that the components $g_{ri}$ is not changed in the infinitesimal transformation.

It is possible to choose $f(r)$ freely so that one can perform the choice $f(r) = r^n$ and the notion of radial conformal weight makes sense. The dependence of $c_A$ on transversal coordinates is constrained by the transversality condition only. In particular, a common scale factor having free dependence on the transversal coordinates is possible meaning that $X^3$- local conformal transformations of $H$ are in question.

2. The equation for $g_{ri}$ gives

$$\partial_r \xi^i = \partial_r c_A h_{kl} j^{Ak} h^{ij} \partial_j h^k \ . \tag{3.3.15}$$

The equation states that $g_{ri}$ are not affected by the symmetry. The radial dependence of $\xi^i$ is fixed by this differential equation. No condition on $\xi^r$ results. These conditions imply that the local gauge transformations are dynamical with the light-like radial coordinate $r$ playing the role of the time variable. One should be able to fix the transformation more or less arbitrarily at the partonic 2-surface $X^2$.

3. The three independent equations for $g_{ij}$ give

$$\xi^\alpha \partial_\alpha g_{ij} + g_{kj}\partial_i \xi^k + g_{ki}\partial_j \xi^k = \partial_i c_A h_{kl} j^{Ak} \partial_j h^l \ . \tag{3.3.16}$$

These are 3 differential equations for 3 functions $\xi^\alpha$ on 2 independent variables $x^i$ with $r$ appearing as a parameter. Note however that the derivatives of $\xi^r$ do not appear in the equation.

At least formally equations are not over-determined so that solutions should exist for arbitrary choices of $c_A$ as functions of $X^3$ coordinates satisfying the orthogonality conditions. If this is the case, the Kac-Moody algebra can be regarded as a local algebra in $X^3$ subject to the orthogonality constraint.

This algebra contains as a subalgebra the analog of Kac-Moody algebra for which all $c_A$ except the one associated with time translation and fixed by the orthogonality condition depends on the radial coordinate $r$ only. The larger algebra decomposes into a direct sum of representations of this algebra.

**Commutators of infinitesimal symmetries**

The commutators of infinitesimal symmetries need not be what one might expect since the vector fields $\xi^\mu$ are functionals $c_A$ and of the induced metric and also $c_A$ depends on induced metric via the orthogonality condition. What this means that $j^{A,k}$ in principle acts also to $\phi_B$ in the commutator $[c_A J^A, c_B J^B]$.

$$\left[c_A J^A, c_B J^B\right] = c_A c_B J^{[A,B]} + J^A \circ c_B J^B - J^B \circ c_A J^A \ , \tag{3.3.17}$$

where $\circ$ is a short hand notation for the change of $c_B$ induced by the effect of the conformal transformation $J^A$ on the induced metric.

Luckily, the conditions in the case $g_{rr} = g_{ir} = 0$ state that the components $g_{rr}$ and $g_{ir}$ of the induced metric are unchanged in the transformation so that the condition for $c_A$ resulting from $g_{rr}$ component of the metric is not affected. Also the conditions coming from $g_{ir} = 0$ remain unchanged. Therefore the commutation relations of local algebra apart from constraint from transversality result.

The commutator algebra of infinitesimal symmetries should also close in some sense. The orthogonality to the light-like tangent vector creates here a problem since the commutator does not obviously satisfy this condition automatically. The problem can be solved by following the recipes of non-covariant quantization of string model.

1. Make a choice of gauge by choosing time translation $P^0$ in a preferred $M^4$ coordinate frame to be the preferred generator $J^{A_0} \equiv P^0$, whose coefficient $\Phi_{A_0} \equiv \Psi(P^0)$ is solved from the orthogonality condition. This assumption is analogous with the assumption that time coordinate is non-dynamical in the quantization of strings. The natural basis for the algebra is obtained by allowing only a single generator $J^A$ besides $P^0$ and putting $d_A = 1$.

2. This prescription must be consistent with the well-defined radial conformal weight for the $J^A \neq P^0$ in the sense that the proportionality of $d_A$ to $r^n$ for $J^A \neq P^0$ must be consistent with commutators. SU(3) part of the algebra is of course not a problem. From the Lorentz vector property of $P^k$ it is clear that the commutators resulting in a repeated commutation have well-defined radial conformal weights only if one restricts $SO(3,1)$ to $SO(3)$ commuting with $P^0$. Also $D$ could be allowed without losing well-defined radial conformal weights but the argument below excludes it. This picture conforms with the earlier identification of the Kac-Moody algebra.

   Conformal algebra contains besides Poincare algebra and the dilation $D = m^k\partial_{m^k}$ the mutually commuting generators $K^k = (m^r m_r \partial_{m^k} - 2m^k m^l \partial_{m^l})/2$. The commutators involving added generators are

$$\begin{array}{ll} \left[D, K^k\right] = -K^k \ , & \left[D, P^k\right] = P^k \ , \\ \left[K^k, K^l\right] = 0 \ , & \left[K^k, P^l\right] = m^{kl} D - M^{kl} \ . \end{array} \tag{3.3.18}$$

   From the last commutation relation it is clear that the inclusion of $K^k$ would mean loss of well-defined radial conformal weights.

3. The coefficient $dm^0/dr$ of $\Psi(P^0)$ in the equation

$$\Psi(P^0)\frac{dm^0}{dr} = -J^{Ak} h_{kl} \partial_r h^l$$

is always non-vanishing due to the light-likeness of $r$. Since $P^0$ commutes with generators of $SO(3)$ (but not with $D$ so that it is excluded!), one can *define* the commutator of two generators as a commutator of the remaining part and identify $\Psi(P^0)$ from the condition above.

4. Of course, also the more general transformations act as Kac-Moody type symmetries but the interpretation would be that the sub-algebra plays the same role as $SO(3)$ in the case of Lorentz group: that is gives rise to generalized spin degrees of freedom whereas the entire algebra divided by this sub-algebra would define the coset space playing the role of orbital degrees of freedom. In fact, also the Kac-Moody type symmetries for which $c_A$ depends on the transversal coordinates of $X^3$ would correspond to orbital degrees of freedom. The presence of these orbital degrees of freedom arranging super Kac-Moody representations into infinite multiplets labeled by function basis for $X^2$ means that the number of degrees of freedom is much larger than in string models.

5. It is possible to replace the preferred time coordinate $m^0$ with a preferred light-like coordinate. There are good reasons to believe that orbifold singularity for phases of matter involving non-standard value of Planck constant corresponds to a preferred light-ray going through the tip of $\delta M^4_\pm$. Thus it would be natural to assume that the preferred $M^4$ coordinate varies along this light ray or its dual. The Kac-Moody group $SO(3)\times E^3$ respecting the radial conformal weights would reduce to $SO(2)\times E^2$ as in string models. $E^2$ would act in tangent plane of $S^2_\pm$ along this ray defining also $SO(2)$ rotation axis.

### 3.3.5 Coset space structure for a symmetric space

The key ingredient in the theory of symmetric spaces is that the Lie-algebra of $G$ has the following decomposition

$$\begin{array}{l} g = h + t \ , \\ [h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ . \end{array}$$

In present case this has highly nontrivial consequences. The commutator of *any* two infinitesimal generators generating nontrivial deformation of 3-surface belongs to $h$ and thus vanishing norm in the configuration space metric at the point which is left invariant by $H$. In fact, this same condition follows from Ricci flatness requirement and guarantees also that $G$ acts as isometries of the configuration space. This generalization is supported by the properties of the unitary representations of Lorentz group at the light cone boundary and by number theoretical considerations.

The algebras suggesting themselves as candidates are symplectic algebra of $\delta M^\pm\times CP_2$ and Kac-Moody algebra mapping light-like 3-surfaces to light-like 3-surfaces to be discussed in the next section.

The identification of the precise form of the coset space structure is however somewhat delicate.

1. The essential point is that both symplectic and Kac-Moody algebras allow representation in terms of $X^3_l$-local Hamiltonians. The general expression for the Hamilton of Kac-Moody algebra is

$$H = \sum \Phi_A(x)H^A \ . \tag{3.3.19}$$

Here $H^A$ are Hamiltonians of $SO(3)\times SU(3)$ acting in $\delta X^3_l\times CP_2$. For symplectic algebra any Hamiltonian is allowed. If $x$ corresponds to any point of $X^3_l$, one must assume a slicing of the causal diamond $CD$ by translates of $\delta M^4_\pm$.

2. For symplectic generators the dependence of form on $r^\Delta$ on light-like coordinate of $\delta X^3_l\times CP_2$ is allowed. $\Delta$ is complex parameter whose modulus squared is interpreted as conformal weight. $\Delta$ is identified as analogous quantum number labeling the modes of induced spinor field.

3. One can wonder whether the choices of the $r_M = constant$ sphere $S^2$ is the only choice. The Hamiltonin-Jacobi coordinate for $X^4_{X^3_l}$ suggest an alternative choice as $E^2$ in the decomposition of $M^4 = M^2(x)\times E^2(x)$ required by number theoretical compactification and present for known extremals of Kähler action with Minkowskian signature of induced metric. In this case $SO(3)$ would be replaced with $SO(2)$. It however seems that the radial light-like coordinate $u$ of $X^4(X^3_l)$ would remain the same since any other curve along light-like boundary would be space-like.

4. The vector fields for representing Kac-Moody algebra must vanish at the partonic 2-surface $X^2 \subset \delta M^4_\pm \times CP_2$. The corresponding vector field must vanish at each point of $X^2$:

$$j^k = \sum \Phi_A(x) J^{kl} H_l^A = 0 \ . \tag{3.3.20}$$

   This means that the vector field corresponds to $SO(2) \times U(2)$ defining the isotropy group of the point of $S^2 \times CP_2$.

   This expression could be deduced from the idea that the surfaces $X^2$ are analogous to origin of $CP_2$ at which $U(2)$ vector fields vanish. Configuration space at $X^2$ could be also regarded as the analog of the origin of local $S^2 \times CP_2$. This interpretation is in accordance with the original idea which however was given up in the lack of proper realization. The same picture can be deduced from braiding in which case the Kac-Moody algebra corresponds to local $SO(2) \times U(2)$ for each point of the braid at $X^2$. The condition that Kac-Moody generators with positive conformal weight annihilate physical states could be interpreted by stating effective 2-dimensionality in the sense that the deformations of $X^3_l$ preserving its light-likeness do not affect the physics. Note however that Kac-Moody type Virasoro generators do not annihilate physical states.

5. Kac-Moody algebra generator must leave induced Kähler form invariant at $X^2$. This is of course trivial since the action leaves each point invariant. The conditions of Cartan decomposition are satisfied. The commutators of the Kac-Moody vector fields with symplectic generators are non-vanishing since the action of symplectic generator on Kac-Moody generator restricted to $X^2$ gives a non-vanishing result belonging to the symplectic algebra. Also the commutators of Kac-Moody generators are Kac-Moody generators.

## 3.4 Complexification

A necessary prerequisite for the Kähler geometry is the complexification of the tangent space in vibrational degrees of freedom. What this means in recent context is non-trivial.

### 3.4.1 Why complexification is needed?

The Minkowskian signature of $M^4$ metric seems however to represent an insurmountable obstacle for the complexification of $M^4$ type vibrational degrees of freedom. On the other hand, complexification seems to have deep roots in the actual physical reality.

1. In the perturbative quantization of gauge fields one associates to each gauge field excitation polarization vector $e$ and massless four-momentum vector $p$ ($p^2 = 0$, $p \cdot e = 0$). These vectors define the decomposition of the tangent space of $M^4$: $M^4 = M^2 \times E^2$, where $M^2$ type polarizations correspond to zero norm states and $E^2$ type polarizations correspond to physical states with non-vanishing norm. Same type of decomposition occurs also in the linearized theory of gravitation. The crucial feature is that $E^2$ allows complexification! The general conclusion is that the modes of massless, linear, boson fields define always complexification of $M^4$ (or its tangent space) by effectively reducing it to $E^2$. Also in string models similar situation is encountered. For a string in D-dimensional space only D-2 transversal Euclidian degrees of freedom are physical.

2. Since symplectically extended isometry generators are expected to create physical states in TGD approach same kind of physical complexification should take place for them, too: this indeed takes place in string models in critical dimension. Somehow one should be able to associate polarization vector and massless four momentum vector to the deformations of a given 3-surface so that these vectors define the decomposition $M^4 = M^2 \times E^2$ for each mode. Configuration space metric should be degenerate: the norm of $M^2$ deformations should vanish as opposed to the norm of $E^2$ deformations.

Consider now the implications of this requirement.

1. In order to associate four-momentum and polarization (or at least the decomposition $M^4 = M^2 \times E^2$) to the deformations of the 3-surface one should have field equations, which determine the time development of the 3-surface uniquely. Furthermore, the time development for small deformations should be such that it makes sense to associate four momentum and polarization or at least the decomposition $M^4 = M^2 \times E^2$ to the deformations in suitable basis.

   The solution to this problem is afforded by the proposed definition of the Kähler function. The definition of the Kähler function indeed associates to a given 3-surface a unique four-surface as the preferred extremal of the Kähler action. Therefore one can associate a unique time development to the deformations of the surface $X^3$ and if TGD describes the observed world this time development should describe the evolution of photon, gluon, graviton, etc. states and so we can hope that tangent space complexification could be defined.

2. We have found that $M^2$ part of the deformation should have zero norm. In particular, the time like vibrational modes have zero norm in configuration space metric. This is true if Kähler function is not only $Diff^3$ invariant but also Diff$^4$ invariant in the sense that Kähler function has same value for all 3-surfaces belonging to the orbit of $X^3$ and related to $X^3$ by diffeomorphism of $X^4$. This is indeed the case.

3. Even this is not enough. One expects the presence of massive modes having also longitudinal polarization and for these states the number of physical vibrational degrees of freedom is 3 so that complexification seems to be impossible by odd dimension.

The reduction to the light cone boundary implied by $Diff^4$ invariance makes possible to identify the complexification. Crucial role is played by the special properties of the boundary of 4-dimensional light cone, which is metrically two-sphere and thus allows generalized complex and Kähler structure.

### 3.4.2 The metric, conformal and symplectic structures of the light cone boundary

The special metric properties of the light cone boundary play a crucial role in the complexification. The point is that the boundary of the light cone has degenerate metric: although light cone boundary is topologically 3-dimensional it is metrically 2-dimensional: effectively sphere. In standard spherical Minkowski coordinates light cone boundary is defined by the equation $r_M = m^0$ and induced metric reads

$$ds^2 = -r_M^2 d\Omega^2 = -r_M^2 dz d\bar{z}/(1+z\bar{z})^2 \ , \tag{3.4.1}$$

and has Euclidian signature. Since $S^2$ allows complexification and thus also Kähler structure (and as a by-product also symplectic structure) there are good hopes of obtaining just the required type of complexification in non-degenerate $M^4$ degrees of freedom: configuration space would effectively inherit its Kähler structure from $S^2 \times CP_2$.

By its effective two-dimensionality the boundary of the four-dimensional light cone has infinite-dimensional group of (local) conformal transformations. Using complex coordinate $z$ for $S^2$ the general local conformal transformation reads

$$\begin{aligned} r &\rightarrow f(r_M, z, \bar{z}) \ , \\ z &\rightarrow g(z) \ , \end{aligned} \tag{3.4.2}$$

where $f$ is an arbitrary real function and $g$ is an arbitrary analytic function with a finite number of poles. The infinitesimal generators of this group span an algebra, call it $C$, analogous to Virasoro algebra. This algebra is semidirect sum of two algebras $L$ and $R$ given by

$$\begin{aligned} C &= L \oplus R \ , \\ [L, R] &\subset R \ , \end{aligned} \tag{3.4.3}$$

where $L$ denotes standard Virasoro algebra of the two- sphere generated by the generators

$$L_n = z^{n+1}d/dz \tag{3.4.4}$$

and $R$ denotes the algebra generated by the vector fields

$$R_n = f_n(z,\bar{z},r_M)\partial_{r_M} \ , \tag{3.4.5}$$

where $f(z,\bar{z},r_M)$ forms complete real scalar function basis for light cone boundary. The vector fields of $R$ have the special property that they have vanishing norm in $M^4$ metric.

This modification of conformal group implies that the Virasoro generator $L_0$ becomes $L_0 = zd/dz - r_M d/dr_M$ so that the scaling momentum becomes the difference $n-m$ or $S^2$ and radial scaling momenta. One could achieve conformal invariance by requiring that $S^2$ and radial scaling quantum numbers compensate each other.

Of crucial importance is that light cone boundary allows infinite dimensional group of isometries! An arbitrary conformal transformation $z \rightarrow f(z)$ induces to the metric a conformal factor given by $|df/dz|^2$. The compensating radial scaling $r_M \rightarrow r_M/|df/dz|$ compensates this factor so that the line element remains invariant.

The Kähler structure of light cone boundary defines automatically symplectic structure. The symplectic form is degenerate and just the area form of $S^2$ given by

$$J = r_M^2 sin(\theta)d\theta \wedge d\phi,$$

in standard spherical coordinates, there is infinite-dimensional group of symplectic transformations leaving the symplectic form of the light cone boundary (that is $S^2$) invariant. These transformations are local with respect to the radial coordinate $r_M$. The symplectic and Kähler structures of light cone boundary are not unique: different structures are labeled by the coset space $SO(3,1)/SO(3)$. One can however associate with a given 3-surface $Y^3$ a unique structure by requiring that the the corresponding subgroup $SO(3)$ of Lorentz group acts as the isotropy group of the conserved classical four-momentum assigned to $Y^3$ by the preferred extremal property.

In case of $\delta M^4_+ \times CP_2$ both the conformal transformations, isometries and symplectic transformations of the light cone boundary can be made local also with respect to $CP_2$. The idea that the infinite-dimensional algebra of symplectic transformations of $\delta M^4_+ \times CP_2$ act as isometries of the configuration space and that radial vector fields having zero norm in the metric of light cone boundary possess zero norm also in configuration space metric, looks extremely attractive.

In the case of $\delta M^4_+ \times CP_2$ one could combine the symplectic and Kähler structures of $\delta M^4_+$ and $CP_2$ to single symplectic/Kähler structure. The symplectic transformations leaving this symplectic structure invariant would be generated by the function algebra of $\delta M^4_+ \times CP_2$ such that a arbitrary function serves as a Hamiltonian of a symplectic transformation. This group serves as a candidate for the isometry group of the configuration space. An alternative identification for the isometry algebra is as symplectic symmetries of $CP_2$ localized with respect to the light cone boundary. Hamiltonians would be also now elements of the function algebra of $\delta M^4_+ \times CP_2$ but their Poisson brackets would be defined using only $CP_2$ symplectic form.

The problem is to decide which option is correct. There is a simple argument fixing the latter option. The symplecticly imbedded $CP_2$ would be left invariant under $\delta M^4_+$ local symplectic transformations of $CP_2$. This seems strange. Under symplectic algebra of $\delta M^4_+ \times CP_2$ also symplecticly imbedded $CP_2$ is deformed and this sounds more realistic. The isometry algebra is therefore assumed to be the group $can(\delta M^4_+ \times CP_2)$ generated by the scalar function basis $S(\delta M^4_+ \times CP_2) = S(\delta M^4_+) \times S(CP_2)$ of the light cone boundary using the Poisson brackets to be discussed in more detail later.

There are some no-go theorems associated with higher-dimensional Abelian extensions [A57] , and although the contexts are quite different, it is interesting to consider the recent situation in light of these theorems.

1. Conformal invariance is an essentially 2-dimensional notion. Light cone boundary is however metrically and conformally 2-sphere, and therefore the conformal algebra is effectively that associated with the 2-sphere. In the same manner, the quaternion conformal algebra associated

with the metrically 2-dimensional elementary particle horizons surrounding wormhole contacts allows the usual Kac Moody algebra and actually also contributes to the configuration space metric.

2. In dimensions $D > 2$ Abelian extensions of the gauge algebra are extensions by an infinite-dimensional Abelian group rather than central extensions by the group $U(1)$. This result has an analog at the level of configuration space geometry. The extension associated with the symplectic algebra of $CP_2$ localized with respect to the light cone boundary is analogous a symplectic extension defined by Poisson bracket $\{p,q\} = 1$. The central extension is the function space associated with $\delta M^4_+$ and indeed infinite-dimensional if only only $CP_2$ symplectic structure induces the Poisson bracket but one-dimensional if $\delta M^4_+ \times CP_2$ Poisson bracket induces the extension. In the latter case the symmetries fix the metric completely at the point corresponding to the origin of symmetric space (presumably the maximum of Kähler function for given values of zero modes).

3. $D > 2$ extensions possess no unitary faithful representations (satisfying certain well motivated physical constraints) [A57] . It might be that the degeneracy of the configuration space metric is the analog for the loss of faithful representations.

### 3.4.3 Complexification and the special properties of the light cone boundary

In case of Kähler metric $G$ and $H$ Lie-algebras must allow complexification so that the isometries can act as holomorphic transformations. Since $G$ and $H$ can be regarded as subalgebras of the vector fields of $\delta M^4_+ \times CP_2$, they inherit in a natural manner the complex structure of the light cone boundary.

There are two candidates for the configuration space complexification. The simplest, and also the correct, alternative is that complexification is induced by natural complexification of vector field basis on $\delta M^4_+ \times CP_2$. In $CP_2$ degrees of freedom there is natural complexification

$$\xi \to \bar{\xi} \ .$$

In $\delta M^4_+$ degrees of freedom this could involve the transformation

$$z \to \bar{z}$$

and certainly involves complex conjugation for complex scalar function basis in the radial direction:

$$f(r_M) \to \overline{f(r_M)} \ ,$$

which turns out to play same role as the function basis of circle in the Kähler geometry of loop groups [A44] .

The requirement that the functions are eigen functions of radial scalings favors functions $(r_M/r_0)^k$, where $k$ is in general a complex number. The function can be expressed as a product of real power of $r_M$ and logarithmic plane wave. It turns out that the radial complexification alternative is the correct manner to obtain Kähler structure. The reason is that symplectic transformations leave the value of $r_M$ invariant. Radial Virasoro invariance plays crucial role in making the complexification possible.

One could consider also a second alternative assumed in the earlier formulation of the configuration space geometry. The close analogy with string models and conformal field theories suggests that for Virasoro generators the complexification must reduce to the hermitian conjugation of the conformal field theories: $L_n \to L_{-n} = L_n^\dagger$. Clearly this complexification is induced from the transformation $z \to \frac{1}{z}$ and differs from the complexification induced by complex conjugation $z \to \bar{z}$. The basis would be polynomial in $z$ and $\bar{z}$. Since radial algebra could be also seen as Virasoro algebra localized with respect to $S^2 \times CP_2$ one could consider the possibility that also in radial direction the inversion $r_M \to \frac{1}{r_M}$ is involved.

The essential prerequisite for the Kähler structure is that both $G$ and $H$ allow same complexification so that the isometries in question can be regarded as holomorphic transformations. In finite-dimensional case this essentially what is needed since metric can be constructed by parallel translation along the orbit of $G$ from $H$-invariant Kähler metric at a representative point. The requirement of $H$-invariance forces the radial complexification based on complex powers $r_M^k$: radial complexification works since symplectic transformations leave $r_M$ invariant.

Some comments on the properties of the proposed complexification are in order.

1. The proposed complexification, which is analogous to the choice of gauge in gauge theories is not Lorentz invariant unless one can fix the coordinates of the light cone boundary apart from $SO(3)$ rotation not affecting the value of the radial coordinate $r_M$ (if the imaginary part of $k$ in $r_M^k$ is always non-vanishing). This is possible as will be explained later.

2. It turns out that the function basis of light-cone boundary multiplying $CP_2$ Hamiltonians corresponds to unitary representations of the Lorentz group at light cone boundary so that the Lorentz invariance is rather manifest.

3. There is a nice connection with the proposed physical interpretation of the complexification. At the moment of the big bang all particles move with the velocity of light and therefore behave as massless particles. To a given point of the light cone boundary one can associate a unique direction of massless four-momentum by semiclassical considerations: at the point $m^k = (m^0, m^i)$ momentum is proportional to the vector $(m^0, -m^i)$. Since the particles are massless only two polarization vectors are possible and these correspond to the tangent vectors to the sphere $m^0 = r_M$. Of course, one must always fix polarizations at some point of tangent space but since massless polarization vectors are not physical this doesn't imply difficulties: different choices correspond to different gauges.

4. Complexification in the proposed manner is not possible except in the case of four-dimensional Minkowski space. Non-zero norm deformations correspond to vector fields of the light cone boundary acting on the sphere $S^{D-2}$ and the decomposition to $(1,0)$ and $(0,1)$ parts is possible only when the sphere in question is two-dimensional since other spheres do allow neither complexification nor Kähler structure.

### 3.4.4 How to fix the complex and symplectic structures in a Lorentz invariant manner?

One can assign to light-cone boundary a symplectic structure since it reduces effectively to $S^2$. The possible symplectic structures of $\delta M^4_+$ are parameterized by the coset space $SO(3,1)/SO(3))$, where $H$ is the isotropy group $SO(3)$ of a time like vector. Complexification also fixes the choice of the spherical coordinates apart from rotations around the quantization axis of angular momentum.

The selection of some preferred symplectic structure in an ad hoc manner breaks manifest Lorentz invariance but is possible if physical theory remains Lorentz invariant. The more natural possibility is that 3-surface $Y^3$ itself fixes in some natural manner the choice of the symplectic structure so that there is unique subgroup $SO(3)$ of $SO(3,1)$ associated with $Y^3$. If configuration space Kähler function corresponds to a preferred extremal of Kähler action, this is indeed the case. One can associate unique conserved four-momentum $P^k(Y^3)$ to the preferred extremal $X^4(Y^3)$ of the Kähler action and the requirement that the rotation group $SO(3)$ leaving the symplectic structure invariant leaves also $P^k(Y^3)$ invariant, fixes the symplectic structure associated with $Y^3$ uniquely.

Therefore configuration space decomposes into a union of symplectic spaces labeled by $SO(3,1)/SO(3)$ isomorphic to $a = constant$ hyperboloid of light cone. The direction of the classical angular momentum vector $w^k = \epsilon^{klmn} P_l J_{mn}$ determined by the classical angular momentum tensor of associated with $Y^3$ fixes one coordinate axis and one can require that $SO(2)$ subgroup of $SO(3)$ acting as rotation around this coordinate axis acts as phase transformation of the complex coordinate $z$ of $S^2$. Other rotations act as nonlinear holomorphic transformations respecting the complex structure.

Clearly, the coordinates are uniquely fixed modulo $SO(2)$ rotation acting as phase multiplication in this case. If $P^k(Y^3)$ is light like, one can only require that the rotation group $SO(2)$ serving as the isotropy group of 3-momentum belongs to the group $SO(3)$ characterizing the symplectic structure and it seems that symplectic structure cannot be uniquely fixed without additional constraints in this case. Probably this has no practical consequences since the 3-surfaces considered have actually infinite size and 4-momentum is most probably time like for them. Note however that the direction of 3-momentum defines unique axis such that $SO(2)$ rotations around this axis are represented as phase multiplication.

Similar almost unique frame exists also in $CP_2$ degrees of freedom and corresponds to the complex coordinates transforming linearly under $U(2)$ acting as isotropy group of the Lie-algebra element defined by classical color charges $Q_a$ of $Y^3$. One can fix unique Cartan subgroup of $U(2)$ by noticing that $SU(3)$ allows completely symmetric structure constants $d_{abc}$ such that $R_a = d_a^{\ bc} Q_b Q_c$ defines

Lie-algebra element commuting with $Q_a$. This means that $R_a$ and $Q_a$ span in generic case $U(1)\times U(1)$ Cartan subalgebra and there are unique complex coordinates for which this subgroup acts as phase multiplications. The space of nonequivalent frames is isomorphic with $CP(2)$ so that one can say that cm degrees of freedom correspond to Cartesian product of $SO(3,1)/SO(3)$ hyperboloid and $CP_2$ whereas coordinate choices correspond to the Cartesian product of $SO(3,1)/SO(2)$ and $SU(3)/U(1)\times U(1)$.

Symplectic transformations leave the value of $\delta M^4_+$ radial coordinate $r_M$ invariant and this implies large number of additional zero modes characterizing the size and shape of the 3-surface. Besides this Kähler magnetic fluxes through the $r_M = constant$ sections of $X^3$ as a function of $r_M$ provide additional invariants, which are functions rather than numbers. The Fourier components for the magnetic fluxes provide infinite number of symplectic invariants. The presence of these zero modes imply that 3-surfaces behave much like classical objects in the sense that neither their shape nor form nor classical Kähler magnetic fields, are subject to Gaussian fluctuations. Of course, quantum superpositions of 3-surfaces with different values of these invariants are possible.

There are reasons to expect that at least certain infinitesimal symplectic transformations correspond to zero modes of the Kähler metric (symplectic transformations act as dynamical symmetries of the vacuum extremals of the Kähler action). If this is indeed the case, one can ask whether it is possible to identify an integration measure for them.

If one can associate symplectic structure with zero modes, the symplectic structure defines integration measure in a standard manner (for 2n-dimensional symplectic manifold the integration measure is just the n-fold wedge power $J\wedge J...\wedge J$ of the symplectic form $J$). Unfortunately, in infinite-dimensional context this is not enough since divergence free functional integral analogous to a Gaussian integral is needed and it seems that it is not possible to integrate in zero modes and that this relates in a deep manner to state function reduction. If all symplectic transformations of $\delta M^4_+ \times CP_2$ are represented as symplectic transformations of the configuration space, then the existence of symplectic structure decomposing into Kähler (and symplectic) structure in complexified degrees of freedom and symplectic (but not Kähler) structure in zero modes, is an automatic consequence.

### 3.4.5 The general structure of the isometry algebra

There are three options for the isometry algebra of configuration space

1. Isometry algebra as the algebra of $CP_2$ symplectic transformations leaving invariant the symplectic form of $CP_2$ localized with respect to $\delta M^4_+$.

2. Certainly the configuration space metric in $\delta M^4_+$ must be non-trivial and actually given by the magnetic flux Hamiltonians defining symplectic invariants. Furthermore, the super-symplectic generators constructed from quarks automatically give as anti-commutators this part of the configuration space metric. One could interpret these symplectic invariants as configuration space Hamiltonians for $\delta M^4_+$ symplectic transformations obtained when $CP_2$ Hamiltonian is constant.

3. Isometry algebra consists of $\delta M^4_+ \times CP_2$ symplectic transformations. In this case a local color transformation involves necessarily a local $S^2$ transformation. Unfortunately, it is difficult to decide at this stage which of these options is correct.

The eigen states of the rotation generator and Lorentz boost in the same direction defining a unitary representation of the Lorentz group at light cone boundary define the most natural function basis for the light cone boundary. The elements of this bases have also well defined scaling quantum numbers and define also a unitary representation of the conformal algebra. The product of the basic functions is very simple in this basis since various quantum numbers are additive.

Spherical harmonics of $S^2$ provide an alternative function basis for the light cone boundary:

$$H^m_{jk} \equiv Y_{jm}(\theta,\phi) r_M^k \quad . \tag{3.4.6}$$

One can criticize this basis for not having nice properties under Lorentz group.

The product of basis functions is given by Glebch-Gordan coefficients for symmetrized tensor product of two representation of the rotation group. Poisson bracket in turn reduces to the Glebch-Gordans of anti-symmetrized tensor product. The quantum numbers $m$ and $k$ are additive. The basis is eigen-function basis for the imaginary part of the Virasoro generator $L_0$ generating rotations around quantization axis of angular momentum. In fact, only the imaginary part of the Virasoro generator $L_0 = zd/dz = \rho\partial_\rho - \frac{2}{2}\partial_\phi$ has global single valued Hamiltonian, whereas the corresponding representation for the transformation induced by the real part of $L_0$, with a compensating radial scaling added, cannot be realized as a global symplectic transformation.

The Poisson bracket of two functions $H^m_{j_1k_1}$ and $H^m_{j_2k_2}$ can be calculated and is of the general form

$$\begin{aligned} \{H^{m_1}_{j_1k}, H^{m_2}_{j_2k_2}\} &\equiv C(j_1m_1j_2m_2|j,m_1+m_2)_A H^{m_1+m_2}_{j,k_1+k_2} \\ . \end{aligned} \tag{3.4.7}$$

The coefficients are Glebch-Gordan coefficients for the anti-symmetrized tensor product for the representations of the rotation group.

The isometries of the light cone boundary correspond to conformal transformations accompanied by a local radial scaling compensating the conformal factor coming from the conformal transformations having parametric dependence of radial variable and $CP_2$ coordinates. It seems however that isometries cannot in general be realized as symplectic transformations. The first difficulty is that symplectic transformations cannot affect the value of the radial coordinate. For rotation algebra the representation as symplectic transformations is however possible.

In $CP_2$ degrees of freedom scalar function basis having definite color transformation properties is desirable. Scalar function basis can be obtained as the algebra generated by the Hamiltonians of color transformations by multiplication. The elements of basis can be typically expressed as monomials of color Hamiltonians $H^A_c$

$$H^A_D = \sum_{\{B_j\}} C^A_{DB_1B_2....B_N} \prod_{B_i} H^{B_i}_c \ , \tag{3.4.8}$$

where summation over all index combinations $\{B_i\}$ is understood. The coefficients $C^A_{DB_1B_2....B_N}$ are Glebch-Gordan coefficients for completely symmetric $N$:th power $8\otimes 8...\otimes 8$ of octet representations. The representation is not unique since $\sum_A H^A_c H^A_c = 1$ holds true. One can however find for each representation $D$ some minimum value of $N$.

The product of Hamiltonians $H^{D_1}_A$ and $H^B_{D_2}$ can be decomposed by Glebch-Gordan coefficients of the symmetrized representation $(D_1\otimes D_2)_S$ as

$$H^A_{D_1} H^B_{D_2} = C^{ABD}_{D_1D_2DC}(S) H^C_D \ , \tag{3.4.9}$$

where $'S'$ indicates that the symmetrized representation is in question. In the similar manner one can decompose the Poisson bracket of two Hamiltonians

$$\{H^A_{D_1}, H^B_{D_2}\} = C^{ABD}_{D_1D_2DC}(A) H^C_D \ . \tag{3.4.10}$$

Here $'A'$ indicates that Glebch-Gordan coefficients for the anti-symmetrized tensor product of the representations $D_1$ and $D_2$ are in question.

One can express the infinitesimal generators of $CP_2$ symplectic transformations in terms of the color isometry generators $J^B_c$ using the expansion of the Hamiltonian in terms of the monomials of color Hamiltonians:

$$\begin{aligned} j^A_{DN} &= F^A_{DB} J^B_c \ , \\ F^A_{DB} &= N \sum_{\{B_j\}} C^A_{DB_1B_2...B_{N-1}B} \prod_j H^{B_j}_c \ , \end{aligned} \tag{3.4.11}$$

where summation over all possible $\{B_j\}$:s appears. Therefore, the interpretation as a color group localized with respect to $CP_2$ coordinates is valid in the same sense as the interpretation of space-time diffeomorphism group as local Poincare group. Thus one can say that TGD color is localized with respect to the entire $\delta M^4_+ \times CP_2$.

A convenient basis for the Hamiltonians of $\delta M^4_+ \times CP_2$ is given by the functions

$$H^{mA}_{jkD} = H^m_{jk} H^A_D \ .$$

The symplectic transformation generated by $H^{mA}_{jkD}$ acts both in $M^4$ and $CP_2$ degrees of freedom and the corresponding vector field is given by

$$J^r \ = \ H^A_D J^{rl}(\delta M^4_+)\partial_l H^m_{jk} + H^m_{jk} J^{rl}(CP_2)\partial_l H^A_D \ . \tag{3.4.12}$$

The general form for their Poisson bracket is:

$$\begin{aligned}
&\{H^{m_1A_1}_{j_1k_1D_1}, H^{m_2A_2}_{j_2k_2D_2}\} = H^{A_1}_{D_1} H^{A_2}_{D_2}\{H^{m_1}_{j_1k_1}, H^{m_2}_{j_2k_2}\} + H^{m_1}_{j_1k_1} H^{m_2}_{j_2k_2}\{H^{A_1}_{D_1}, H^{A_2}_{D_2}\} \\
&= \left[C^{A_1A_2A}_{D_1D_2D}(S)C(j_1m_1j_2m_2|jm)_A + C^{A_1A_2A}_{D_1D_2D}(A)C(j_1m_1j_2m_2|jm)_S\right] H^{mA}_{j,k_1+k_2,D} \ .
\end{aligned} \tag{3.4.13}$$

What is essential that radial 'momenta' and angular momentum are additive in $\delta M^4_+$ degrees of freedom and color quantum numbers are additive in $CP_2$ degrees of freedom.

### 3.4.6 Representation of Lorentz group and conformal symmetries at light cone boundary

A guess deserving testing is that the representations of the Lorentz group at light cone boundary might provide natural building blocks for the construction of the configuration space Hamiltonians. In the following the explicit representation of the Lorentz algebra at light cone boundary is deduced, and a function basis giving rise to the representations of Lorentz group and having very simple properties under modified Poisson bracket of $\delta M^4_+$ is constructed.

#### Explicit representation of Lorentz algebra

It is useful to write the explicit expressions of Lorentz generators using complex coordinates for $S^2$. The expression for the $SU(2)$ generators of the Lorentz group are

$$\begin{aligned}
J_x &= (z^2-1)d/dz + c.c. = L_1 - L_{-1} + c.c. \ , \\
J_y &= (iz^2+1)d/dz + c.c. = iL_1 + iL_{-1} + c.c. \ , \\
J_z &= iz\frac{d}{dz} + c.c. = iL_z + c.c. \ .
\end{aligned} \tag{3.4.14}$$

The expressions for the generators of Lorentz boosts can be derived easily. The boost in $m^3$ direction corresponds to an infinitesimal transformation

$$\begin{aligned}
\delta m^3 &= -\varepsilon r_M \ , \\
\delta r_M &= -\varepsilon m^3 = -\varepsilon\sqrt{r_M^2 - (m^1)^2 - (m^2)^2} \ .
\end{aligned} \tag{3.4.15}$$

The relationship between complex coordinates of $S^2$ and $M^4$ coordinates $m^k$ is given by stereographic projection

$$\begin{aligned} z &= \frac{(m^1+im^2)}{(r_M-\sqrt{r_M^2-(m^1)^2-(m^2)^2})} \\ &= \frac{sin(\theta)(cos\phi+isin\phi)}{(1-cos\theta)} \ , \\ cot(\theta/2) &= \rho=\sqrt{z\bar{z}} \ , \\ tan(\phi) &= \frac{m^2}{m^1} \ . \end{aligned} \quad (3.4.16)$$

This implies that the change in $z$ coordinate doesn't depend at all on $r_M$ and is of the following form

$$\delta z \ = \ -\frac{\varepsilon}{2}(1+\frac{z(z+\bar{z})}{2})(1+z\bar{z}) \ . \quad (3.4.17)$$

The infinitesimal generator for the boosts in $z$-direction is therefore of the following form

$$L_z \ = \ [\frac{2z\bar{z}}{(1+z\bar{z})}-1]r_M\frac{\partial}{\partial_{r_M}}-iJ_z \ . \quad (3.4.18)$$

Generators of $L_x$ and $L_y$ are most conveniently obtained as commutators of $[L_z,J_y]$ and $[L_z,J_x]$. For $L_y$ one obtains the following expression:

$$L_y \ = \ 2\frac{(z\bar{z}(z+\bar{z})+i(z-\bar{z}))}{(1+z\bar{z})^2}r_M\frac{\partial}{\partial_{r_M}}-iJ_y \ , \quad (3.4.19)$$

For $L_x$ one obtains analogous expressions. All Lorentz boosts are of the form $L_i=-iJ_i+local\ radial\ scaling$ and of zeroth degree in radial variable so that their action on the general generator $X^{klm}\propto z^k\bar{z}^lr_M^m$ doesn't change the value of the label $m$ being a mere local scaling transformation in radial direction. If radial scalings correspond to zero norm isometries this representation is metrically equivalent with the representations of Lorentz boosts as Möbius transformations.

### Representations of the Lorentz group reduced with respect to $SO(3)$

The ordinary harmonics of $S^2$ define in a natural manner infinite series of representation functions transformed to each other in Lorentz transformations. The inner product defined by the integration measure $r_M^2d\Omega dr_M/r_M$ remains invariant under Lorentz boosts since the scaling of $r_M$ induced by the Lorentz boost compensates for the conformal scaling of $d\Omega$ induced by a Lorentz transformation represented as a Möbius transformation. Thus unitary representations of Lorentz group are in question.

The unitary main series representations of the Lorentz group are characterized by half-integer $m$ and imaginary number $k_2=i\rho$, where $\rho$ is any real number [A50] . A natural guess is that $m=0$ holds true for all representations realizable at the light cone boundary and that radial waves are of form $r_M^k$, $k=k_1+ik_2=-1+i\rho$ and thus eigen states of the radial scaling so that the action of Lorentz boosts is simple in the angular momentum basis. The inner product in radial degrees of freedom reduces to that for ordinary plane waves when $log(r_M)$ is taken as a new integration variable. The complexification is well-defined for non-vanishing values of $\rho$.

It is also possible to have non-unitary representations of the Lorentz group and the realization of the symmetric space structure suggests that one must have $k=k_1+ik_2$, $k_1$ half-integer. For these representations unitarity fails because the inner product in the radial degrees of freedom is non-unitary. A possible physical interpretation consistent with the general ideas about conformal invariance is that the representations $k=-1+i\rho$ correspond to the unitary ground state representations and $k=-1+n/2+i\rho$, $n=\pm1,\pm2,...$, to non-unitary representations. The general view about conformal invariance suggests that physical states constructed as tensor products satisfy the condition $\sum_i n_i=0$ completely analogous to Virasoro conditions.

### Representations of the Lorentz group with $E^2 \times SO(2)$ as isotropy group

One can construct representations of Lorentz group and conformal symmetries at the light cone boundary. Since $SL(2,C)$ is the group generated by the generators $L_0$ and $L_\pm$ of the conformal algebra, it is clear that infinite-dimensional representations of Lorentz group can be also regarded as representations of the conformal algebra. One can require that the basis corresponds to eigen functions of the rotation generator $J_z$ and corresponding boost generator $L_z$. For functions which do not depend on $r_M$ these generators are completely analogous to the generators $L_0$ generating scalings and $iL_0$ generating rotations. Also the generator of radial scalings appears in the formulas and one must consider the possibility that it corresponds to the generator $L_0$.

In order to construct scalar function eigen basis of $L_z$ and $J_z$, one can start from the expressions

$$\begin{aligned} L_3 &\equiv i(L_z + L_{\bar{z}}) = 2i[\frac{2z\bar{z}}{(1+z\bar{z})} - 1]r_M\frac{\partial}{\partial_{r_M}} + i\rho\partial_\rho \ , \\ J_3 &\equiv iL_z - iL_{\bar{z}} = i\partial_\phi \ . \end{aligned} \tag{3.4.20}$$

If the eigen functions do not depend on $r_M$, one obtains the usual basis $z^n$ of Virasoro algebra, which however is not normalizable basis. The eigenfunctions of the generators $L_3, J_3$ and $L_0 = ir_M d/dr_M$ satisfying

$$\begin{aligned} J_3 f_{m,n,k} &= m f_{m,n,k} \ , \\ L_3 f_{m,n,k} &= n f_{m,n,k} \ , \\ L_0 f_{m,n,k} &= k f_{m,n,k} \ . \end{aligned} \tag{3.4.21}$$

are given by

$$f_{m,n,k} = e^{im\phi}\frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k \ . \tag{3.4.22}$$

$n = n_1 + in_2$ and $k = k_1 + ik_2$ are in general complex numbers. The condition

$$n_1 - k_1 \geq 0$$

is required by regularity at the origin of $S^2$ The requirement that the integral over $S^2$ defining norm exists (the expression for the differential solid angle is $d\Omega = \frac{\rho}{(1+\rho^2)^2}d\rho d\phi$) implies

$$n_1 < 3k_1 + 2 \ .$$

From the relationship $(cos(\theta), sin(\theta)) = (\rho^2-1)/(\rho^2+1), 2\rho/(\rho^2+1))$ one can conclude that for $n_2 = k_2 = 0$ the representation functions are proportional to f $sin(\theta)^{n-k}(cos(\theta)-1)^{n-k}$. Therefore they have in their decomposition to spherical harmonics only spherical harmonics with angular momentum $l < 2(n-k)$. This suggests that the condition

$$|m| \leq 2(n-k) \tag{3.4.23}$$

is satisfied quite generally.

The emergence of the three quantum numbers $(m,n,k)$ can be understood. Light cone boundary can be regarded as a coset space $SO(3,1)/E^2 \times SO(2)$, where $E^2 \times SO(2)$ is the group leaving the light like vector defined by a particular point of the light cone invariant. The natural choice of the Cartan group is therefore $E^2 \times SO(2)$. The three quantum numbers $(m,n,k)$ have interpretation as quantum numbers associated with this Cartan algebra.

The representations of the Lorentz group are characterized by one half-integer valued and one complex parameter. Thus $k_2$ and $n_2$, which are Lorentz invariants, might not be independent parameters, and the simplest option is $k_2 = n_2$.

The nice feature of the function basis is that various quantum numbers are additive under multiplication:

$$f(m_a, n_a, k_a) \times f(m_b, n_b, k_b) = f(m_a + m_b, n_a + n_b, k_a + k_b) \ .$$

These properties allow to cast the Poisson brackets of the symplectic algebra of the configuration space into an elegant form.

The Poisson brackets for the $\delta M^4_+$ Hamiltonians defined by $f_{mnk}$ can be written using the expression $J^{\rho\phi} = (1+\rho^2)/\rho$ as

$$\begin{aligned} \{f_{m_a,n_a,k_a}, f_{m_b,n_b,k_b}\} &= i\left[(n_a - k_a)m_b - (n_b - k_b)m_a\right] \times f_{m_a+m_b,n_a+n_b-2,k_a+k_b} \\ &+ 2i\left[(2-k_a)m_b - (2-k_b)m_a\right] \times f_{m_a+m_b,n_a+n_b-1,k_a+k_b-1} \ . \end{aligned} \tag{3.4.24}$$

**Can one find unitary light-like representations of Lorentz group?**

It is interesting to compare the representations in question to the unitary representations of Lorentz group discussed in [A50] .

1. The unitary representations discussed in [A50] are characterized by are constructed by deducing the explicit representations for matrix elements of the rotation generators $J_x, J_y, J_z$ and boost generators $L_x, L_y, L_z$ by decomposing the representation into series of representations of $SU(2)$ defining the isotropy subgroup of a time like momentum. Therefore the states are labeled by eigenvalues of $J_z$. In the recent case the isotropy group is $E^2 \times SO(2)$ leaving light like point invariant. States are therefore labeled by three different quantum numbers.

2. The representations of [A50] are realized the space of complex valued functions of complex coordinates $\xi$ and $\overline{\xi}$ labeling points of complex plane. These functions have complex degrees $n_+ = m/2 - 1 + l_1$ with respect to $\xi$ and $n_- = -m/2 - 1 + l_1$ with respect to $\overline{\xi}$. $l_0$ is complex number in the general case but for unitary representations of main series it is given by $l_1 = i\rho$ and for the representations of supplementary series $l_1$ is real and satisfies $0 < |l_1| < 1$. The main series representation is derived from a representation space consisting of homogenous functions of variables $z^0, z^1$ of degree $n_+$ and of $\overline{z}^0$ and $\overline{z}^1$ of degrees $n_\pm$. One can separate express these functions as product of $(z^1)^{n^+}$ $(\overline{z}^1)^{n_-}$ and a polynomial of $\xi = z^1/z^2$ and $\overline{\xi}$ with degrees $n_+$ and $n_-$. Unitarity reduces to the requirement that the integration measure of complex plane is invariant under the Lorentz transformations acting as Moebius transformations of the complex plane. Unitarity implies $l_1 = -1 + i\rho$.

3. For the representations at $\delta M^4_+$ formal unitarity reduces to the requirement that the integration measure of $r_M^2 d\Omega dr_M/r_M$ of $\delta M^4_+$ remains invariant under Lorentz transformations. The action of Lorentz transformation on the complex coordinates of $S^2$ induces a conformal scaling which can be compensated by an $S^2$ local radial scaling. At least formally the function space of $\delta M^4_+$ thus defines a unitary representation. For the function basis $f_{mnk}$ $k = -1 + i\rho$ defines a candidate for a unitary representation since the logarithmic waves in the radial coordinate are completely analogous to plane waves for $k_1 = -1$. This condition would be completely analogous to the vanishing of conformal weight for the physical states of super conformal representations. The problem is that for $k_1 = -1$ guaranteing square integrability in $S^2$ implies $-2 < n_1 < -2$ so that unitarity is possible only for the function basis consisting of spherical harmonics.

   There is no deep reason against non-unitary representations and symmetric space structure indeed requires that $k_1$ is half-integer valued. First of all, configuration space spinor fields are analogous to ordinary spinor fields in $M^4$, which also define non-unitary representations of Lorentz group. Secondly, if 3-surfaces at the light cone boundary are finite-sized, the integrals defined by $f_{mnk}$ over 3-surfaces $Y^3$ are always well-defined. Thirdly, the continuous spectrum of $k_2$ could be transformed to a discrete spectrum when $k_1$ becomes half-integer valued.

Hermitian form for light cone Hamiltonians involves also the integration over $S^2$ degrees of freedom and the non-unitarity of the inner product reflects itself as non-orthogonality of the the eigen function basis. Introducing the variable $u = \rho^2 + 1$ as a new integration variable, one can express the inner product in the form

$$\begin{aligned}\langle m_a, n_a, k_a | m_b, n_b, k_b \rangle &= \pi\delta(k_{2a} - k_{2b}) \times \delta_{m_1,m_2} \times I \ , \\ I &= \int_1^\infty f(u)du \ , \\ f(u) &= \frac{(u-1)^{\frac{(N-K)+i\Delta}{2}}}{u^{K+2}} \ . \end{aligned} \tag{3.4.25}$$

The integrand has cut from $u = 1$ to infinity along real axis. The first thing to observe is that for $N = K$ the exponent of the integral reduces to very simple form and integral exists only for $K = k_{1a} + k_{1b} > -1$. For $k_{1i} = -1/2$ the integral diverges.

The discontinuity of the integrand due to the cut at the real axis is proportional to the integrand and given by

$$\begin{aligned} f(u) - f(e^{i2\pi}u) &= \left[1 - e^{-\pi\Delta}\right] f(u) \ , \\ \Delta &= n_{1a} - k_{1a} - n_{1b} + k_{1b} \ . \end{aligned} \tag{3.4.26}$$

This means that one can transform the integral to an integral around the cut. This integral can in turn completed to an integral over closed loop by adding the circle at infinity to the integration path. The integrand has $K + 1$-fold pole at $u = 0$.

Under these conditions one obtains

$$\begin{aligned} I &= \frac{2\pi i}{1 - e^{-\pi\Delta}} \times R \times (R-1) .... \times (R - K - 1) \times (-1)^{\frac{N-K}{2} - K - 1} \ , \\ R &\equiv \frac{N-K}{2} + i\Delta \ . \end{aligned} \tag{3.4.27}$$

This expression is non-vanishing for $\Delta \neq 0$. Thus it is not possible to satisfy orthogonality conditions without the un-physical $n = k, k_1 = 1/2$ constraint. The result is finite for $K > -1$ so that $k_1 > -1/2$ must be satisfied and if one allows only half-integers in the spectrum, one must have $k_1 \geq 0$, which is very natural if real conformal weights which are half integers are allowed.

### 3.4.7 How the complex eigenvalues of the radial scaling operator relate to conformal weights?

Complexified Hamiltonians can be chosen to be eigenmodes of the radial scaling operator $r_M d/dr_M$, and the first guess was that the correct interpretation is as conformal weights. The problem is however that the eigenvalues are complex. Second problem is that general arguments are not enough to fix the spectrum of eigenvalues. There should be a direct connection to the dynamics defined by Kähler action with instanton term included and the modified Dirac action defined by it.

The construction of configuration space spinor structure in terms of second quantized induced spinor fields [K9] leads to the conclusion that the modes of induced spinor fields are labeled by generalized eigenvalues $\lambda$ such that $|\lambda|^2$ has interpretation as a conformal weight and $\lambda$ itself is analogous to Higgs expectation value. Coset construction requires that super-symplectic and super Kac-Moody conformal weights $|\lambda|^2$ are same. This is achieved if the Hamiltonians are generalized eigen modes of $D = \gamma^x d/dx$, $x = log(r/r_0)$, satisfying $DH = \lambda\gamma^x H$ and thus of form $exp(\lambda x) = (r/r_0)^\lambda$ with the same spectrum of complex eigenvalues $\lambda$ as associated with the modified Dirac operator. That $log(r/r_0)$ naturally corresponds to the coordinate $u$ assignable to the generalized eigen modes of modified Dirac operator supports this interpretation.

If the Kähler action and modified Dirac action involve also the CP breaking instanton term, the eigenvalues $\lambda$ are complex and complexity relates directly also to the breaking of time reversal invariance.

## 3.5 Magnetic and electric representations of the configuration space Hamiltonians

Symmetry considerations lead to the hypothesis that configuration space Hamiltonians are apart from a factor depending on symplectic invariants equal to magnetic flux Hamiltonians. On the other hand, the hypothesis that Kähler function corresponds to a preferred extremal of Kähler action leads to the hypothesis that configuration space Hamiltonians corresponds to classical charges associated with the Hamiltonians of the light cone boundary. These charges are very much like electric charges. The requirement that two approaches are equivalent leads to the hypothesis that magnetic and electric Hamiltonians are identical apart from a factor depending on isometry invariants. At the level of $CP_2$ corresponding duality corresponds to the self-duality of Kähler form stating that the magnetic and electric parts of Kähler form are identical.

### 3.5.1 Radial symplectic invariants

All $\delta M^4_+ \times CP_2$ symplectic transformations leave invariant the value of the radial coordinate $r_M$. Therefore the radial coordinate $r_M$ of $X^3$ regarded as a function of $S^2 \times CP_2$ coordinates serves as height function. The number, type, ordering and values for the extrema for this height function in the interior and boundary components are isometry invariants. These invariants characterize not only the topology but also the size and shape of the 3-surface. The result implies that configuration space metric indeed differentiates between 3-surfaces with the size of Planck length and with the size of galaxy. The characterization of these invariants reduces to Morse theory. The extrema correspond to topology changes for the two-dimensional (one-dimensional) $r_M = constant$ section of 3-surface (boundary of 3-surface). The height functions of sphere and torus serve as a good illustrations of the situation. A good example about non-topological extrema is provided by a sphere with two horns.

There are additional symplectic invariants. The 'magnetic fluxes' associated with the $\delta M^4_+$ symplectic form

$$J_{S^2} = r_M^2 sin(\theta) d\theta \wedge d\phi$$

over any $X^2 \subset X^3$ are symplectic invariants. In particular, the integrals over $r_M = constant$ sections (assuming them to be 2-dimensional) are symplectic invariants. They give simply the solid angle $\Omega(r_M)$ spanned by $r_M = constant$ section and thus $r_M^2\Omega(r_M)$ characterizes transversal geometric size of the 3-surface. A convenient manner to discretize these invariants is to consider the Fourier components of these invariants in radial logarithmic plane wave basis discussed earlier:

$$\Omega(k) = \int_{r_{min}}^{r_{max}} (r_M/r_{max})^k \Omega(r_M) \frac{dr_M}{r_M} \ , \ k = k_1 + ik_2 \ , \ perk_1 \geq 0 \ . \quad (3.5.1)$$

One must take into account that for each section in which the topology of $r_M = constant$ section remains constant one must associate invariants with separate components of the two-dimensional section. For a given value of $r_M$, $r_M$ constant section contains several components (to visualize the situation consider torus as an example).

Also the quantities

$$\Omega^+(X^2) = \int_{X^2} |J| \equiv \int |\epsilon^{\alpha\beta} J_{\alpha\beta}| \sqrt{g_2} d^2x$$

are symplectic invariants and provide additional geometric information about 3-surface. These fluxes are non-vanishing also for closed surfaces and give information about the geometry of the boundary components of 3-surface (signed fluxes vanish for boundary components unless they enclose the dip of the light cone).

Since zero norm generators remain invariant under complexification, their contribution to the Kähler metric vanishes. It is not at all obvious whether the configuration space integration measure in these degrees of freedom exists at all. A localization in zero modes occurring in each quantum jump seems a more plausible and under suitable additional assumption it would have interpretation as a state function reduction. In string model similar situation is encountered; besides the functional integral determined by string action, one has integral over the moduli space.

If the effective 2-dimensionality implied by the strong form of general coordinate invariance discussed in the introduction is accepted, there is no need to integrate over the variable $r_M$ and just the fluxes over the 2-surfaces $X_i^2$ identified as intersections of light like 3-D causal determinants with $X^3$ contain the data relevant for the construction of the configuration space geometry. Also the symplectic invariants associated with these surfaces are enough.

### 3.5.2 Kähler magnetic invariants

The Kähler magnetic fluxes defined both the normal component of the Kähler magnetic field and by its absolute value

$$\begin{aligned} Q_m(X^2) &= \int_{X^2} J_{CP_2} = J_{\alpha\beta}\epsilon^{\alpha\beta}\sqrt{g_2}d^2x \ , \\ Q_m^+(X^2) &= \int_{X^2} |J_{CP_2}| \equiv \int_{X^2} |J_{\alpha\beta}\epsilon^{\alpha\beta}|\sqrt{g_2}d^2x \ , \end{aligned} \quad (3.5.2)$$

over suitably defined 2-surfaces are invariants under both Lorentz isometries and the symplectic transformations of $CP_2$ and can be calculated once $X^3$ is given.

For a closed surface $Q_m(X^2)$ vanishes unless the homology equivalence class of the surface is nontrivial in $CP_2$ degrees of freedom. In this case the flux is quantized. $Q_M^+(X^2)$ is non-vanishing for closed surfaces, too. Signed magnetic fluxes over non-closed surfaces depend on the boundary of $X^2$ only:

$$\begin{array}{l} \int_{X^2} J = \int_{\delta X^2} A \ . \\ J = dA \ . \end{array}$$

Un-signed fluxes can be written as sum of similar contributions over the boundaries of regions of $X^2$ in which the sign of $J$ remains fixed.

$$\begin{aligned} Q_m(X^2) &= \int_{X^2} J_{CP_2} = J_{\alpha\beta}\epsilon^{\alpha\beta}\sqrt{g_2}d^2x \ , \\ Q_m^+(X^2) &= \int_{X^2} |J_{CP_2}| \equiv \int_{X^2} |J_{\alpha\beta}\epsilon^{\alpha\beta}|\sqrt{g_2}d^2x \ , \end{aligned} \quad (3.5.3)$$

There are also symplectic invariants, which are Lorentz covariants and defined as

$$\begin{aligned} Q_m(K,X^2) &= \int_{X^2} f_K J_{CP_2} \ , \\ Q_m^+(K,X^2) &= \int_{X^2} f_K |J_{CP_2}| \ , \\ f_{K\equiv(s,n,k)} &= e^{is\phi} \times \frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k \end{aligned} \quad (3.5.4)$$

These symplectic invariants transform like an infinite-dimensional unitary representation of Lorentz group.

There must exist some minimal number of symplectically non-equivalent 2-surfaces of $X^3$, and the magnetic fluxes over the representatives these surfaces give thus good candidates for zero modes.

1. If effective 2-dimensionality is accepted, the surfaces $X_i^2$ defined by the intersections of light like 3-D causal determinants $X_l^3$ and $X^3$ provide a natural identification for these 2-surfaces.

2. Without effective 2-dimensionality the situation is more complex. Since symplectic transformations leave $r_M$ invariant, a natural set of 2-surfaces $X^2$ appearing in the definition of fluxes are separate pieces for $r_M = \textit{constant}$ sections of 3-surface. For a generic 3-surface, these surfaces are 2-dimensional and there is continuum of them so that discrete Fourier transforms of these invariants are needed. One must however notice that $r_M = \textit{constant}$ surfaces could be be 3-dimensional in which case the notion of flux is not well-defined.

### 3.5.3 Isometry invariants and spin glass analogy

The presence of isometry invariants implies coset space decomposition $\cup_i G/H$. This means that quantum states are characterized, not only by the vacuum functional, which is just the exponential $exp(K)$ of Kähler function (Gaussian in lowest approximation) but also by a wave function in vacuum modes. Therefore the functional integral over the configuration space decomposes into an integral over zero modes for approximately Gaussian functionals determined by $exp(K)$. The weights for the various vacuum mode contributions are given by the probability density associated with the zero modes. The integration over the zero modes is a highly problematic notion and it could be eliminated if a localization in the zero modes occurs in quantum jumps. The localization would correspond to a state function reduction and zero modes would be effectively classical variables correlated in one-one manner with the quantum numbers associated with the quantum fluctuating degrees of freedom.

For a given orbit $K$ depends on zero modes and thus one has mathematical similarity with spin glass phase for which one has probability distribution for Hamiltonians appearing in the partition function $exp(-H/T)$. In fact, since TGD Universe is also critical, exact similarity requires that also the temperature is critical for various contributions to the average partition function of spin glass phase. The characterization of isometry invariants and zero modes of the Kähler metric provides a precise characterization for how TGD Universe is quantum analog of spin glass.

The spin glass analogy has been the basic starting point in the construction of p-adic field theory limit of TGD. The ultra-metric topology for the free energy minima of spin glass phase motivates the hypothesis that effective quantum average space-time possesses ultra-metric topology. This approach leads to excellent predictions for elementary particle masses and predicts even new branches of physics [K32, K57] . As a matter fact, an entire fractal hierarchy of copies of standard physics is predicted.

### 3.5.4 Magnetic flux representation of the symplectic algebra

Accepting the strong form of general coordinate invariance implying effective two-dimensionality WCW Hamiltonians correspond to the fluxes associated with various 2-surfaces $X^2_i$ defined by the intersections of light-like light-like 3-surfaces $X^3_{l,i}$ with $X^3$ at the boundaries of $CD$ considered. Bearing in mind that zero energy ontology is the correct approach, one can restrict the consideration on fluxes at $\delta M^4_+ \times CP_2$ One must also remember that if the proposed symmetries hold true, it is in principle choose any partonic 2-surface in the conjectured slicing of the Minkowskian space-time sheet to partonic 2-surfaces parametrized by the points of stringy world sheets.vA physically attractive realization of the slicings of space-time surface by 3-surfaces and string world sheets is discussed in [K25] by starting from the observation that TGD could define a natural realization of braids, braid cobordisms, and 2-knots.

#### Generalized magnetic fluxes

Isometry invariants are just special case of the fluxes defining natural coordinate variables for the configuration space. Symplectic transformations of $CP_2$ act as $U(1)$ gauge transformations on the Kähler potential of $CP_2$ (similar conclusion holds at the level of $\delta M^4_+ \times CP_2$).

One can generalize these transformations to local symplectic transformations by allowing the Hamiltonians to be products of the $CP_2$ Hamiltonians with the real and imaginary parts of the functions $f_{m,n,k}$ (see Eq. 3.4.22) defining the Lorentz covariant function basis $H_A$, $A \equiv (a, m, n, k)$ at the light cone boundary: $H_A = H_a \times f(m, n, k)$, where $a$ labels the Hamiltonians of $CP_2$.

One can associate to any Hamiltonian $H^A$ of this kind both signed and unsigned magnetic flux via the following formulas:

$$\begin{aligned} Q_m(H_A|X^2) &= \int_{X^2} H_A J \ , \\ Q^+_m(H_A|X^2) &= \int_{X^2} H_A|J| \ . \end{aligned} \tag{3.5.5}$$

Here $X^2$ corresponds to any surface $X^2_i$ resulting as intersection of $X^3$ with $X^3_{l,i}$. Both signed and unsigned magnetic fluxes and their superpositions

$$Q_m^{\alpha,\beta}(H_A|X^2) \quad = \quad \alpha Q_m(H_A|X^2) + \beta Q_m^+(H_A|X^2) \ , \quad A \equiv (a,s,n,k) \tag{3.5.6}$$

provide representations of Hamiltonians. Note that symplectic invariants $Q_m^{\alpha,\beta}$ correspond to $H^A = 1$ and $H^A = f_{s,n,k}$. $H^A = 1$ can be regarded as a natural central term for the Poisson bracket algebra. Therefore, the isometry invariance of Kähler magnetic and electric gauge fluxes follows as a natural consequence.

The obvious question concerns about the correct values of the parameters $\alpha$ and $\beta$. One possibility is that the flux is an unsigned flux so that one has $\alpha = 0$. This option is favored by the construction of the configuration space spinor structure involving the construction of the fermionic super charges anti-commuting to configuration space Hamiltonians: super charges contain the square root of flux, which must be therefore unsigned. Second possibility is that magnetic fluxes are signed fluxes so that $\beta$ vanishes.

One can define also the electric counterparts of the flux Hamiltonians by replacing $J$ in the defining formulas with its dual $*J$

$$*J_{\alpha\beta} = \epsilon_{\alpha\beta}^{\ \ \gamma\delta} J_{\gamma\delta} .$$

For $H_A = 1$ these fluxes reduce to ordinary Kähler electric fluxes. These fluxes are however not symplectic covariants since the definition of the dual involves the induced metric, which is not symplectic invariant. The electric gauge fluxes for Hamiltonians in various representations of the color group ought to be important in the description of hadrons, not only as string like objects, but quite generally. These degrees of freedom would be identifiable as non-perturbative degrees of freedom involving genuinely classical Kähler field whereas quarks and gluons would correspond to the perturbative degrees of freedom, that is the interactions between $CP_2$ type extremals.

### Poisson brackets

From the symplectic invariance of the radial component of Kähler magnetic field it follows that the Lie-derivative of the flux $Q_m^{\alpha,\beta}(H_A)$ with respect to the vector field $X(H_B)$ is given by

$$X(H_B) \cdot Q_m^{\alpha,\beta}(H_A) \quad = \quad Q_m^{\alpha,\beta}(\{H_B, H_A\}) \ . \tag{3.5.7}$$

The transformation properties of $Q_m^{\alpha,\beta}(H_A)$ are very nice if the basis for $H_B$ transforms according to appropriate irreducible representation of color group and rotation group. This in turn implies that the fluxes $Q_m^{\alpha,\beta}(H_A)$ as functionals of 3-surface on given orbit provide a representation for the Hamiltonian as a functional of 3-surface. For a given surface $X^3$, the Poisson bracket for the two fluxes $Q_m^{\alpha,\beta}(H_A)$ and $Q_m^{\alpha,\beta}(H_B)$ can be defined as

$$\begin{aligned} \{Q_m^{\alpha,\beta}(H_A), Q_m^{\alpha,\beta}(H_B)\} \quad &\equiv \quad X(H_B) \cdot Q_m^{\alpha,\beta}(H_A) \\ &= \quad Q_m^{\alpha,\beta}(\{H_A, H_B\}) = Q_m^{\alpha,\beta}(\{H_A, H_B\}) \ . \end{aligned} \tag{3.5.8}$$

The study of configuration space gamma matrices identifiable as symplectic super charges demonstrates that the supercharges associated with the radial deformations vanish identically so that radial deformations correspond to zero norm degrees of freedom as one might indeed expect on physical grounds. The reason is that super generators involve the invariants $j^{ak}\gamma_k$ which vanish by $\gamma_{r_M} = 0$.

The natural central extension associated with the symplectic group of $CP_2$ ($\{p,q\} = 1!$) induces a central extension of this algebra. The central extension term resulting from $\{H_A, H_B\}$ when $CP_2$ Hamiltonians have $\{p,q\} = 1$ equals to the symplectic invariant $Q_m^{\alpha,\beta}(f(m_a + m_b, n_a + n_b, k_a + k_b))$ on the right hand side. This extension is however anti-symmetric in symplectic degrees of freedom rather than in loop space degrees of freedom and therefore does not lead to the standard Kac Moody type algebra.

Quite generally, the Virasoro and Kac Moody algebras of string models are replaced in TGD context by much larger symmetry algebras. Kac Moody algebra corresponds to the the deformations of light-like 3-surfaces respecting their light-likeness and leaving partonic 2-surfaces at $\delta CD$ intact and are highly relevant to the elementary particle physics. This algebra allows a representation in terms of $X^3_l$ local Hamiltonians generating isometries of $\delta M^4_\pm \times CP_2$. Hamiltonian representation is essential for super-symmetrization since fermionic super charges anti-commute to Hamiltonians rather than vector fields: this is one of the deep differences between TGD and string models. Kac-Moody algebra does not contribute to configuration space metric since by definition the generators vanish at partonic 2-surfaces. This is essential for the coset space property.

A completely new algebra is the $CP_2$ symplectic algebra localized with respect to the light cone boundary and relevant to the configuration space geometry. This extends to $S^2 \times CP_2$ -or rather $\delta M^4_\pm \times CP_2$ symplectic algebra and this gives the strongest predictions concerning configuration space metric. The local radial Virasoro localized with respect to $S^2 \times CP_2$ acts in zero modes and has automatically vanishing norm with respect to configuration space metric defined by super charges.

### 3.5.5 Symplectic transformations of $\delta M^4_\pm \times CP_2$ as isometries and electric-magnetic duality

According to the construction of Kähler metric, symplectic transformations of $\delta M^4_\pm \times CP_2$ act as isometries whereas radial Virasoro algebra localized with respect to $CP_2$ has zero norm in the configuration space metric.

Hamiltonians can be organized into light like unitary representations of $so(3,1) \times su(3)$ and the symmetry condition $Zg(X,Y) = 0$ requires that the component of the metric is $so(3,1) \times su(3)$ invariant and this condition is satisfied if the component of metric between two different representations $D_1$ and $D_2$ of $so(3,1) \times su(3)$ is proportional to Glebch-Gordan coefficient $C_{D_1 D_2, D_S}$ between $D_1 \otimes D_2$ and singlet representation $D_S$. In particular, metric has components only between states having identical $so(3,1) \times su(3)$ quantum numbers.

Magnetic representation of configuration space Hamiltonians means the action of the symplectic transformations of the light cone boundary as configuration space isometries is an intrinsic property of the light cone boundary. If electric-magnetic duality holds true, the preferred extremal property only determines the conformal factor of the metric depending on zero modes. This is precisely as it should be if the group theoretical construction works. Hence it should be possible by a direct calculation check whether the metric defined by the magnetic flux Hamiltonians as half Poisson brackets in complex coordinates is invariant under isometries. Symplectic invariance of the metric means that matrix elements of the metric are left translates of the metric along geodesic lines starting from the origin of coordinates, which now naturally corresponds to the preferred extremal of Kähler action. Since metric derives from symplectic form this means that the matrix elements of symplectic form given by Poisson brackets of Hamiltonians must be left translates of their values at origin along geodesic line. The matrix elements in question are given by flux Hamiltonians and since symplectic transforms of flux Hamiltonian is flux Hamiltonian for the symplectic transform of Hamiltonian, it seems that the conditions are satisfied.

## 3.6 General expressions for the symplectic and Kähler forms

One can derive general expressions for symplectic and Kähler forms as well as Kähler metric of the configuration space. The fact that these expressions involve only first variation of the Kähler action implies huge simplification of the basic formulas. Duality hypothesis leads to further simplifications of the formulas.

### 3.6.1 Closedness requirement

The fluxes of Kähler magnetic and electric fields for the Hamiltonians of $\delta M^4_+ \times CP_2$ suggest a general representation for the components of the symplectic form of the configuration space. The basic requirement is that Kähler form satisfies the defining condition

$$X \cdot J(Y,Z) + J([X,Y],Z) + J(X,[Y,Z]) = 0 \ , \tag{3.6.1}$$

where $X, Y, Z$ are now vector fields associated with Hamiltonian functions defining configuration space coordinates.

### 3.6.2 Matrix elements of the symplectic form as Poisson brackets

Quite generally, the matrix element of $J(X(H_A), X(H_B))$ between vector fields $X(H_A))$ and $X(H_B))$ defined by the Hamiltonians $H_A$ and $H_B$ of $\delta M^4_+ \times CP_2$ isometries is expressible as Poisson bracket

$$J^{AB} \quad = \quad J(X(H_A), X(H_B)) = \{H_A, H_B\} \ . \tag{3.6.2}$$

$J^{AB}$ denotes contravariant components of the symplectic form in coordinates given by a subset of Hamiltonians. The magnetic flux Hamiltonians $Q^{\alpha,\beta}_m(H_{A,k})$ of Eq. 4.7.1 provide an explicit representation for the Hamiltonians at the level of configuration space so that the components of the symplectic form of the configuration space are expressible as classical charges for the Poisson brackets of the Hamiltonians of the light cone boundary:

$$J(X(H_A), X(H_B)) \quad = \quad Q^{\alpha,\beta}_m(\{H_A, H_B\}) \ . \tag{3.6.3}$$

Recall that the superscript $\alpha, \beta$ refers the coefficients of $J$ and $|J|$ in the superposition of these Kähler magnetic fluxes. Note that $Q^{\alpha,\beta}_m$ contains unspecified conformal factor depending on symplectic invariants characterizing $Y^3$ and is unspecified superposition of signed and unsigned magnetic fluxes.

This representation does not carry information about the tangent space of space-time surface at the partonic 2-surface, which motivates the proposal that also electric fluxes are present and proportional to magnetic fluxes with a factor $K$, which is symplectic invariant so that commutators of flux Hamiltonians come out correctly. This would give

$$Q^{\alpha,\beta}_m(H_A)_{em} \quad = \quad Q^{\alpha,\beta}_e(H_A) + Q^{\alpha,\beta}_m(H_A) = (1+K)Q^{\alpha,\beta}_m(H_A) \ . \tag{3.6.4}$$

Since Kähler form relates to the standard field tensor by a factor $e/\hbar$, flux Hamiltonians are dimensionless so that commutators do not involve $\hbar$. The commutators would come as

$$Q^{\alpha,\beta}_{em}(\{H_A, H_B\}) \to (1+K)Q^{\alpha,\beta}_m(\{H_A, H_B\}) \ . \tag{3.6.5}$$

The factor $1 + K$ plays the same role as Planck constant in the commutators.

WCW Hamiltonians vanish for the extrema of the Kähler function as variational derivatives of the Kähler action. Hence Hamiltonians are good candidates for the coordinates appearing as coordinates in the perturbative functional integral around extrema (with maxima giving dominating contribution). It is clear that configuration space coordinates around a given extremum include only those Hamiltonians, which vanish at extremum (that is those Hamiltonians which span the tangent space of $G/H$) In Darboux coordinates the Poisson brackets reduce to the symplectic form

$$\begin{aligned} \{P^I, Q^J\} \quad &= \quad J^{IJ} = J_I \delta^{I,J} \ . \\ J_I \quad &= \quad 1 \ . \end{aligned} \tag{3.6.6}$$

It is not clear whether Darboux coordinates with $J_I = 1$ are possible in the recent case: probably the unit matrix on right hand side of the defining equation is replaced with a diagonal matrix depending on symplectic invariants so that one has $J_I \neq 1$. The integration measure is given by the symplectic volume element given by the determinant of the matrix defined by the Poisson brackets of the Hamiltonians appearing as coordinates. The value of the symplectic volume element is given by the matrix formed by the Poisson brackets of the Hamiltonians and reduces to the product

$$Vol = \prod_I J_I$$

in generalized Darboux coordinates.

Kähler potential (that is gauge potential associated with Kähler form) can be written in Darboux coordinates as

$$A \quad = \quad \sum_I J_I P_I dQ^I \ . \tag{3.6.7}$$

### 3.6.3 General expressions for Kähler form, Kähler metric and Kähler function

The expressions of Kähler form and Kähler metric in complex coordinates can obtained by transforming the contravariant form of the symplectic form from symplectic coordinates provided by Hamiltonians to complex coordinates:

$$J^{Z^i\bar{Z}^j} = iG^{Z^i\bar{Z}^j} \quad = \quad \partial_{H^A}Z^i\partial_{H^B}\bar{Z}^jJ^{AB} \ , \tag{3.6.8}$$

where $J^{AB}$ is given by the classical Kahler charge for the light cone Hamiltonian $\{H^A, H^B\}$. Complex coordinates correspond to linear coordinates of the complexified Lie-algebra providing exponentiation of the isometry algebra via exponential mapping. What one must know is the precise relationship between allowed complex coordinates and Hamiltonian coordinates: this relationship is in principle calculable. In Darboux coordinates the expressions become even simpler:

$$J^{Z^i\bar{Z}^j} = iG^{Z^i\bar{Z}^j} \quad = \quad \sum_I J(I)(\partial_{P^i}Z^i\partial_{Q^I}\bar{Z}^j - \partial_{Q^I}Z^i\partial_{P^I}\bar{Z}^j) \ . \tag{3.6.9}$$

Kähler function can be formally integrated from the relationship

$$\begin{aligned} A_{Z^i} &= i\partial_{Z^i}K \ , \\ A_{\bar{Z}^i} &= -i\partial_{Z^i}K \ . \end{aligned} \tag{3.6.10}$$

holding true in complex coordinates. Kähler function is obtained formally as integral

$$K \quad = \quad \int_0^Z (A_{Z^i}dZ^i - A_{\bar{Z}^i}d\bar{Z}^i) \ . \tag{3.6.11}$$

### 3.6.4 $Diff(X^3)$ invariance and degeneracy and conformal invariances of the symplectic form

$J(X(H_A), X(H_B))$ defines symplectic form for the coset space $G/H$ only if it is $Diff(X^3)$ degenerate. This means that the symplectic form $J(X(H_A), X(H_B))$ vanishes whenever Hamiltonian $H_A$ or $H_B$ is such that it generates diffeomorphism of the 3-surface $X^3$. If effective 2-dimensionality holds true, $J(X(H_A), X(H_B))$ vanishes if $H_A$ or $H_B$ generates two-dimensional diffeomorphism $d(H_A)$ at the surface $X_i^2$.

One can always write

$$J(X(H_A), X(H_B)) = X(H_A)Q(H_B|X_i^2) \ .$$

If $H_A$ generates diffeomorphism, the action of $X(H_A)$ reduces to the action of the vector field $X_A$ of some $X_i^2$-diffeomorphism. Since $Q(H_B|r_M)$ is manifestly invariant under the diffemorphisms of $X^2$, the result is vanishing:

$$X_AQ(H_B|X_i^2) = 0 \ ,$$

so that $Diff^2$ invariance is achieved.

The radial diffeomorphisms possibly generated by the radial Virasoro algebra do not produce trouble. The change of the flux integrand $X$ under the infinitesimal transformation $r_M \rightarrow r_M + \epsilon r_M^n$ is given by $r_M^n dX/dr_M$. Replacing $r_M$ with $r_M^{-n+1}/(-n+1)$ as variable, the integrand reduces to a total divergence $dX/du$ the integral of which vanishes over the closed 2-surface $X_i^2$. Hence radial Virasoro generators having zero norm annihilate all matrix elements of the symplectic form. The induced metric of $X_i^2$ induces a unique conformal structure and since the conformal transformations of $X_i^2$ can be interpreted as a mere coordinate changes, they leave the flux integrals invariant.

### 3.6.5 Complexification and explicit form of the metric and Kähler form

The identification of the Kähler form and Kähler metric in symplectic degrees of freedom follows trivially from the identification of the symplectic form and definition of complexification. The requirement that Hamiltonians are eigen states of angular momentum (and possibly Lorentz boost generator), isospin and hypercharge implies physically natural complexification. In order to fix the complexification completely one must introduce some convention fixing which states correspond to 'positive' frequencies and which to 'negative frequencies' and which to zero frequencies that is to decompose the generators of the symplectic algebra to three sets $Can_+$, $Can_-$ and $Can_0$. One must distinguish between $Can_0$ and zero modes, which are not considered here at all. For instance, $CP_2$ Hamiltonians correspond to zero modes.

The natural complexification relies on the imaginary part of the radial conformal weight whereas the real part defines the $g = t + h$ decomposition naturally. The wave vector associated with the radial logarithmic plane wave corresponds to the angular momentum quantum number associated with a wave in $S^1$ in the case of Kac Moody algebra. One can imagine three options.

1. It is quite possible that the spectrum of $k_2$ does not contain $k_2 = 0$ at all so that the sector $Can_0$ could be empty. This complexification is physically very natural since it is manifestly invariant under $SU(3)$ and $SO(3)$ defining the preferred spherical coordinates. The choice of $SO(3)$ is unique if the classical four-momentum associated with the 3-surface is time like so that there are no problems with Lorentz invariance.

2. If $k_2 = 0$ is possible one could have

$$\begin{aligned} Can_+ &= \{H^a_{m,n,k=k1_+ik_2}, k_2 > 0\} \ , \\ Can_- &= \{H^a_{m,n,k}, k_2 < 0\} \ , \\ Can_0 &= \{H^a_{m,n,k}, k_2 = 0\} \ . \end{aligned} \tag{3.6.12}$$

3. If it is possible to $n_2 \neq 0$ for $k_2 = 0$, one could define the decomposition as

$$\begin{aligned} Can_+ &= \{H^a_{m,n,k}, k_2 > 0 \ \text{ or } \ k_2 = 0, n_2 > 0\} \ , \\ Can_- &= \{H^a_{m,n,k}, k_2 < 0 \ \text{ or} k_2 = 0, n_2 < 0\} \ , \\ Can_0 &= \{H^a_{m,n,k}, k_2 = n_2 = 0\} \ . \end{aligned} \tag{3.6.13}$$

   In this case the complexification is unique and Lorentz invariance guaranteed if one can fix the $SO(2)$ subgroup uniquely. The quantization axis of angular momentum could be chosen to be the direction of the classical angular momentum associated with the 3-surface in its rest system.

The only thing needed to get Kähler form and Kähler metric is to write the half Poisson bracket defined by Eq. 3.6.15

$$\begin{aligned} J_f(X(H_A), X(H_B)) &= 2Im\,(iQ_f(\{H_A, H_B\}_{-+})) \ , \\ G_f(X(H_A), X(H_B)) &= 2Re\,(iQ_f(\{H_A, H_B\}_{-+})) \ . \end{aligned} \tag{3.6.14}$$

Symplectic form, and thus also Kähler form and Kähler metric, could contain a conformal factor depending on the isometry invariants characterizing the size and shape of the 3-surface. At this stage one cannot say much about the functional form of this factor.

### 3.6.6 Comparison of $CP_2$ Kähler geometry with configuration space geometry

The explicit discussion of the role of $g = t + h$ decomposition of the tangent space of the configuration space provides deep insights to the metric of the symmetric space. There are indeed many questions to be answered. To what point of configuration space (that is 3-surface) the proposed $g = t + h$ decomposition corresponds to? Can one derive the components of the metric and Kähler form from the Poisson brackets of complexified Hamiltonians? Can one characterize the point in question in terms of the properties of configuration space Hamiltonians? Does the central extension of the configuration space reduce to the symplectic central extension of the symplectic algebra or can one consider also other options?

#### Cartan decomposition for $CP_2$

A good manner to gain understanding is to consider the $CP_2$ metric and Kähler form at the origin of complex coordinates for which the sub-algebra $h = u(2)$ defines the Cartan decomposition.

1. $g = t + h$ decomposition depends on the point of the symmetric space in general. In case of $CP_2$ $u(2)$ sub-algebra transforms as $g \circ u(2) \circ g^{-1}$ when the point $s$ is replaced by $gsg^{-1}$. This is expected to hold true also in case of configuration space (unless it is flat) so that the task is to identify the point of the configuration space at which the proposed decomposition holds true.

2. The Killing vector fields of $h$ sub-algebra vanish at the origin of $CP_2$ in complex coordinates. The corresponding Hamiltonians need not vanish but their Poisson brackets must vanish. It is possible to add suitable constants to the Hamiltonians in order to guarantee that they vanish at origin.

3. It is convenient to introduce complex coordinates and decompose isometry generators to holomorphic components $J^a_+ = j^{ak}\partial_k$ and $j^a_- = j^{a\bar{k}}\partial_{\bar{k}}$. One can introduce what might be called half Poisson bracket and half inner product defined as

$$\begin{aligned} \{H^a, H^b\}_{-+} &\equiv \partial_{\bar{k}} H^a J^{\bar{k}l} \partial_l H^b \\ &= j^{ak} J_{k\bar{l}} j^{b\bar{l}} = -i(j^a_+, j^b_-) \ . \end{aligned} \tag{3.6.15}$$

One can express Poisson bracket of Hamiltonians and the inner product of the corresponding Killing vector fields in terms of real and imaginary parts of the half Poisson bracket:

$$\begin{aligned} \{H^a, H^b\} &= 2Im\left(i\{H^a, H^b\}_{-+}\right) \ , \\ (j^a, j^b) &= 2Re\left(i(j^a_+, j^b_-)\right) = 2Re\left(i\{H^a, H^b\}_{-+}\right) \ . \end{aligned} \tag{3.6.16}$$

What this means that Hamiltonians and their half brackets code all information about metric and Kähler form. Obviously this is of utmost importance in the case of the configuration space metric whose symplectic structure and central extension are derived from those of $CP_2$.

Consider now the properties of the metric and Kähler form at the origin.

1. The relations satisfied by the half Poisson brackets can be written symbolically as

$$\begin{aligned} &\{h, h\}_{-+} = 0 \ , \\ &Re\left(i\{h, t\}_{-+}\right) = 0 \ , \quad Im\left(i\{h, t\}_{-+}\right) = 0 \ , \\ &Re\left(i\{t, t\}_{-+}\right) \neq 0 \ , \quad Im\left(i\{t, t\}_{-+}\right) \neq 0 \ . \end{aligned} \tag{3.6.17}$$

2. The first two conditions state that $h$ vector fields have vanishing inner products at the origin. The first condition states also that the Hamiltonians for the commutator algebra $[h,h] = SU(2)$ vanish at origin whereas the Hamiltonian for $U(1)$ algebra corresponding to the color hyper charge need not vanish although it can be made vanishing. The third condition implies that the Hamiltonians of $t$ vanish at origin.

3. The last two conditions state that the Kähler metric and form are non-vanishing between the elements of $t$. Since the Poisson brackets of $t$ Hamiltonians are Hamiltonians of $h$, the only possibility is that $\{t,t\}$ Poisson brackets reduce to a non-vanishing $U(1)$ Hamiltonian at the origin or that the bracket at the origin is due to the symplectic central extension. The requirement that all Hamiltonians vanish at origin is very attractive aesthetically and forces to interpret $\{t,t\}$ brackets at origin as being due to a symplectic central extension. For instance, for $S^2$ the requirement that Hamiltonians vanish at origin would mean the replacement of the Hamiltonian $H = cos(\theta)$ representing a rotation around z-axis with $H_3 = cos(\theta) - 1$ so that the Poisson bracket of the generators $H_1$ and $H_2$ can be interpreted as a central extension term.

4. The conditions for the Hamiltonians of $u(2)$ sub-algebra state that their variations with respect to $g$ vanish at origin. Thus $u(2)$ Hamiltonians have extremum value at origin.

5. Also the Kähler function of $CP_2$ has extremum at the origin. This suggests that in the case of the configuration space the counterpart of the origin corresponds to the maximum of the Kähler function.

**Cartan algebra decomposition at the level of configuration space**

The discussion of the properties of $CP_2$ Kähler metric at origin provides valuable guide lines in an attempt to understand what happens at the level of the configuration space. The use of the half bracket for the configuration space Hamiltonians in turn allows to calculate the matrix elements of the configuration space metric and Kähler form explicitly in terms of the magnetic or electric flux Hamiltonians.

The earlier construction was rather tricky and formula-rich and not very convincing physically. Cartan decomposition had to be assigned with something and in lack of anything better it was assigned with Super Virasoro algebra, which indeed allows this kind of decompositions but without any strong physical justification. The realization that super-symplectic and super Kac-Moody symmetries define coset construction at the level of basic quantum TGD, and that this construction provides a realization of Equivalence Principle at microscopic level, forced eventually the realization that also the coset space decomposition of configuration space realizes Equivalence Principle geometrically.

It must be however emphasized that holography implying effective 2-dimensionality of 3-surfaces in some length scale resolution is absolutely essential for this construction since it allows to effectively reduce Kac-Moody generators associated with $X^3_l$ to $X^2 = X^3_l \cap \delta M^4_\pm \times CP_2$. In the similar manner super-symplectic generators can be dimensionally reduced to $X^2$. Number theoretical compactification forces the dimensional reduction and the known extremals are consistent with it [K5] . The construction of configuration space spinor structure and metric in terms of the second quantized spinor fields [K9] relies to this picture as also the recent view about $M$-matrix [K12] .

In this framework the coset space decomposition becomes trivial.

1. The algebra $g$ is labeled by color quantum numbers of $CP_2$ Hamiltonians and by the label $(m,n,k)$ labeling the function basis of the light cone boundary. Also a localization with respect to $X^2$ is needed. This is a new element as compared to the original view.

2. Super Kac-Moody algebra is labeled by color octet Hamiltonians and function basis of $X^2$. Since Lie-algebra action does not lead out of irreps, this means that Cartan algebra decomposition is satisfied.

### 3.6.7 Comparison with loop groups

It is useful to compare the recent approach to the geometrization of the loop groups consisting of maps from circle to Lie group G [A44] , which served as the inspirer of the configuration space geometry approach but later turned out to not apply as such in TGD framework.

In the case of loop groups the tangent space $T$ corresponds to the local Lie-algebra $T(k,A) = exp(ik\phi)T_A$, where $T_A$ generates the finite-dimensional Lie-algebra $g$ and $\phi$ denotes the angle variable of circle; $k$ is integer. The complexification of the tangent space corresponds to the decomposition

$$T = \{X(k>0,A)\} \oplus \{X(k<0,A)\} \oplus \{X(k=0,A)\} = T_+ \oplus T_- \oplus T_0$$

of the tangent space. Metric corresponds to the central extension of the loop algebra to Kac Moody algebra and the Kähler form is given by

$$J(X(k_1<0,A),X(k_2>0,B)) = k_2\delta(k_1+k_2)\delta(A,B) \ .$$

In present case the finite dimensional Lie algebra $g$ is replaced with the Lie-algebra of the symplectic transformations of $\delta M^4_+ \times CP_2$ centrally extended using symplectic extension. The scalar function basis on circle is replaced with the function basis on an interval of length $\Delta r_M$ with periodic boundary conditions; effectively one has circle also now.

The basic difference is that one can consider two kinds of central extensions now.

1. Central extension is most naturally induced by the natural central extension ($\{p,q\}=1$) defined by Poisson bracket. This extension is anti-symmetric with respect to the generators of the symplectic group: in the case of the Kac Moody central extension it is symmetric with respect to the group $G$. The symplectic transformations of $CP_2$ might correspond to non-zero modes also because they are not exact symmetries of Kähler action. The situation is however rather delicate since $k=0$ light cone harmonic has a diverging norm due to the radial integration unless one poses both lower and upper radial cutoffs although the matrix elements would be still well defined for typical 3-surfaces. For Kac Moody group $U(1)$ transformations correspond to the zero modes. Light cone function algebra can be regarded as a local $U(1)$ algebra defining central extension in the case that only $CP_2$ symplectic transformations local with respect to $\delta M^4_+$ act as isometries: for Kac Moody algebra the central extension corresponds to an ordinary $U(1)$ algebra. In the case that entire light cone symplectic algebra defines the isometries the central extension reduces to a $U(1)$ central extension.

### 3.6.8 Symmetric space property implies Ricci flatness and isometric action of symplectic transformations

The basic structure of symmetric spaces is summarized by the following structural equations

$$\begin{aligned} &g = h + t \ , \\ &[h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ . \end{aligned} \tag{3.6.18}$$

In present case the equations imply that all commutators of the Lie-algebra generators of $Can(\neq 0)$ having non-vanishing integer valued radial quantum number $n_2$, possess zero norm. This condition is extremely strong and guarantees isometric action of $Can(\delta M^4_+ \times CP_2)$ as well as Ricci flatness of the configuration space metric.

The requirement $[t,t] \subset h$ and $[h,t] \subset t$ are satisfied if the generators of the isometry algebra possess generalized parity $P$ such that the generators in $t$ have parity $P=-1$ and the generators belonging to $h$ have parity $P=+1$. Conformal weight $n$ must somehow define this parity. The first possibility to come into mind is that odd values of $n$ correspond to $P=-1$ and even values to $P=1$. Since $n$ is additive in commutation, this would automatically imply $h \oplus t$ decomposition with the required properties. This assumption looks however somewhat artificial. TGD however forces a generalization of Super Algebras and N-S and Ramond type algebras can be combined to a larger algebra containing also Virasoro and Kac Moody generators labeled by half-odd integers. This suggests strongly that isometry generators are labeled by half integer conformal weight and that half-odd integer conformal weight corresponds to parity $P=-1$ whereas integer conformal weight corresponds to parity $P=1$. Coset space would structure would state conformal invariance of the theory since super-symplectic generators with integer weight would correspond to zero modes.

Quite generally, the requirement that the metric is invariant under the flow generated by vector field $X$ leads together with the covariant constancy of the metric to the Killing conditions

$$X \cdot g(Y,Z) = 0 = g([X,Y],Z) + g(Y,[X,Z]) \ . \tag{3.6.19}$$

If the commutators of the complexified generators in $Can(\neq 0)$ have zero norm then the two terms on the right hand side of Eq. (3.6.19) vanish separately. This is true if the conditions

$$Q_m^{\alpha,\beta}(\{H^A,\{H^B,H^C\}\}) = 0 \ , \tag{3.6.20}$$

are satisfied for all triplets of Hamiltonians in $Can_{\neq 0}$. These conditions follow automatically from the $[t,t] \subset h$ property and guarantee also Ricci flatness as will be found later.

It must be emphasized that for Kähler metric defined by purely magnetic fluxes, one cannot pose the conditions of Eq. (3.6.20) as consistency conditions on the initial values of the time derivatives of imbedding space coordinates whereas in general case this is possible. If the consistency conditions are satisfied for a single surface on the orbit of symplectic group then they are satisfied on the entire orbit. Clearly, isometry and Ricci flatness requirements and the requirement of time reversal invariance might well force Kähler electric alternative.

### 3.6.9 How to find Kähler function?

If one has found the expansion of configuration space Kähler form in terms of electric fluxes one can solve also the Kähler function from the defining partial differential equations $J_{k\bar{l}} = \partial k \partial_{\bar{l}} K$. The solution is not unique since the equation allows the symmetry

$$K \rightarrow K + f(z^k) + \overline{f(z^k)} \ ,$$

where $f$ is arbitrary holomorphic function of $z^k$. This non-uniqueness is probably eliminated by the requirement that Kähler function vanishes for vacuum extremals. This in turn makes in principle possible to find the maxima of Kähler function and to perform functional integration perturbatively around them.

Electric-magnetic duality implies that, apart from conformal factor depending on isometry invariants, one can solve Kähler metric without any knowledge on the initial values of the time derivatives of the imbedding space coordinates. Apart from conformal factor the resulting geometry is purely intrinsic to $\delta CH$. The role of Kähler action is only to to define $Diff^4$ invariance and give the rule how the metric is translated to metric on arbitrary point of $CH$. The degeneracy of the preferred extrema also implies that configuration space has multi-sheeted structure analogous to that encountered in case of Riemann surfaces.

As shown in [K24] , very general assumptions inspired by the light-likeness of Kähler current for the known extremals combined with electric-magnetic duality imply the reduction of Kähler action for the preferred extremals to Chern-Simons terms at the ends of $CD$ and at wormhole throats plus boundary term depending on induced metric so that one has almost topological QFT. The latter is due to the possibility to choose the gauge for Kähler potential to code information about conserved quantum numbers to the Kähler function and is the counterpart for the measurement interaction term in Dirac action. This term should correspond to a real part of a holomorphic function so that it does not contribute to the Kähler metric.

Also a promising concrete construction recipe for Kähler function is in terms of the modified Dirac operator [K9] . The recipe is described briefly in the introduction. If the conjecture that Dirac determinant coincides with the exponent of Kähler action for a preferred extremal is correct, the value of the Kähler coupling strength follows as a prediction of the theory. From the construction it is clear that Dirac determinant involves only a finite number of eigenvalues of the modified Dirac operator and can thus be an algebraic or even rational number if eigenvalues have this property. The most satisfactory property of the construction is that one can use the intuition from the behavior of 2-D magnetic systems.

## 3.7 Ricci flatness and divergence cancelation

Divergence cancelation in configuration space integration requires Ricci flatness and in this section the arguments in favor of Ricci flatness are discussed in detail.

### 3.7.1 Inner product from divergence cancelation

Forgetting the delicacies related to the non-determinism of the Kähler action, the inner product is given by integrating the usual Fock space inner product defined at each point of the configuration space over the reduced configuration space containing only the 3-surfaces $Y^3$ belonging to $\delta H = \delta M^4_+ \times CP_2$ ('lightcone boundary') using the exponent $exp(K)$ as a weight factor:

$$\begin{aligned} \langle \Psi_1 | \Psi_2 \rangle &= \int \overline{\Psi}_1(Y^3)\Psi_2(Y^3) exp(K)\sqrt{G} dY^3 \ , \\ \overline{\Psi_1}(Y^3)\Psi_2(Y^3) &\equiv \langle \Psi_1(Y^3)|\Psi_2(Y^3)\rangle_{Fock} \ . \end{aligned} \tag{3.7.1}$$

The degeneracy for the preferred extremals of Kähler action implies additional summation over the degenerate extremals associated with $Y^3$. The restriction of the integration on light cone boundary is Diff$^4$ invariant procedure and resolves in elegant manner the problems related to the integration over Diff$^4$ degrees of freedom. A variant of the inner product is obtained dropping the bosonic vacuum functional $exp(K)$ from the definition of the inner product and by assuming that it is included into the spinor fields themselves. Probably it is just a matter of taste how the necessary bosonic vacuum functional is included into the inner product: what is essential that the vacuum functional $exp(K)$ is somehow present in the inner product.

The unitarity of the inner product follows from the unitary of the Fock space inner product and from the unitarity of the standard $L^2$ inner product defined by configuration space integration in the set of the $L^2$ integrable scalar functions. It could well occur that $Diff^4$ invariance implies the reduction of the configuration space integration to $C(\delta H)$.

Consider next the bosonic integration in more detail. The exponent of the Kähler function appears in the inner product also in the context of the finite dimensional group representations. For the representations of the noncompact groups (say $SL(2,R)$) in coset spaces (now $SL(2,R)/U(1)$ endowed with Kähler metric) the exponent of Kähler function is necessary in order to get square integrable representations [B42]. The scalar product for two complex valued representation functions is defined as

$$(f,g) = \int \overline{f} g exp(nK)\sqrt{g} dV \ . \tag{3.7.2}$$

By unitarity, the exponent is an integer multiple of the Kähler function. In the present case only the possibility $n = 1$ is realized if one requires a complete cancelation of the determinants. In finite dimensional case this corresponds to the restriction to single unitary representation of the group in question.

The sign of the action appearing in the exponent is of decisive importance in order to make theory stable. The point is that the theory must be well defined at the limit of infinitely large system. Minimization of action is expected to imply that the action of infinitely large system is bound from above: the generation of electric Kähler fields gives negative contributions to the action. This implies that at the limit of the infinite system the average action per volume is non-positive. For systems having negative average density of action vacuum functional $exp(K)$ vanishes so that only configurations with vanishing average action per volume have significant probability. On the other hand, the choice $exp(-K)$ would make theory unstable: probability amplitude would be infinite for all configurations having negative average action per volume. In the fourth part of the book it will be shown that the requirement that average Kähler action per volume cancels has important cosmological consequences.

Consider now the divergence cancelation in the bosonic integration. One can develop the Kähler function as a Taylor series around maximum of Kähler function and use the contravariant Kähler metric as a propagator. Gaussian and metric determinants cancel each other for a unique vacuum functional. Ricci flatness guarantees that metric determinant is constant in complex coordinates so that one avoids divergences coming from it. The non-locality of the Kähler function as a functional of the 3-surface serves as an additional regulating mechanism: if $K(X^3)$ were a local functional of $X^3$ one would encounter divergences in the perturbative expansion.

The requirement that quantum jump corresponds to a quantum measurement in the sense of quantum field theories implies that quantum jump involves localization in zero modes. Localization in the zero modes implies automatically p-adic evolution since the decomposition of the configuration space into sectors $D_P$ labeled by the infinite primes $P$ is determined by the corresponding decomposition in zero modes. Localization in zero modes would suggest that the calculation of the physical predictions does not involve integration over zero modes: this would dramatically simplify the calculational apparatus of the theory. Probably this simplification occurs at the level of practical calculations if $U$-matrix separates into a product of matrices associated with zero modes and fiber degrees of freedom.

One must also calculate the predictions for the ratios of the rates of quantum transitions to different values of zero modes and here one cannot actually avoid integrals over zero modes. To achieve this one is forced to define the transition probabilities for quantum jumps involving a localization in zero modes as

$$P(x,\alpha\rightarrow y,\beta)=\sum_{r,s}|S(r,\alpha\rightarrow s,\beta)|^2|\Psi_r(x)|^2|\Psi_s(y)|^2 \ \ ,$$

where $x$ and $y$ correspond to the zero mode coordinates and $r$ and $s$ label a complete state functional basis in zero modes and $S(r,m\rightarrow s,n)$ involves integration over zero modes. In fact, only in this manner the notion of the localization in the zero modes makes mathematically sense at the level of S-matrix. In this case also unitarity conditions are well-defined. In zero modes state function basis can be freely constructed so that divergence difficulties could be avoided. An open question is whether this construction is indeed possible.

Some comments about the actual evaluation of the bosonic functional integral are in order.

1. Since configuration space metric is degenerate and the bosonic propagator is essentially the contravariant metric, bosonic integration is expected to reduce to an integration over the zero modes. For instance, isometry invariants are variables of this kind. These modes are analogous to the parameters describing the conformal equivalence class of the orbit of the string in string models.

2. $\alpha_K$ is a natural small expansion parameter in configuration space integration. It should be noticed that $\alpha_K$, when defined by the criticality condition, could also depend on the coordinates parameterizing the zero modes.

3. Semiclassical approximation, which means the expansion of the functional integral as a sum over the extrema of the Kähler function, is a natural approach to the calculation of the bosonic integral. Symmetric space property suggests that for the given values of the zero modes there is only single extremum and corresponds to the maximum of the Kähler function. There are theorems (Duistermaat-Hecke theorem) stating that semiclassical approximation is exact for certain systems (for example for integrable systems [A42] ). Symmetric space property suggests that Kähler function might possess the properties guaranteing the exactness of the semiclassical approximation. This would mean that the calculation of the integral $\int exp(K)\sqrt{G}dY^3$ and even more complex integrals involving configuration space spinor fields would be completely analogous to a Gaussian integration of free quantum field theory. This kind of reduction actually occurs in string models and is consistent with the criticality of the Kähler coupling constant suggesting that all loop integrals contributing to the renormalization of the Kähler action should vanish. Also the condition that configuration space integrals are continuable to p-adic number fields requires this kind of reduction.

### 3.7.2 Why Ricci flatness

It has been already found that the requirement of divergence cancelation poses extremely strong constraints on the metric of the configuration space. The results obtained hitherto are the following.

1. If the vacuum functional is the exponent of Kähler function one gets rid of the divergences resulting from the Gaussian determinants and metric determinants: determinants cancel each other.

2. The non-locality of the Kähler action gives good hopes of obtaining divergence free perturbation theory.

The following arguments show that Ricci flatness of the metric is a highly desirable property.

1. Dirac operator should be a well defined operator. In particular its square should be well defined. The problem is that the square of Dirac operator contains curvature scalar, which need not be finite since it is obtained via two infinite-dimensional trace operations from the curvature tensor. In case of loop spaces [A44] the Kähler property implies that even Ricci tensor is only conditionally convergent. In fact, loop spaces with Kähler metric are Einstein spaces (Ricci tensor is proportional to metric) and Ricci scalar is infinite.

   In 3-dimensional case situation is even worse since the trace operation involves 3 summation indices instead of one! The conclusion is that Ricci tensor had better to vanish in vibrational degrees of freedom.

2. For Ricci flat metric the determinant of the metric is constant in geodesic complex coordinates as is seen from the expression for Ricci tensor [A69]

$$R_{k\bar{l}} = \partial_k \partial_{\bar{l}} ln(det(g)) \tag{3.7.3}$$

   in Kähler metric. This obviously simplifies considerably functional integration over the configuration space: one obtains just the standard perturbative field theory in the sense that metric determinant gives no contributions to the functional integration.

3. The constancy of the metric determinant results not only in calculational simplifications: it also eliminates divergences. This is seen by expanding the determinant as a functional Taylor series with respect to the coordinates of the configuration space. In local complex coordinates the first term in the expansion of the metric determinant is determined by Ricci tensor

$$\delta\sqrt{g} \propto R_{k\bar{l}} z^k \bar{z}^l \quad . \tag{3.7.4}$$

   In configuration space integration using standard rules of Gaussian integration this term gives a contribution proportional to the contraction of the propagator with Ricci tensor. But since the propagator is just the contravariant metric one obtains Ricci scalar as result. So, in order to avoid divergences, Ricci scalar must be finite: this is certainly guaranteed if Ricci tensor vanishes.

4. The following group theoretic argument suggests that Ricci tensor either vanishes or is divergent. The holonomy group of the configuration space is a subgroup of $U(n = \infty)$ ($D = 2n$ is the dimension of the Kähler manifold) by Kähler property and Ricci flatness is guaranteed if the $U(1)$ factor is absent from the holonomy group. In fact Ricci tensor is proportional to the trace of the $U(1)$ generator and since this generator corresponds to an infinite dimensional unit matrix the trace diverges: therefore given element of the Ricci tensor is either infinite or vanishes. Therefore the vanishing of the Ricci tensor seems to be a mathematical necessity. This naive argument doesn't hold true in the case of loop spaces, for which Kähler metric with finite non-vanishing Ricci tensor exists [A44] . Note however that also in this case the sum defining Ricci tensor is only conditionally convergent.

There are indeed good hopes that Ricci tensor vanishes. By the previous argument the vanishing of the Ricci tensor is equivalent with the absence of divergences in configuration space integration. That divergences are absent is suggested by the non-locality of the Kähler function as a functional of 3-surface: the divergences of local field theories result from the locality of interaction vertices. Ricci flatness in vibrational degrees of freedom is not only necessary mathematically. It is also appealing physically: one can regard Ricci flat configuration space as a vacuum solution of Einstein's equations $G^{\alpha\beta} = 0$.

### 3.7.3 Ricci flatness and Hyper Kähler property

Ricci flatness property is guaranteed if configuration space geometry is Hyper Kähler [A65, A31] (there exists 3 covariantly constant antisymmetric tensor fields, which can be regarded as representations of quaternionic imaginary units). Hyper Kähler property guarantees Ricci flatness because the contractions of the curvature tensor appearing in the components of the Ricci tensor transform to traces over Lie algebra generators, which are $SU(n)$ generators instead of $U(n)$ generators so that the traces vanish. In the case of the loop spaces left invariance implies that Ricci tensor in the vibrational degrees is a multiple of the metric tensor so that Ricci scalar has an infinite value. This is basically due to the fact that Kac-Moody algebra has $U(1)$ central extension.

Consider now the arguments in favor of Ricci flatness of the configuration space.

1. The symplectic algebra of $\delta M^4_+$ takes effectively the role of the $U(1)$ extension of the loop algebra. More concretely, the $SO(2)$ group of the rotation group $SO(3)$ takes the role of $U(1)$ algebra. Since volume preserving transformations are in question, the traces of the symplectic generators vanish identically and in finite-dimensional this should be enough for Ricci flatness even if Hyper Kähler property is not achieved.

2. The comparison with $CP_2$ allows to link Ricci flatness with conformal invariance. The elements of the Ricci tensor are expressible in terms of traces of the generators of the holonomy group $U(2)$ at the origin of $CP_2$, and since $U(1)$ generator is non-vanishing at origin, the Ricci tensor is non-vanishing. In recent case the origin of $CP_2$ is replaced with the maximum of Kähler function and holonomy group corresponds to super-symplectic generators labelled by integer valued real parts $k_1$ of the conformal weights $k = k_1 + i\rho$. If generators with $k_1 = n$ vanish at the maximum of the Kähler function, the curvature scalar should vanish at the maximum and by the symmetric space property everywhere. These conditions correspond to Virasoro conditions in super string models.

   A possible source of difficulties are the generators having $k_1 = 0$ and resulting as commutators of generators with opposite real parts of the conformal weights. It might be possible to assume that only the conformal weights $k = k_1 + i\rho$, $k_1 = 0, 1, ...$ are possible since it is the imaginary part of the conformal weight which defines the complexification in the recent case. This would mean that the commutators involve only positive values of $k_1$.

3. In the infinite-dimensional case the Ricci tensor involves also terms which are non-vanishing even when the holonomy algebra does not contain $U(1)$ factor. It will be found that symmetric space property guarantees Ricci flatness even in this case and the reason is essentially the vanishing of the generators having $k_1 = n$ at the maximum of Kähler function.

There are also arguments in favor of the Hyper Kähler property.

1. The dimensions of the imbedding space and space-time are 8 and 4 respectively so that the dimension of configuration space in vibrational modes is indeed multiple of four as required by Hyper Kähler property. Hyper Kähler property requires a quaternionic structure in the tangent space of the configuration space. Since any direction on the sphere $S^2$ defined by the linear combinations of quaternionic imaginary units with unit norm defines a particular complexification physically, Hyper Kähler property means the possibility to perform complexification in $S^2$-fold manners.

2. $S^2$-fold degeneracy is indeed associated with the definition of the complex structure of the configuration space. First of all, the direction of the quantization axis for the spherical harmonics or for the eigen states of Lorentz Cartan algebra at $\delta M^4_+$ can be chosen in $S^2$-fold manners. Quaternion conformal invariance means Hyper Kähler property almost by definition and the $S^2$-fold degeneracy for the complexification is obvious in this case.

If these naive arguments survive a more critical inspection, the conclusion would be that the effective 2-dimensionality of light like 3-surfaces implying generalized conformal and symplectic symmetries would also imply Hyper Kähler property of the configuration space and make the theory well-defined mathematically. This obviously fixes the dimension of space-time surfaces as well as the dimension of Minkowski space factor of the imbedding space.

In the sequel we shall show that Ricci flatness is guaranteed provided that the holonomy group of the configuration space is isomorphic to some subgroup of $SU(n=\infty)$ instead of $U(n=\infty)$ ($n$ is the complex dimension of the configuration space) implied by the Kähler property of the metric. We also derive an expression for the Ricci tensor in terms of the structure constants of the isometry algebra and configuration space metric. The expression for the Ricci tensor is formally identical with that obtained by Freed for loop spaces: the only difference is that the structure constants of the finite-dimensional group are replaced with the group $Can(\delta H)$. Also the arguments in favor of Hyper Kähler property are discussed in more detail.

### 3.7.4 The conditions guaranteing Ricci flatness

In the case of Kähler geometry Ricci flatness condition can be characterized purely Lie-algebraically: the holonomy group of the Riemann connection, which in general is subgroup of $U(n)$ for Kähler manifold of complex dimension $n$, must be subgroup of $SU(n)$ so that the Lie-algebra of this group consists of traceless matrices. This condition is easy to derive using complex coordinates. Ricci tensor is given by the following expression in complex vielbein basis

$$R^{A\bar{B}} = R^{A\bar{C}B}_{\quad\ \ \bar{C}} \ , \tag{3.7.5}$$

where the latter summation is only over the antiholomorphic indices $\bar{C}$. Using the cyclic identities

$$\sum_{cycl\ \bar{C}B\bar{D}} R^{A\bar{C}B\bar{D}} = 0 \ , \tag{3.7.6}$$

the expression for Ricci tensor reduces to the form

$$R^{A\bar{B}} = R^{A\bar{B}C}_{\quad\ \ C} \ , \tag{3.7.7}$$

where the summation is only over the holomorphic indices $C$. This expression can be regarded as a trace of the curvature tensor in the holonomy algebra of the Riemann connection. The trace is taken over holomorphic indices only: the traces over holomorphic and anti-holomorphic indices cancel each other by the antisymmetry of the curvature tensor. For Kähler manifold holonomy algebra is subalgebra of $U(n)$, when the complex dimension of manifold is $n$ and Ricci tensor vanishes if and only if the holonomy Lie-algebra consists of traceless matrices, or equivalently: holonomy group is subgroup of $SU(n)$. This condition is expected to generalize also to the infinite-dimensional case.

We shall now show that if configuration space metric is Kähler and possesses infinite-dimensional isometry algebra with the property that its generators form a complete basis for the tangent space (every tangent vector is expressible as a superposition of the isometry generators plus zero norm vector) it is possible to derive a representation for the Ricci tensor in terms of the structure constants of the isometry algebra and of the components of the metric and its inverse in the basis formed by the isometry generators and that Ricci tensor vanishes identically for the proposed complexification of the configuration space provided the generators $\{H_{A,m\neq 0}, H_{B,n\neq 0}\}$ correspond to zero norm vector fields of configuration space.

The general definition of the curvature tensor as an operator acting on vector fields reads

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \ . \tag{3.7.8}$$

If the vector fields considered are isometry generators the covariant derivative operator is given by the expression

$$\begin{aligned} \nabla_X Y &= (Ad_X Y - Ad^*_X Y - Ad^*_Y X)/2 \ , \\ (Ad^*_X Y, Z) &= (Y, Ad_X Z) \ , \end{aligned} \tag{3.7.9}$$

where $Ad_X Y = [X,Y]$ and $Ad^*_X$ denotes the adjoint of $Ad_X$ with respect to configuration space metric.

In the sequel we shall assume that the vector fields in question belong to the basis formed by the isometry generators. The matrix representation of $Ad_X$ in terms of the structure constants $C_{X,Y:Z}$ of the isometry algebra is given by the expression

$$\begin{aligned} Ad^m_{Xn} &= C_{X,Y:Z}\hat{Y}_n Z^m \ , \\ [X,Y] &= C_{X,Y:Z}Z \ , \\ \hat{Y} &= g^{-1}(Y,V)V \ , \end{aligned} \tag{3.7.10}$$

where the summation takes place over the repeated indices and $\hat{Y}$ denotes the dual vector field of $Y$ with respect to the configuration space metric. From its definition one obtains for $Ad^*_X$ the matrix representation

$$\begin{aligned} Ad^{*m}_{Xn} &= C_{X,Y:Z}\hat{Y}^m Z_n \ , \\ Ad^*_X Y &= C_{X,U:V}g(Y,U)g^{-1}(V,W)W = g(Y,U)g^{-1}([X,U],W)W \ , \end{aligned} \tag{3.7.11}$$

where the summation takes place over the repeated indices.

Using the representations of $\nabla_X$ in terms of $Ad_X$ and its adjoint and the representations of $Ad_X$ and $Ad^*_X$ in terms of the structure constants and some obvious identities (such as $C_{[X,Y],Z:V} = C_{X,Y:U}C_{U,Z:V}$ ) one can by a straightforward but tedious calculation derive a more detailed expression for the curvature tensor and Ricci tensor. Straightforward calculation of the Ricci tensor has however turned to be very tedious even in the case of the diagonal metric and in the following we shall use a more convenient representation [A44] of the curvature tensor applying in case of the Kähler geometry.

The expression of the curvature tensor is given in terms of the so called Toeplitz operators $T_X$ defined as linear operators in the "positive energy part" $G_+$ of the isometry algebra spanned by the $(1,0)$ parts of the isometry generators. In present case the positive and negative energy parts and cm part of the algebra can be defined just as in the case of loop spaces:

$$\begin{aligned} G_+ &= \{H^{Ak}|k>0\} \ , \\ G_- &= \{H^{Ak}|k<0\} \ , \\ G_0 &= \{H^{Ak}|k=0\} \ . \end{aligned} \tag{3.7.12}$$

Here $H^{Ak}$ denote the Hamiltonians generating the symplectic transformations of $\delta H$. The positive energy generators with non-vanishing norm have positive radial scaling dimension: $k \geq 0$, which corresponds to the imaginary part of the scaling momentum $K = k_1 + i\rho$ associated with the factors $(r_M/r_0)^K$. A priori the spectrum of $\rho$ is continuous but it is quite possible that the spectrum of $\rho$ is discrete and $\rho = 0$ does not appear at all in the spectrum in the sense that the flux Hamiltonians associated with $\rho = 0$ elements vanish for the maximum of Kähler function which can be taken to be the point where the calculations are done.

$T_X$ differs from $Ad_X$ in that the negative energy part of $Ad_X Y = [X,Y]$ is dropped away:

$$\begin{aligned} T_X : G_+ &\rightarrow G_+ \ , \\ Y &\rightarrow [X,Y]_+ \ . \end{aligned} \tag{3.7.13}$$

Here " + " denotes the projection to "positive energy" part of the algebra. Using Toeplitz operators one can associate to various isometry generators linear operators $\Phi(X_0)$, $\Phi(X_-)$ and $\Phi(X_+)$ acting on $G_+$:

$$\begin{aligned} \Phi(X_0) &= T_{X_0} \ , \ X_0 \varepsilon G_0 \ , \\ \Phi(X_-) &= T_{X_-} \ , \ X_- \varepsilon G_- \ , \\ \Phi(X_+) &= -T^*_{X_-} \ , \ X_+ \varepsilon G_+ \ . \end{aligned} \tag{3.7.14}$$

Here "*" denotes hermitian conjugate in the diagonalized metric: the explicit representation $\Phi(X_+)$ is given by the expression [A44]

$$
\begin{aligned}
\Phi(X_+) &= D^{-1}T_{X_-}D \ , \\
DX_+ &= d(X)X_- \ , \\
d(X) &= g(X_-,X_+) \ .
\end{aligned}
\tag{3.7.15}
$$

Here $d(X)$ is just the diagonal element of metric assumed to be diagonal in the basis used. denotes the conformal factor associated with the metric.

The representations for the action of ,$\Phi(X_0)$, $\Phi(X_-)$ and $\Phi(X_+)$ in terms of metric and structure constants of the isometry algebra are in the case of the diagonal metric given by the expressions

$$
\begin{aligned}
\Phi(X_0)Y_+ &= C_{X_0,Y_+:U_+}U_+ \ , \\
\Phi(X_-)Y_+ &= C_{X_-,Y_+:U_+}U_+ \ , \\
\Phi(X_+)Y_+ &= \frac{d(Y)}{d(U)}C_{X_-,Y_-:U_-}U_+ \ .
\end{aligned}
\tag{3.7.16}
$$

The expression for the action of the curvature tensor in positive energy part $G_+$ of the isometry algebra in terms of the these operators is given as [A44] :

$$
R(X,Y)Z_+ = \{[\Phi(X),\Phi(Y)] - \Phi([X,Y])\}Z_+ \ . \tag{3.7.17}
$$

The calculation of the Ricci tensor is based on the observation that for Kähler manifolds Ricci tensor is a tensor of type $(1,1)$, and therefore it is possible to calculate Ricci tensor as the trace of the curvature tensor with respect to indices associated with $G_+$.

$$
Ricci(X_+,Y_-) = (\hat{Z}_+, R(X_+,Y_-)Z_+) \equiv Trace(R(X_+,Y_-)) \ , \tag{3.7.18}
$$

where the summation over $Z_+$ generators is performed.

Using the explicit representations of the operators $\Phi$ one obtains the following explicit expression for the Ricci tensor

$$
\begin{aligned}
Ricci(X_+,Y_-) &= Trace\{[D^{-1}T_{X_+}D, T_{Y_-}] - T_{[X_+,Y_-]_{|G_0+G_-}} \\
&- D^{-1}T_{[X_+,Y_-]_{|G_+}}D\} \ .
\end{aligned}
\tag{3.7.19}
$$

This expression is identical to that encountered in case of loop spaces and the following arguments are repetition of those applying in the case of loop spaces.

The second term in the Ricci tensor is the only term present in the finite-dimensional case. This term vanishes if the Lie-algebra in question consists of traceless matrices. Since symplectic transformations are volume-preserving the traces of Lie-algebra generators vanish so that this term is absent. The last term gives a non-vanishing contribution to the trace for the same reason.

The first term is quadratic in structure constants and does not vanish in case of loop spaces. It can be written explicitly using the explicit representations of the various operators appearing in the formula:

$$
\begin{aligned}
Trace\{[D^{-1}T_{X_-}D, T_{Y_-}]\} &= \sum_{Z_+,U_+}[C_{X_-,U_-:Z_-}C_{Y_-,Z_+:U_+}\frac{d(U)}{d(Z)} \\
&- C_{X_-,Z_-:U_-}C_{Y_-,U_+:Z_+}\frac{d(Z)}{d(U)}] \ .
\end{aligned}
\tag{3.7.20}
$$

Each term is antisymmetric under the exchange of $U$ and $Z$ and one might fail to conclude that the sum vanishes identically. This is not the case. By the diagonality of the metric with respect to radial quantum number, one has $m(X_-) = m(Y_-)$ for the non-vanishing elements of the Ricci tensor. Furthermore, one has $m(U) = m(Z) - m(Y)$, which eliminates summation over $m(U)$ in the first term and summation over $m(Z)$ in the second term. Note however, that summation over other labels related to symplectic algebra are present.

By performing the change $U \rightarrow Z$ in the second term one can combine the sums together and as a result one has finite sum

$$\begin{aligned}
\sum_{0<m(Z)<m(X)} [C_{X_-,U_-:Z_-} C_{Y_-,Z_+:U_+} \frac{d(U)}{d(Z)} &= C \sum_{0<m(Z)<m(X)} \frac{m(X)}{m(Z)-m(X)} \ , \\
C &= \sum_{Z,U} C_{X,U:Z} C_{Y,Z:U} \frac{d_0(U)}{d_0(Z)} \ . \qquad (3.7.21)
\end{aligned}$$

Here the dependence of $d(X) = |m(X)|d_0(X)$ on $m(X)$ is factored out; $d_0(X)$ does not depend on $k_X$. The dependence on $m(X)$ in the resulting expression factorizes out, and one obtains just the purely group theoretic term $C$, which should vanish for the space to be Ricci flat.

The sum is quadratic in structure constants and can be visualized as a loop sum. It is instructive to write the sum in terms of the metric in the symplectic degrees of freedom to see the geometry behind the Ricci flatness:

$$C = \sum_{Z,U} g([Y,Z],U) g^{-1}([X,U],Z) \ . \qquad (3.7.22)$$

Each term of this sum involves a commutator of two generators with a non-vanishing norm. Since tangent space complexification is inherited from the local coset space, the non-vanishing commutators in complexified basis are always between generators in $Can_{\neq 0}$; that is they do not not belong to rigid $su(2) \times su(3)$.

The condition guaranteing Ricci flatness at the maximum of Kähler function and thus everywhere is simple. All elements of type $[X_{\neq 0}, Y_{\neq 0}]$ vanish or have vanishing norm. In case of $CP_2$ Kähler geometry this would correspond to the vanishing of the $U(2)$ generators at the origin of $CP_2$ (note that the holonomy group is $U(2)$ in case of $CP_2$). At least formally stronger condition is that the algebra generated by elements of this type, the commutator algebra associated with $Can_{\neq 0}$, consist of elements of zero norm. Already the (possibly) weaker condition implies that adjoint map $Ad_{X \neq 0}$ and its hermitian adjoint $Ad^*_{X_{\neq 0}}$ create zero norm states. Since isometry conditions involve also adjoint action the condition also implies that $Can_{\neq 0}$ acts as isometries. More concrete form for the condition is that all flux factors involving double Poisson bracket and three generators in $Can_{\neq 0}$ vanish:

$$Q_e(\{H_A, \{H_B, H_C\}\}) = 0 \ , \ \text{for} \ H_A, H_B, H_C \ \text{in} \ Can_{\neq 0} \ . \qquad (3.7.23)$$

The vanishing of fluxes involving two Poisson brackets and three Hamiltonians guarantees isometry invariance and Ricci flatness and, as found in [K10] , is implied by the $[t,t] \subset h$ property of the Lie-algebra of coset space $G/H$ having symmetric space structure.

The conclusion is that the mere existence of the proposed isometry group (guaranteed by the symmetric space property) implies the vanishing of the Ricci tensor and vacuum Einstein equations. The existence of the infinite parameter isometry group in turn follows basically from the condition guaranteing the existence of the Riemann connection. Therefore vacuum Einstein equations seem to arise, not only as a consequence of a physically motivated variational principle but as a mathematical consistency condition in infinite dimensional Kähler geometry. The flux representation seems to provide elegant manner to formulate and solve these conditions and isometry invariance implies Ricci flatness.

### 3.7.5 Is configuration space metric Hyper Kähler?

The requirement that configuration space integral integration is divergence free implies that configuration space metric is Ricci flat. The so called Hyper-Kähler metrics [A65, A31] , [B25] are particularly nice representatives of Ricci flat metrics. In the following the basic properties of Hyper-Kähler metrics are briefly described and the problem whether Hyper Kähler property could realized in case of $M^4_+ \times CP_2$ is considered.

### Hyper-Kähler property

Hyper-Kähler metric is a generalization of the Kähler metric. For Kähler metric metric tensor and Kähler form correspond to the complex numbers 1 and i and therefore define complex structure in the tangent space of the manifold. For Hyper Kähler metric tangent space allows three closed Kähler forms $I, J, K$, which with respect to the multiplication obey the algebra of quaternionic imaginary units and have square equal to - 1, which corresponds to the metric of Hyper Kähler space.

$$I^2 \;=\; J^2 = K^2 = -1 \;\; IJ = -JI = K, \text{ etc. } . \tag{3.7.24}$$

To define Kähler structure one must choose one of the Kähler forms or any linear combination of $I, J$ and $K$ with unit norm. The group $SO(3)$ rotates different Kähler structures to each other playing thus the role of quaternion automorphisms. This group acts also as coordinate transformations in Hyper Kähler manifold but in general fails to act as isometries.

If $K$ is chosen to define complex structure then $K$ is tensor of type $(1,1)$ in complex coordinates, $I$ and $J$ being tensors of type $(2,0)+(0,2)$. The forms $I+iJ$ and $I-iJ$ are holomorphic and anti-holomorphic forms of type $(2,0)$ and $(0,2)$ respectively and defined standard step operators $I_+$ and $I_-$ of $SU(2)$ algebra. The holonomy group of Hyper-Kähler metric is always $Sp(k)$, $k \leq dimM/4$, the group of $k \times k$ unitary matrices with quaternionic entries. This group is indeed subgroup of SU(2k), so that its generators are traceless and Hyper Kähler metric is therefore Ricci flat.

Hyper Kähler metrics have been encountered in the context of 3-dimensional super symmetric sigma models: a necessary prerequisite for obtaining $N = 4$ super-symmetric sigma model is that target space allows Hyper Kähler metric [B25, B12] . In particular, it has been found that Hyper Kähler property is decisive for the divergence cancelation.

Hyper-Kähler metrics arise also in monopole and instanton physics [A31] . The moduli spaces for monopoles have Hyper Kähler property. This suggests that Hyper Kähler property is characteristic for the configuration (or moduli) spaces of 4-dimensional Yang Mills types systems. Since YM action appears in the definition of configuration space metric there are hopes that also in present case the metric possesses Hyper-Kähler property.

$CP_2$ allows what might be called almost Hyper-Kähler structure known as quaternionion structure. This means that the Weil tensor of $CP_2$ consists of three components in one-one correspondence with components of iso-spin and only one of them- the one corresponding to Kähler form- is covariantly constant. The physical interpretation is in terms of electroweak symmetry breaking selecting one isospin direction as a favored direction.

### Does the 'almost' Hyper-Kähler structure of $CP_2$ lift to a genuine Hyper-Kähler structure in configuration space?

The Hyper-Kähler property of configuration space metric does not seem to be in conflict with the general structure of TGD.

1. In string models the dimension of the "space-time" is two and Weyl invariance and complex structures play a decisive role in the theory. In present case the dimension of the space-time is four and one therefore might hope that quaternions play a similar role. Indeed, Weyl invariance implies YM action in dimension 4 and as already mentioned moduli spaces of instantons and monopoles enjoy the Hyper Kähler property.

2. Also the dimension of the imbedding space is important. The dimension of Hyper Kähler manifold must be multiple of 4. The dimension of configuration space is indeed infinite multiple of 8: each vibrational mode giving one "8".

3. The complexification of the configuration space in symplectic degrees of freedom is inherited from $S^2 \times CP_2$ and $CP_2$ Kähler form defines the symplectic form of configuration space. The point is that $CP_2$ Weyl tensor has 3 covariantly constant components, having as their square

metric apart from sign. One of them is Kähler form, which is closed whereas the other two are non-closed forms and therefore fail to define Kähler structure. The group $SU(2)$ of electro-weak isospin rotations rotate these forms to each other. It would not be too suprising if one could identify the configuration space counterparts of these forms as representations of quaternionic units at the level of configuration space. The failure of the Hyper Kähler property at the level of $CP_2$ geometry is due to the electro-weak symmetry breaking and physical intuition (in particular, p-adic mass calculations [K34] ) suggests that electro-weak symmetry might not be broken at the level of configuration space geometry).

A possible topological obstruction for the Hyper Kähler property is related to the cohomology of the configuration space: the three Kähler forms must be co-homologically trivial as is clear from the following argument. If any of 3 quaternionic 2-form is cohomologically nontrivial then by $SO(3)$ symmetry rotating Kähler forms to each other all must be co-homologically nontrivial. On the other hand, electro-weak isospin rotation leads to a linear combination of 3 Kähler forms and the flux associated with this form is in general not integer valued. The point is however that Kähler form forms only the $(1,1)$ part of the symplectic form and must be co-homologically trivial whereas the zero mode part is same for all complexifications and can be co-homologically nontrivial. The co-homological non-triviality of the zero mode part of the symplectic form is indeed a nice feature since it fixes the normalization of the Kähler function apart from a multiplicative integer. On the other hand the hypothesis that Kähler coupling strength is analogous to critical temperature provides a dynamical (and perhaps equivalent) manner to fix the normalization of the Kähler function.

Since the properties of the configuration space metric are inherited from $M^4_+ \times CP_2$ then also the Hyper Kähler property should be understandable in terms of the imbedding space geometry. In particular, the complex structure in $CP_2$ vibrational degrees of freedom is inherited from $CP_2$. Hyper Kähler property implies the existence of a continuum (sphere $S^2$) of complex structures: any linear superposition of 3 independent Kähler forms defines a respectable complex structure. Therefore also $CP_2$ should have this continuum of complex structures and this is certainly not the case.

Indeed, if we had instead of $CP_2$ Hyper Kähler manifold with 3 covariantly constant 2-forms then it would be easy to understand the Hyper Kähler structure of configuration space. Given the Kähler structure of the configuration space would be obtained by replacing induced Kähler electric and magnetic fields in the definition of flux factors $Q(H_{A,m})$ with the appropriate component of the induced Weyl tensor. $CP_2$ indeed manages to be very nearly Hyper Kähler manifold!

How $CP_2$ fails to be Hyper Kähler manifold can be seen in the following manner. The Weyl tensor of $CP_2$ allows three independent components, which are self dual as 2-forms and rotated to each other by vielbein rotations.

$$\begin{aligned} W_{03} &= W_{12} \equiv 2I_3 = 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ W_{01} &= W_{23} \equiv I_1 = -e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ W_{02} &= W_{31} \equiv I_2 = -e^0 \wedge e^2 - e^3 \wedge e^1 \ . \end{aligned} \tag{3.7.25}$$

The component $I_3$ is just the Kähler form of $CP_2$. Remaining components are covariantly constant only with respect to spinor connection and not closed forms so that they cannot be interpreted as Maxwell fields. Their squares equal however apart from sign with the metric of $CP_2$, when appropriate normalization factor is used. If these forms were covariantly constant Kähler action defined by any linear superposition of these forms would indeed define Kähler structure in configuration space and the group $SO(3)$ would rotate these forms to each other. The projections of the components of the Weyl tensor on 3-surface define 3 vector fields as their duals and only one of these vector fields (Kähler magnetic field) is divergenceless. One might regard these 3 vector fields as counter parts of quaternion units associated with the broken Hyper Kähler structure, that is quaternion structure. The interpretation in terms of electro-weak symmetry breaking is obvious.

One cannot exclude the possibility that the symplectic invariance of the induced Kähler electric field implies that the electric parts of the other two components of induced Weyl tensor are symplectic invariants. This is the minimum requirement. What is however obvious is that the magnetic parts cannot be closed forms for arbitrary 3-surfaces at light cone boundary. One counter example is enough and $CP_2$ type extremals seem to provide this counter example: the components of the induced Weyl tensor are just the same as they are for $CP_2$ and clearly not symplecticly invariant.

Thus it seems that configuration space could allow Hyper Kähler structure broken by electro-weak interactions but it cannot be inherited from $CP_2$. An open question is whether it allows genuine quaternionic structure. Good prospects for obtaining quaternionic structure are provided by the quaternionic counterpart $QP_2$ of $CP_2$, which is 8-dimensional and has coset space structure $QP_2 = Sp(3)/Sp(2) \times Sp(1)$. This choice does not seem to be consistent with the symmetries of the standard model. Note however that the over all symmetry group is obtained by replacing complex numbers with quaternions on the matrix representation of the standard model group.

**Could different complexifications for $M^4_+$ and light like surfaces induce Hyper Kähler structure for configuration space?**

Quaternionic structure means also the existence of a family of complex structures parameterized by a sphere $S^2$. The complex structure of the configuration space is inherited from the complex structure of some light like surface.

In the case of the light cone boundary $\delta M^4_+$ the complex structure corresponds to the choice of quantization axis of angular momentum for the sphere $r_M = constant$ so that the coordinates orthogonal to the quantization axis define a complex coordinate: the sphere $S^2$ parameterizes these choices. Thus there is a temptation to identify the choice of quantization axis with a particular imaginary unit and Hyper Kähler structure would directly relate to the properties rotation group. This would bring an additional item to the list of miraculous properties of light like surfaces of 4-dimensional space-times.

This might relate to the fact that configuration space geometry is not determined by the symplectic algebra of $CP_2$ localized with respect to the light cone boundary as one might first expect but consists of $M^4_+ \times CP_2$ Hamiltonians so that infinitesimal symplectic transformation of $CP_2$ involves always also $M^4_+$-symplectic transformation. $M^4_+$ Hamiltonians are defined by a function basis generated as products of the Hamiltonians $H_3$ and $H_1 \pm iH_2$ generating rotations with respect to three orthogonal axes, and two of these Hamiltonians are complexified.

Also the light like 3-surfaces $X^3_l$ associated with quaternion conformal invariance are determined by some 2-surface $X^2$ and the choice of complex coordinates and if $X^2$ is sphere the choices are labelled by $S^2$. In this case, the presence of quaternion conformal structure would be almost obvious since it is possible to choose some complex coordinate in several manners and the choices are labelled by $S^2$. The choice of the complex coordinate in turn fixes 2-surface $X^2$ as a surface for which the remaining coordinates are constant. $X^2$ need not however be located at the elementary particle horizon unless one poses additional constraint. One might hope that different choices of $X^2$ resulting in this manner correspond to all possible different selections of the complex structure and that this choice could fix uniquely the conformal equivalence class of $X^2$ appearing as argument in elementary particle vacuum functionals. If $X^2$ has a more complex topology the identification is not so clear but since conformal algebra SL(2,C) containing algebra of rotation group is involved, one might argue that the choice of quantization axis also now involves $S^2$ degeneracy. If these arguments are correct one could conclude that Hyper Kähler structure is implicitly involved and guarantees Ricci flatness of the configuration space metric.

## 3.8 Consistency conditions on metric

In this section various consistency conditions on the configuration space metric are discussed. In particular, it will be found that the conditions guaranteing the existence of Riemann connection in the set of all(!) vector fields (including zero norm vector fields) gives very strong constraints on the general form of the metric and that these constraints are indeed satisfied for the proposed metric.

### 3.8.1 Consistency conditions on Riemann connection

To study the consequences of the consistency conditions, it is most convenient to consider matrix elements of the metric in the basis formed by the isometry generators themselves. The consistency conditions state the covariant constancy of the metric tensor

$$\nabla_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = Z \cdot g(X,Y) \quad . \tag{3.8.1}$$

$Z \cdot g(X,Y)$ vanishes, when $Z$ generates isometries so that conditions state the covariant constancy of the matrix elements in this case. It must be emphasized that the ill defined-ness of the inner products of form $g(\nabla_Z X, Y)$ is just the reason for requiring infinite-dimensional isometry group. The point is that $\nabla_Z X$ need not to belong to the Hilbert space spanned by the tangent vector fields since the terms of type $Zg(X,Y)$ do not necessarily exist mathematically [A44] . The elegant solution to the problem is that all tangent space vector fields act as isometries so that these quantities vanish identically.

The conditions of Eq. (3.8.1) can be written explicitly by using the general expression for the covariant derivative

$$\begin{aligned} g(\nabla_Z X, Y) &= [Zg(X,Y) + Xg(Z,Y) - Yg(Z,X) \\ &+ g(Ad_Z X - Ad^*_Z X - Ad^*_X Z, Y)]/2 \ . \end{aligned} \tag{3.8.2}$$

What happens is that the terms depending on the derivatives of the matrix elements (terms of type $Zg(X,Y)$ ) cancel each other (these terms vanish for the metric invariant under isometries), and one obtains the following consistency conditions

$$g(Ad_Z X - Ad^*_Z X - Ad^*_X Z, Y) + g(X, Ad_Z Y - Ad^*_Z Y - Ad^*_Y Z) = 0 \ . \tag{3.8.3}$$

Using the explicit representations of $Ad_Z X$ and $Ad*_Z X$ in terms of structure constants

$$\begin{aligned} Ad_Z X &= [Z,X] = C_{Z,X:U}U \ . \\ Ad^*_Z X &= C_{Z,U:V}g(X,V)g^{-1}(U,W)W = g(X,[Z,U])g^{-1}(U,W)W \ . \end{aligned} \tag{3.8.4}$$

where the summation over repeated "indices" is performed, one finds that consistency conditions are identically satisfied provided the generators $X$ and $Y$ have a non-vanishing norm. The reason is that the contributions coming from $\nabla_Z X$ and $\nabla_Z Y$ cancel each other.

When one of the generators, say $X$, appearing in the inner product has a vanishing norm so that one has $g(X,Y) = 0$, for any generator $Y$, situation changes! The contribution of $\nabla_Z Y$ term to the consistency conditions drops away and using Eqs. (3.8.3) and (3.8.4) one obtains the following consistency conditions

$$C_{Z,X:U}g(U,Y) + C_{X,Y:U}g(U,Z) = -X \cdot g(Z,Y) \ . \tag{3.8.5}$$

Note that summation over $U$ is carried out. If $X$ is isometry generator (this need not be the case always) the condition reduces to a simpler form:

$$C_{X,Z:U}g(U,Y) + C_{X,Y:U}g(Z,U) = g([X,Z].Y) + g(Z,[X,Y]) = 0 \ . \tag{3.8.6}$$

These conditions have nice geometric interpretation. If the matrix elements are regarded as ordinary Hilbert space products between the isometry generators the conditions state that the metric defining the inner product behaves as a scalar in the general case.

### 3.8.2 Consistency conditions for the radial Virasoro algebra

The action of the radial Virasoro in nontrivial manner in the zero modes. Therefore isometry interpretation is excluded and consistency conditions do not make sense in this case. One can however consider the possibility that metric is invariant or suffers only an overall scaling under the action of the radial scaling generated by $L_0 = r_M d/dr_M$. Since the radial integration measure is scaling invariant and only powers of $r_M/r_0$ appear in Hamiltonians, the effect of the scaling $r_M \rightarrow \lambda r_M$ on the matrix elements of the metric is a scaling by $\lambda^{k_a+\bar{k}_b}$). One can interpret this by saying that scaling changes the values of zero modes and hence leads outside the symmetric space in question.

Invariance of reduced matrix element obtained by dividing away the powers of the scaling factor is achieved if the metric contains the conformal factor

$$S = \frac{1}{\Delta u} f(\frac{r_i}{r_j}) \ , \tag{3.8.7}$$

where $r_i$ are the extrema of $r_M$ interpreted as height function of $X^3$ and $f$ is a priori arbitrary positive definite function. Since the presence of $f$ presumably gives rise to renormalization corrections depending on the size and shape of 3-surface by scaling the propagator defined by the contravariant metric, the dependence on the ratios $r_i/r_j$ should be slow, logarithmic dependence. Also the dependence on the Fourier components of the solid angles $\Omega(r_M)$ associated with the $r_M = constant$ sections is possible.

### 3.8.3 Explicit conditions for the isometry invariance

The identification of the Lie-algebra of isometry generators has been proposed but cannot provide any proof for the existence of the infinite parameter symmetry group at this stage. What one can do at this stage is to formulate explicitly the conditions guaranteing isometry invariance of the metric and try to see whether there are any hopes that these conditions are satisfied. It has been already found that the expression of the metric reduces for light cone alternative to the sum of two boundary terms coming from infinite future and from the boundary of the light cone. If the contribution from infinitely distant future vanishes, as one might expect, then only the contribution from the boundary of the light cone remains.

A tedious but straightforward evaluation of the second variation (see Appendix of the book) for Kähler action implies the following form for the second variation of the Kähler action

$$\delta^2 S = \int_{a=0}^{a=\infty} I^{n\beta}_{kl} \delta h^k D_\beta \delta h^l \ , \tag{3.8.8}$$

where the tensor $I^{\alpha\beta}_{kl}$ is defined as partial derivatives of the Kähler Lagrangian with respect to the derivatives $\partial_\alpha h^k$

$$I^{\alpha\beta}_{kl} = \partial_{\partial_\alpha h^k} \partial_{\partial_\beta h^l} L_M \ . \tag{3.8.9}$$

If the upper limit $a = \sqrt{(m^0)^2 - r_M^2} = \infty$ in the substitution vanishes then one can calculate second variation and therefore metric from the knowledge of the time derivatives $\partial_n h^k$ and $\partial_n \delta h^k$ on the boundary of the light cone only.

Kähler metric can be identified as the $(1,1)$ part of the second variation. This means that one can express the deformation as an element of the isometry algebra plus a arbitrary deformation in radial direction of the light cone boundary interpretable as conformal transformation of the light cone boundary. Radial contributions to the second variation are dropped (by definition of Kähler metric) and what remains is essentially a deformation in $S^2$ degrees of freedom.

The left invariance of the metric under the deformations of the isometry algebra implies an infinite number of conditions of the form

$$J^C g(J^A, J^B) = 0 \ , \tag{3.8.10}$$

where $J^A, J^B$ and $J^C$ denote the generators of the isometry group. These conditions ought to fix completely the time derivatives of the coordinates $h^k$ for each 3-surface at light cone boundary and therefore in principle the whole minimizing four-surface provided the initial value problem associated with the Kähler action possesses a unique solution. What is nice that the requirement of isometry invariance in principle would provides solution to the problem of finding preferred extremals of the Kähler action.

These conditions, when written explicitly give infinite number of conditions for the time derivative of the generator $J^C$ (we assume for a moment that $C$ is held fixed and let $A$ and $B$ run) at the boundary of the light cone. Time derivatives are in principle determined also by the requirement that deformed surface corresponds to an absolute minimum of the Kähler action. The basis of $\delta H$ scalar functions respecting color and rotational symmetries is the most promising one.

### 3.8.4 Direct consistency checks

If duality holds true, the most general form of the configuration space metric is defined by the fluxes $Q_m^{\alpha,\beta}$, where $\alpha$ and $\beta$ are the coefficients of signed and unsigned magnetic fluxes. Present is also a conformal factor depending on those zero modes, which do not appear in the symplectic form and which characterize the size and shape of the 3-surface. $[t,t] \subset h$ property implying Ricci flatness and isometry property of symplectic transformations, requires the vanishing of the fluxes $Q_m^{\alpha,\beta}(\{H_{A,m\neq 0}, \{H_{B,n\neq 0}, H_{C,p\neq 0}\}\})$ associated with double commutators and poses strong consistency conditions on the metric. If $n$ labelling symplectic generators has half integer values then the conditions simply state conformal invariance: generators labelled by integers have vanishing norm whereas half-odd integers correspond to non-vanishing norm. Isometry invariance gives additional conditions on fluxes $Q_m^{\alpha,\beta}$. Lorentz invariance strengthens these conditions further. It could be that these conditions fix the initial values of the imbedding space coordinates completely.

# Chapter 4

# Configuration Space Spinor Structure

## 4.1 Introduction

Quantum TGD should be reducible to the classical spinor geometry of the configuration space. In particular, physical states should correspond to the modes of the configuration space spinor fields. The immediate consequence is that configuration space spinor fields cannot, as one might naively expect, be carriers of a definite spin and unit fermion number. Concerning the construction of the configuration space spinor structure there are some important clues.

### 4.1.1 Geometrization of fermionic statistics in terms of configuration space spinor structure

The great vision has been that the second quantization of the induced spinor fields can be understood geometrically in terms of the configuration space spinor structure in the sense that the anti-commutation relations for configuration space gamma matrices require anti-commutation relations for the oscillator operators for free second quantized induced spinor fields.

1. One must identify the counterparts of second quantized fermion fields as objects closely related to the configuration space spinor structure. [B40] has as its basic field the anti-commuting field $\Gamma^k(x)$, whose Fourier components are analogous to the gamma matrices of the configuration space and which behaves like a spin 3/2 fermionic field rather than a vector field. This suggests that the are analogous to spin 3/2 fields and therefore expressible in terms of the fermionic oscillator operators so that their naturally derives from the anti-commutativity of the fermionic oscillator operators.

   As a consequence, configuration space spinor fields can have arbitrary fermion number and there would be hopes of describing the whole physics in terms of configuration space spinor field. Clearly, fermionic oscillator operators would act in degrees of freedom analogous to the spin degrees of freedom of the ordinary spinor and bosonic oscillator operators would act in degrees of freedom analogous to the 'orbital' degrees of freedom of the ordinary spinor field.

2. The classical theory for the bosonic fields is an essential part of the configuration space geometry. It would be very nice if the classical theory for the spinor fields would be contained in the definition of the configuration space spinor structure somehow. The properties of the associated with the induced spinor structure are indeed very physical. The modified massless Dirac equation for the induced spinors predicts a separate conservation of baryon and lepton numbers. Contrary to the long held belief it seems that covariantly constant right handed neutrino does not generate . The differences between quarks and leptons result from the different couplings to the $CP_2$ Kähler potential. In fact, these properties are shared by the solutions of massless Dirac equation of the imbedding space.

3. Since TGD should have a close relationship to the ordinary quantum field theories it would be highly desirable that the second quantized free induced spinor field would somehow appear in the definition of the configuration space geometry. This is indeed true if the complexified configuration space gamma matrices are linearly related to the oscillator operators associated with the second quantized induced spinor field on the space-time surface and its boundaries. There is actually no deep reason forbidding the gamma matrices of the configuration space to be spin half odd-integer objects whereas in the finite-dimensional case this is not possible in general. In fact, in the finite-dimensional case the equivalence of the spinorial and vectorial vielbeins forces the spinor and vector representations of the vielbein group $SO(D)$ to have same dimension and this is possible for $D = 8$-dimensional Euclidian space only. This coincidence might explain the success of 10-dimensional super string models for which the physical degrees of freedom effectively correspond to an 8-dimensional Euclidian space.

4. It took a long time to realize that the ordinary definition of the gamma matrix algebra in terms of the anti-commutators $\{\gamma_A, \gamma_B\} = 2g_{AB}$ must in TGD context be replaced with

$$\{\gamma_A^\dagger, \gamma_B\} = iJ_{AB} \ ,$$

where $J_{AB}$ denotes the matrix elements of the Kähler form of the configuration space. The presence of the Hermitian conjugation is necessary because configuration space gamma matrices carry fermion number. This definition is numerically equivalent with the standard one in the complex coordinates. The realization of this delicacy is necessary in order to understand how the square of the configuration space Dirac operator comes out correctly.

5. TGD as a generalized number theory vision leads to the understanding of how the second quantization of the induced spinor fields should be carried out and space-time conformal symmetries allow to explicitly solve the Dirac equation associated with the modified Dirac action in the interior and at the 3-D light like causal determinants. An essentially new element is the notion of number theoretic braid forced by the fact that the modified Dirac operator allows only finite number of generalized eigen modes so that the number of fermionic oscillator operators is finite. As a consequence, anticommutation relations can be satisfied only for a finite set of points defined by the number theoretic braid, which is uniquely identifiable. The interpretation is in terms of finite measurement resolution. The finite Clifford algebra spanned by the fermionic oscillator operators is interpreted as the factor space $\mathcal{M}/\mathcal{N}$ of infinite hyper-finite factors of type $II_1$ defined by configuration space Clifford algebra $\mathcal{N}$ and included Clifford algebra $\mathcal{M} \subset \mathcal{N}$ interpreted as the characterizer of the finite measurement resolution. Note that the finite number of eigenvalues guarantees that Dirac determinant identified as the exponent of Kähler function is finite. Finite number of eigenvalues is also essential for number theoretic universality.

### 4.1.2 Modified Dirac equation for induced classical spinor fields

The earlier approach to the definition of the configuration space spinor structure relied on the second quantized ordinary massless Dirac action for the induced spinors. This action had some anomalous looking features. The first anomaly was the appearance of the effective tachyonic mass term proportional to the trace of the second fundamental form vanishing only for minimal surfaces. The breaking of $N = 2$ super symmetry generated by right-handed neutrinos for other than minimal surfaces was the second anomalous feature. It became also clear that the divergences of the fermionic isometry currents can have a non-vanishing c-number anomaly unless one varies Dirac action also with respect to the configuration space coordinates. This anomaly obviously might destroy the definition of the configuration space spinor structure.

The vision about quantum TGD as a generalized number theory [K53, K54, K52] comes in rescue here. One of its outcomes was the realization that, in order to achieve exact super-symmetry, one must modify Dirac action so that its variation with respect to the imbedding space coordinates gives the field equations derivable from the action principle in question. By taking the modified Dirac action as the fundamental action, one can identify vacuum functional as the Dirac determinant. If this determinant equals to exponent of Kähler action for the preferred extremal containing partonic 3-surfaces, one can predict even the value of the Kähler coupling constant.

**Chern-Simons - or Kähler Dirac action?**

Two alternative choices represented themselves as candidates for the modified Dirac action: either the 3-D Chern-Simons Dirac action or 4-D Kähler action. Eventually came the realization that the addition of a measurement interaction term to either Chern-Simons action or Kähler action is needed to resolve a bundle of conceptual problems. It took still some time to conclude that Kähler action with instanton term is the correct choice since the measurement interaction term assigned to Chern-Simons-Dirac action creates more problems than it solves.

1. *Basic implications*

1. A correlation between 4-D geometry of space-time sheet and quantum numbers is achieved by the identification of exponent of Kähler function as Dirac determinant making possible the entanglement of classical degrees of freedom in the interior of space-time sheet with quantum numbers.

2. Cartan algebra plays a key role not only at quantum level but also at the level of space-time geometry since quantum critical conserved currents vanish for Cartan algebra of isometries and the measurement interaction terms giving rise to conserved currents are possible only for Cartan algebras. Furthermore, modified Dirac equation makes sense only for eigen states of Cartan algebra generators. The hierarchy of Planck constants realized in terms of the book like structure of the generalized imbedding space assigns to each $CD$ (causal diamond) preferred Cartan algebra: in case of Poincare algebra there are two of them corresponding to linear and cylindrical $M^4$ coordinates.

3. Quantum holography and dimensional reduction hierarchy in which partonic 2-surface defined fermionic sources for 3-D fermionic fields at light-like 3-surfaces $Y_l^3$ in turn defining fermionic sources for 4-D spinors find an elegant realization. Effective 2-dimensionality is achieved if the replacement of light-like wormhole throat $X_l^3$ with light-like 3-surface $Y_l^3$ "parallel" with it in the definition of Dirac determinant corresponds to the $U(1)$ gauge transformation $K \to K + f + \overline{f}$ for Kähler function of WCW so that WCW Kähler metric is not affected. Here $f$ is holomorphic function of WCW ("world of classical worlds") complex coordinates and arbitrary function of zero mode coordinates.

4. An elegant description of the interaction between super-conformal representations realized at partonic 2-surfaces and dynamics of space-time surfaces is achieved since the values of Cartan charges are feeded to the 3-D Dirac equation which also receives mass term at the same time. Almost topological QFT at wormhole throats results at the limit when four-momenta vanish: this is in accordance with the original vision about TGD as almost topological QFT.

5. A detailed view about the physical role of quantum criticality results. Quantum criticality fixes the values of Kähler coupling strength as the analog of critical temperature. Quantum criticality implies that second variation of Kähler action vanishes for critical deformations and the existence of conserved current except in the case of Cartan algebra of isometries. Quantum criticality allows to fix the values of couplings appearing in the measurement interaction by using the condition $K \to K + f + \overline{f}$. p-Adic coupling constant evolution can be understood also and corresponds to scale hierarchy for the sizes of causal diamonds ($CD$s).

6. The inclusion of imaginary instanton term to the definition of the modified gamma matrices is not consistent with the conjugation of the induced spinor fields. Measurement interaction can be however assigned to both Kähler action and its instanton term. CP breaking, irreversibility and the space-time description of dissipation are closely related and the CP and T oddness of the instanton part of the measurement interaction term could provide first level description for dissipative effects. It must be however emphasized that the mere addition of instanton term to Kähler function could be enough.

7. A radically new view about matter antimatter asymmetry based on zero energy ontology emerges and one could understand the experimental absence of antimatter as being due to the fact antimatter corresponds to negative energy states. The identification of bosons as wormhole contacts is the only possible option in this framework.

8. Almost stringy propagators and a consistency with the identification of wormhole throats as lines of generalized Feynman diagrams is achieved. The notion of bosonic emergence leads to a long sought general master formula for the $M$-matrix elements. The counterpart for fermionic loop defining bosonic inverse propagator at QFT limit is wormhole contact with fermion and cutoffs in mass squared and hyperbolic angle for loop momenta of fermion and antifermion in the rest system of emitting boson have precise geometric counterpart.

*2. Hyper-quaternionicity and quantum criticality*

The conjecture that quantum critical space-time surfaces are hyper-quaternionic in the sense that the modified gamma matrices span a quaternionic subspace of complexified octonions at each point of the space-time surface is consistent with what is known about preferred extremals. The condition that both the modified gamma matrices and spinors are quaternionic at each point of the space-time surface leads to a precise ansatz for the general solution of the modified Dirac equation making sense also in the real context. The octonionic version of the modified Dirac equation is very simple since $SO(7,1)$ as vielbein group is replaced with $G_2$ acting as automorphisms of octonions so that only the neutral Abelian part of the classical electro-weak gauge fields survives the map.

Octonionic gamma matrices provide also a non-associative representation for the 8-D version of Pauli sigma matrices and encourage the identification of 8-D twistors as pairs of octonionic spinors conjectured to be highly relevant also for quantum TGD. Quaternionicity condition implies that octo-twistors reduce to something closely related to ordinary twistors.

**Super-conformal symmetries of modified Dirac action**

The modified Dirac equation allows large number of super-conformal gauge symmetries as zero modes of $D_K(Y_l^3)$ and are interpreted as generators of exact $N=4$ super-conformal gauge symmetries in both quark and lepton sectors. These super-symmetries correspond to pure super gauge transformations and state the the effective 3-dimensionality of space-time dynamics.

Super-symplectic and super Kac-Moody transformations respecting the light-likeness of light-like 3-surfaces define dynamical super conformal symmetries with covariantly constant right handed neutrino spinor serving as the generator of super symmetries. These are crucial for p-adic thermodynamics. No spartners of ordinary particles are predictedin particular $N=2$ space-time super-symmetry is generated by the righthanded neutrino is absent contrary to the earlier beliefs. There is no need to emphasize the experimental implications of this finding.

An essential difference with respect to the standard super-conformal symmetries is that Majorana condition is not satisfied and the usual super-space formalism does not apply. The notion of super-space is un-necessary since fermionic super-generators do not anticommute to vector fields of symmetries but to their Hamiltonians.

**Identification of configuration space gamma matrices**

Configuration space gamma matrices identified as super generators of super-symplectic or super Kac-Moody algebras (depending on $CH$ coordinates used) are expressible in terms of the oscillator operators associated with the eigen modes of the modified Dirac operator. Super-symplectic and super Kac-Moody charges are expressible as integrals over 2-dimensional partonic surfaces $X^2$ and interior degrees of freedom of $X^4$ can be regarded as zero modes representing classical variables in one-one correspondence with quantal degrees of freedom at $X_l^3$ as indeed required by quantum measurement theory. The resulting situation is highly reminiscent of WZW model and the results imply that at technical level the methods of 2-D conformal field theories should allow to construct quantum TGD.

### 4.1.3 The exponent of Kähler function as Dirac determinant for the modified Dirac action?

Although quantum criticality in principle predicts the possible values of Kähler coupling strength, one might hope that there exists even more fundamental approach involving no coupling constants and predicting even quantum criticality and realizing quantum gravitational holography.

1. The Dirac determinant defined by the product of Dirac determinants associated with the light-like partonic 3-surfaces $X_l^3$ associated with a given space-time sheet $X^4$ is the simplest candidate for vacuum functional identifiable as the exponent of the Kähler function. One can of course worry about the finiteness of the Dirac determinant. p-Adicization requires that the eigenvalues belong to a given algebraic extension of rationals. This restriction would imply a hierarchy of physics corresponding to different extensions and could automatically imply the finiteness and algebraic number property of the Dirac determinants if only finite number of eigenvalues would contribute. The regularization would be performed by physics itself if this were the case.

2. The basic problem has been how to feed in the information about the preferred extremal of Kähler action to the eigenvalue spectrum of the Dirac operator in question. The identification of the preferred extremal associated with $X_l^3$ became possible via the boundary conditions at $X_l^3$ dictated by number theoretical compactification, which also predicted the dual slicings of the $M^4$ projection of space-time surface by string world sheets and partonic 2-surfaces. The basic observation is that the Dirac equation associated with the 4-D Dirac operator $D_K$ associated with by Kähler action can be seen as a conservation law for a super current. The slicing of $X^4(X_l^3)$ by the parallel light-like 3-surfaces $Y_l^3$ allows solutions for which the super current flows along $Y_l^3$ and has no component in normal direction. The zero modes of $D_K$ reducing to effectively 3-D solutions of $D_K$ at each $Y_l^3$ give a family of holographic copies of $X_l^3$. The effective 3-dimensionality is due to the super-conformal gauge invariance in the direction of light-like coordinate $u$ labeling the 3-surfaces $Y_l^3$.

   A physically attractive unique realization of the slicings of space-time surface by 3-surfaces and string world sheets is discussed in [K25] by starting from the observation that TGD could define a natural realization of braids, braid cobordisms, and 2-knots.

3. The spectrum of eigenvalues corresponds to the "energy" spectrum of $D_K$ and the product of the eigenvalues defines the Dirac determinant in standard manner. If the eigenmodes are restricted to those localized to regions of strong induced electro-weak magnetic field, the number of eigen modes is finite and therefore also Dirac determinant is finite.

4. The requirement that the Noether currents associated with Dirac Kähler action are conserved is that preferred extremals of Kähler action correspond to extremals for which the second variation of Kähler action vanishes at least for the deformations associated with the conserved currents. Obviously this is nothing but the formulation of quantum criticality at space-time level!

5. The physical analog is energy spectrum for Dirac operator in external magnetic field. The effective metric appearing in the modified Dirac operator corresponds to $\hat{g}^{\alpha\beta} = \partial L_K/\partial h^k_\alpha \partial L_K/\partial h^l_\beta h_{kl}$, and vanishes at the boundaries of regions carrying non-vanishing Kähler magnetic field. Hence the modes must be localized to regions $X^3_{l,i}$ containing a non-vanishing Kähler magnetic field. Cyclotron states in constant magnetic field serve as a good analog for the situation and only a finite number of cyclotron states are possible since for higher cyclotron states the wave function -essentially harmonic oscillator wave function- would concentrate outside $X^3_{l,i}$.

6. A more precise argument goes as follows. Assume that it is induced Kähler magnetic field $B_K$ that matters. The vanishing of the effective contravariant metric near the boundary of $X^3_{l,i}$ corresponds to an infinite effective mass for massive particle in constant magnetic field so that the counterpart for the cyclotron frequency scale $eB/m$ reduces to zero. The radius of the cyclotron orbit is proportional to $1/\sqrt{eB}$ and approaches to infinity. Hence the required localization is not possible only for cyclotron states for which the cyclotron radius is below that the transversal size scale of $X^3_{l,i}$.

7. It remains to be proven that the product of eigenvalues gives rise to the exponent of Kähler action for the preferred extremal of Kähler action. At this moment the only justification for the conjecture is that this the only thing that one can imagine.

### 4.1.4 Super-conformal symmetries

The almost topological QFT property of partonic formulation based on modified Dirac Kähler action allows a rich structure of $N = 4$ super-conformal symmetries. In particular, the generalized

Kac-Moody symmetries leave corresponding $X^3$-local isometries respecting the light-likeness condition. A rather detailed view about various aspects of super-conformal symmetries emerge leading to identification of fermionic anti-commutation relations and explicit expressions for configuration space gamma matrices and Kähler metric. This picture is consistent with the conditions posed by p-adic mass calculations.

The relationship between super-symplectic ($SC$) and Super Kac-Moody ($SKM$) symmetries has been one of the central themes in the development of TGD. The progress in the understanding of the number theoretical aspects of TGD gives good hopes of lifting $SKMV$ ($V$ denotes Virasoro) to a subalgebra of $SCV$ so that coset construction works meaning that the differences of $SCV$ and $SKMV$ generators annihilate physical states. This condition has interpretation in terms of Equivalence Principle with coset Super Virasoro conditions defining a generalization of Einstein's equations in TGD framework. Also p-adic thermodynamics finds a justification since the expectation values of SKM conformal weights can be non-vanishing in physical states.

Number theoretical considerations play a key role and lead to the picture in which effective discretization occurs so that partonic two-surface is effectively replaced by a discrete set of algebraic points belonging to the intersection of the real partonic 2-surface and its p-adic counterpart obeying the same algebraic equations. This implies effective discretization of super-conformal field theory giving N-point functions defining vertices via discrete versions of stringy formulas.

Before continuing I must represent apologies for the reader. This chapter is just now under updating due to the dramatic simplifications related to identification of the eigenvalue spectrum of the modified Dirac operator and the definition of the Dirac determinant. The new vision is briefly discussed but a lot of mammoth bones remains to be eliminated.

## 4.2 Configuration space spinor structure: general definition

The basic problem in constructing configuration space spinor structure is clearly the construction of the explicit representation for the gamma matrices of the configuration space. One should be able to identify the space, where these gamma matrices act as well as the counterparts of the "free" gamma matrices, in terms of which the gamma matrices would be representable using generalized vielbein coefficients.

### 4.2.1 Defining relations for gamma matrices

The ordinary definition of the gamma matrix algebra is in terms of the anti-commutators

$$\{\gamma_A, \gamma_B\} = 2g_{AB} \quad .$$

This definition served implicitly also as a basic definition of the gamma matrix algebra in TGD context until the difficulties related to the understanding of the configuration space d'Alembertian defined in terms of the square of the Dirac operator forced to reconsider the definition. If configuration space allows Kähler structure, the most general definition allows to replace the metric any covariantly constant Hermitian form. In particular, $g_{AB}$ can be replaced with

$$\{\Gamma^{\dagger}_A, \Gamma_B\} = iJ_{AB} \quad , \tag{4.2.1}$$

where $J_{AB}$ denotes the matrix element of the Kähler form of the configuration space. The reason is that gamma matrices carry fermion number and are non-hermitian in all coordinate systems. This definition is numerically equivalent with the standard one in the complex coordinates but in arbitrary coordinates situation is different since in general coordinates $iJ_{kl}$ is a nontrivial positive square root of $g_{kl}$. The realization of this delicacy is necessary in order to understand how the square of the configuration space Dirac operator comes out correctly. Obviously, what one must do is the equivalent of replacing $D^2 = (\Gamma^k D_k)^2$ with $D\hat{D}$ with $\hat{D}$ defined as

$$\hat{D} = iJ^{kl}\Gamma^{\dagger}_l D_k \quad .$$

### 4.2.2 General vielbein representations

There are two ideas, which make the solution of the problem obvious.

1. Since the classical time development in bosonic degrees of freedom (induced gauge fields) is coded into the geometry of the configuration space it seems natural to expect that same applies in the case of the spinor structure. The time development of the induced spinor fields dictated by TGD counterpart of the massless Dirac action should be coded into the definition of the configuration space spinor structure. This leads to the challenge of defining what classical spinor field means.

2. Since classical scalar field in the configuration space corresponds to second quantized boson fields of the imbedding space same correspondence should apply in the case of the fermions, too. The spinor fields of configuration space should correspond to second quantized fermion field of the imbedding space and the space of the configuration space spinors should be more or less identical with the Fock space of the second quantized fermion field of imbedding space or $X^4(X^3)$. Since classical spinor fields at space-time surface are obtained by restricting the spinor structure to the space-time surface, one might consider the possibility that life is really simple: the second quantized spinor field corresponds to the free spinor field of the imbedding space satisfying the counterpart of the massless Dirac equation and more or less standard anti-commutation relations. Unfortunately life is not *so* simple as the construction of configuration space spinor structure demonstrates: second quantization must be performed for induced spinor fields.

It is relatively simple to fill in the details once these basic ideas are accepted.

1. The only natural candidate for the second quantized spinor field is just the on $X^4$. Since this field is free field, one can indeed perform second quantization and construct fermionic oscillator operator algebra with unique anti-commutation relations. The space of the configuration space spinors can be identified as the associated with these oscillator operators. This space depends on 3-surface and strictly speaking one should speak of the Fock bundle having configuration space as its base space.

2. The gamma matrices of the configuration space (or rather fermionic Kac Moody generators) are representable as super positions of the fermionic oscillator algebra generators:

$$\begin{aligned} \Gamma_A^+ &= E_A^n a_n^\dagger \\ \Gamma_A^- &= \bar{E}_A^n a_n \\ iJ_{A\bar{B}} &= \sum_n E_A^n \bar{E}_B^n \end{aligned} \tag{4.2.2}$$

   where $E_A^n$ are the vielbein coefficients. Induced spinor fields can possess zero modes and there is no oscillator operators associated with these modes. Since oscillator operators are spin 1/2 objects, configuration space gamma matrices are analogous to spin 3/2 spinor fields (in a very general sense). Therefore the generalized vielbein and configuration space metric is analogous to the pair of spin 3/2 and spin 2 fields encountered in super gravitation! Notice that the contractions $j^{Ak}\Gamma_k$ of the complexified gamma matrices with the isometry generators are genuine spin 1/2 objects labeled by the quantum numbers labeling isometry generators. In particular, in $CP_2$ degrees of freedom these fermions are color octets.

3. A further great idea inspired by the symplectic and Kähler structures of the configuration space is that configuration gamma matrices are actually generators of super-symplectic symmetries. This simplifies enormously the construction allows to deduce explicit formulas for the gamma matrices.

### 4.2.3 Inner product for configuration space spinor fields

The conjugation operation for configuration space spinors corresponds to the standard $ket \rightarrow bra$ operation for the states of the Fock space:

$$\begin{aligned} \Psi &\leftrightarrow |\Psi\rangle \\ \bar{\Psi} &\leftrightarrow \langle\Psi| \end{aligned} \tag{4.2.3}$$

The inner product for configuration space spinors at a given point of the configuration space is just the standard Fock space inner product, which is unitary.

$$\bar{\Psi}_1(X^3)\Psi_2(X^3) = \langle\Psi_1|\Psi_2\rangle_{|X^3} \tag{4.2.4}$$

Configuration space inner product for two configuration space spinor fields is obtained as the integral of the Fock space inner product over the whole configuration space using the vacuum functional $exp(K)$ as a weight factor

$$\langle\Psi_1|\Psi_2\rangle = \int \langle\Psi_1|\Psi_2\rangle_{|X^3} exp(K)\sqrt{G}dX^3 \tag{4.2.5}$$

This inner product is obviously unitary. A modified form of the inner product is obtained by including the factor $exp(K/2)$ in the definition of the spinor field. In fact, the construction of the central extension for the isometry algebra leads automatically to the appearance of this factor in vacuum spinor field.

The inner product differs from the standard inner product for, say, Minkowski space spinors in that integration is over the entire configuration space rather than over a time= constant slice of the configuration space. Also the presence of the vacuum functional makes it different from the finite dimensional inner product. These are not un-physical features. The point is that (apart from classical non-determinism forcing to generalized the concept of 3-surface) $Diff^4$ invariance dictates the behavior of the configuration space spinor field completely: it is determined form its values at the moment of the big bang. Therefore there is no need to postulate any Dirac equation to determine the behavior and therefore no need to use the inner product derived from dynamics.

### 4.2.4 Holonomy group of the vielbein connection

Generalized vielbein allows huge gauge symmetry. An important constraint on physical observables is that they do not depend at all on the gauge chosen to represent the gamma matrices. This is indeed achieved using vielbein connection, which is now quadratic in fermionic oscillator operators. The holonomy group of the vielbein connection is the configuration space counterpart of the electro-weak gauge group and its algebra is expected to have same general structure as the algebra of the configuration space isometries. In particular, the generators of this algebra should be labeled by conformal weights like the elements of Kac Moody algebras. In present case however conformal weights are complex as the construction of the configuration space geometry demonstrates.

### 4.2.5 Realization of configuration space gamma matrices in terms of super symmetry generators

In string models super symmetry generators behave effectively as gamma matrices and it is very tempting to assume that configuration space gamma matrices can be regarded as generators of the symplectic algebra extended to super-symplectic Kac Moody type algebra. The experience with string models suggests also that radial Virasoro algebra extends to Super Virasoro algebra. There are good reasons to expect that configuration space Dirac operator and its square give automatically a realization of this algebra. It this is indeed the case, then configuration space spinor structure as well as Dirac equation reduces to mere group theory.

One can actually guess the general form of the super-symplectic algebra. The form is a direct generalization of the ordinary super Kac Moody algebra. The complexified super generators $S_A$ are identifiable as configuration space gamma matrices:

$$\Gamma_A = S_A \ . \tag{4.2.6}$$

The anti-commutators $\{\Gamma_A^\dagger, \Gamma_B\}_+ = i2J_{A,B}$ define a Hermitian matrix, which is proportional to the Kähler form of the configuration space rather than metric as usually. Only in complex coordinates the anti-commutators equal to the metric numerically. This is, apart from the multiplicative constant $n$, is expressible as the Poisson bracket of the configuration space Hamiltonians $H_A$ and $H_B$. Therefore one should be able to identify super generators $S_A(r_M)$ for each values of $r_M$ as the counterparts of fluxes. The anti-commutators between the super generators $S_A$ and their Hermitian conjugates should read as

$$\{S_A, S_B^\dagger\}_+ = iQ_m(H_{[A,B]}) \ . \tag{4.2.7}$$

and should be induced directly from the anti-commutation relations of free second quantized spinor fields of the imbedding space restricted to the light cone boundary.

The commutation relations between s and super generators follow solely from the transformation properties of the super generators under symplectic transformations, which are same as for the Hamiltonians themselves

$$\{H_{Am}, S_{Bn}\}_- = S_{[Am,Bn]} \ , \tag{4.2.8}$$

and are of the same form as in the case of Super-Kac-Moody algebra.

The task is to derive an explicit representation for the super generators $S_A$ in both cases. For obvious reason the spinor fields restricted to the 3-surfaces on the light cone boundary $\delta M^4_+ \times CP_2$ can be used. Leptonic/quark like oscillator operators are used to construct Ramond/NS type algebra.

What is then the strategy that one should follow?

1. Configuration space Hamiltonians correspond to either magnetic or electric flux Hamiltonians and the conjecture is that these representations are equivalent. It turns out that this electric-magnetic duality generalizes to the level of super charges. It also turns out that quark representation is the only possible option whereas leptonic super charges super-symmetrize the ordinary function algebra of the light cone boundary.

2. The simplest option would be that second quantized imbedding space spinors could be used in the definition of super charges. This turns out to not work and one must second quantize the induced spinor fields.

3. The task is to identify a super-symmetric variational principle for the induced spinors: ordinary Dirac action does not work. It turns out that in the most plausible scenario the modified Dirac action varied with respect to *both* imbedding space coordinates and spinor fields is the fundamental action principle. The c-number parts of the conserved symplectic charges associated with this action give rise to bosonic conserved charges defining configuration space Hamiltonians. The second quantization of the spinor fields reduces to the requirement that super charges and Hamiltonians generate super-symplectic algebra determining the anti-commutation relations for the induced spinor fields.

### 4.2.6 Central extension as symplectic extension at configuration space level

The earlier attempts to understand the emergence of central extension of super-symplectic algebra were based on the notion of symplectic extension. This general view is not given up although it seems that this abstract approach is not very practical. Symplectic extension emerged originally in the attempts to construct formal expression for the configuration space Dirac equation. The rather obvious idea was that the Dirac equation reduces to super Virasoro conditions with Super Virasoro generators involving the Dirac operator of the imbedding space. The basic difficulty was the necessity to assign to the gamma matrices of the imbedding space fermion number. In the recent formulation the Dirac operator of $H$ does not appear in in the Super Virasoro conditions so that this problem disappears.

The proposal that Super Virasoro conditions should replaced with conditions stating that the commutator of super-symplectic and super Kac-Moody algebras annihilates physical states, looks rather feasible. One could call these conditions as configuration space Dirac equation but at this moment I feel that this would be just play with words and mask the group theoretical content of these conditions. In any case, the formulas for the symplectic extension and action of isometry generators on configuration space spinor deserve to be summarized.

### Symplectic extension

The Abelian extension of the super-symplectic algebra is obtained by an extremely simple trick. Replace the ordinary derivatives appearing in the definition of, say spinorial isometry generator, by the covariant derivatives defined by a coupling to a multiple of the Kähler potential.

$$\begin{aligned} j^{Ak}\partial_k &\rightarrow j^{Ak}D_k \ , \\ D_k &= \partial_k + ikA_k/2 \ . \end{aligned} \tag{4.2.9}$$

where $A_k$ denotes Kähler potential. The reality of the parameter $k$ is dictated by the Hermiticity requirement and also by the requirement that Abelian extension reduces to the standard form in Cartan algebra. $k$ is expected to be integer also by the requirement that covariant derivative corresponds to connection (quantization of magnetic charge).

The commutation relations for the centrally extended generators $J^A$ read:

$$[J^A, J^B] = J^{[A,B]} + ikj^{Ak}J_{kl}j^{Bl} \equiv J^{[A,B]} + ikJ_{AB} \ . \tag{4.2.10}$$

Since Kähler form defines symplectic structure in configuration space one can express Abelian extension term as a Poisson bracket of two Hamiltonians

$$J_{AB} \equiv j^{Ak}J_{kl}j^{Bl} = \{H^A, H^B\} \ . \tag{4.2.11}$$

Notice that Poisson bracket is well defined also when Kähler form is degenerate.

The extension indeed has acceptable properties:

1. Jacobi-identities reduce to the form

$$\sum_{cyclic} H^{[A,[B,C]]} = 0 \ , \tag{4.2.12}$$

   and therefore to the Jacobi identities of the original Lie- algebra in Hamiltonian representation.

2. In the Cartan algebra Abelian extension reduces to a constant term since the Poisson bracket for two commuting generators must be a multiple of a unit matrix. This feature is clearly crucial for the non-triviality of the Abelian extension and is encountered already at the level of ordinary $(q,p)$ Poisson algebra: although the differential operators $\partial_p$ and $\partial_q$ commute the Poisson bracket of the corresponding Hamiltonians $p$ and $q$ is nontrivial: $\{p,q\} = 1$. Therefore the extension term commutes with the generators of the Cartan subalgebra. Extension is also local $U(1)$ extension since Poisson algebra differs from the Lie-algebra of the vector fields in that it contains constant Hamiltonian ("1" in the commutator), which commutes with all other Hamiltonians and corresponds to a vanishing vector field.

3. For the generators not belonging to Cartan sub-algebra of $CH$ isometries Abelian extension term is not annihilated by the generators of the original algebra and in this respect the extension differs from the standard central extension for the loop algebras. It must be however emphasized that for the super-symplectic algebra generators correspond to products of $\delta M^4_+$ and $CP_2$ Hamiltonians and this means that generators of say $\delta M^4_+$-local $SU(3)$ Cartan algebra are non-commuting and the commutator is completely analogous to central extension term since it is symmetric with respect to $SU(3)$ generators.

4. The proposed method yields a trivial extension in the case of Diff$^4$. The reason is the (four-dimensional!) Diff degeneracy of the Kähler form. Abelian extension term is given by the contraction of the Diff$^4$ generators with the Kähler potential

$$j^{Ak}J_{kl}j^{Bl} = 0 \ , \tag{4.2.13}$$

which vanishes identically by the Diff degeneracy of the Kähler form. Therefore neither 3- or 4-dimensional Diff invariance is not expected to cause any difficulties. Recall that 4-dimensional Diff degeneracy is what is needed to eliminate time like vibrational excitations from the spectrum of the theory. By the way, the fact that the loop space metric is not Diff degenerate makes understandable the emergence of Diff anomalies in string models [B40, B22] .

5. The extension is trivial also for the other zero norm generators of the tangent space algebra, in particular for the $k_2 = Im(k) = 0$ symplectic generators possible present so that these generators indeed act as genuine $U(1)$ transformations.

6. Concerning the solution of configuration space Dirac equation the maximum of Kähler function is expected to be special, much like origin of Minkowski space and symmetric space property suggests that the construction of solutions reduces to this point. At this point the generators and Hamiltonians of the algebra $h$ in the defining Cartan decomposition $g = h+t$ should vanish. $h$ corresponds to integer values of $k_1 = Re(k)$ for Cartan algebra of super-symplectic algebra and integer valued conformal weights $n$ for Super Kac-Moody algebra. The algebra reduces at the maximum to an exceptionally simple form since only central extension contributes to the metric and Kähler form. In the ideal case the elements of the metric and Kähler form could be even diagonal. The degeneracy of the metric might of course pose additional complications.

#### Super symplectic action on configuration space spinors

The generators of symplectic transformations are obtained in the spinor representation of the isometry group of the configuration space by the following formal construction. Take isometry generator in the spinor representation and add to the covariant derivative $D_k$ defined by vielbein connection the coupling to the multiple of the Kähler potential: $D_k \rightarrow D_k + ikAk/2$.

$$\begin{aligned} J^A &= j^{Ak}D_k + D_l j_k \Sigma^{kl}/2 \ , \\ &\rightarrow \hat{J}^A = j^{Ak}(D_k + ikA_k/2) + D_l j^A_k \Sigma^{kl}/2 \ , \end{aligned} \tag{4.2.14}$$

This induces the required central term to the commutation relations. Introduce complex coordinates and define bosonic creation and annihilation operators as $(1,0)$ and $(0,1)$ parts of the modified isometry generators

$$\begin{aligned} B^\dagger_A &= J^A_+ = j^{Ak}(D_k + ... \ , \\ B_A &= J^A_- = j^{A\bar{k}}(D_{\bar{k}} + ... \ . \end{aligned} \tag{4.2.15}$$

where "$k$" refers now to complex coordinates and "$\bar{k}$" to their conjugates.

Fermionic generators are obtained as the contractions of the complexified gamma matrices with the isometry generators

$$\begin{aligned} \Gamma_A^\dagger &= j^{Ak}\Gamma_k \ , \\ \Gamma_A &= j^{A\bar{k}}\Gamma_{\bar{k}} \ . \end{aligned} \tag{4.2.16}$$

Notice that the bosonic Cartan algebra generators obey standard oscillator algebra commutation relations and annihilate fermionic Cartan algebra generators. Hermiticity condition holds in the sense that creation type generators are hermitian conjugates of the annihilation operator type generators. There are two kinds of representations depending on whether one uses leptonic or quark like oscillator operators to construct the gammas. These will be assumed to correspond to Ramond and NS type generators with the radial plane waves being labeled by integer and half odd integer indices respectively.

The non-vanishing commutators between the Cartan algebra bosonic generators are given by the matrix elements of the Kähler form in the basis of formed by the isometry generators

$$[B_A^\dagger, B_B] = J(j^{A\dagger}, j^B) \equiv J_{\bar{A}B} \ . \tag{4.2.17}$$

and are isometry invariant quantities. The commutators between local $SU(3)$ generators not belonging to Cartan algebra are just those of the local gauge algebra with Abelian extension term added.

The anti-commutators between the fermionic generators are given by the elements of the metric (as opposed to Kähler form in the case of bosonic generators) in the basis formed by the isometry generators

$$\{\Gamma_A\dagger, \Gamma_B\} = 2g(j^{A\dagger}, j^B) \equiv 2g_{\bar{A}B} \ . \tag{4.2.18}$$

and are invariant under isometries. Numerically the commutators and anti-commutators differ only the presence of the imaginary unit and the scale factor $R$ relating the metric and Kähler form to each other (the factor $R$ is same for $CP_2$ metric and Kähler form).

The commutators between bosonic and fermionic generators are given by

$$[B_A, \Gamma_B] = \Gamma_{[A,B]} \ . \tag{4.2.19}$$

The presence of vielbein and rotation terms in the representation of the isometry generators is essential for obtaining these nice commutations relations. The commutators vanish identically for Cartan algebra generators. From the commutation relations it is clear that Super Kac Moody algebra structure is directly related to the Kähler structure of the configuration space: the anti-commutator of fermionic generators is proportional to the metric and the commutator of the bosonic generators is proportional to the Kähler form. It is this algebra, which should generate the solutions of the field equations of the theory.

The vielbein and rotational parts of the bosonic isometry generators are quadratic in the fermionic oscillator operators and this suggests the interpretation as the fermionic contribution to the isometry currents. This means that the action of the bosonic generators is essentially non-perturbative since it creates fermion antifermion pairs besides exciting bosonic degrees of freedom.

### 4.2.7 Configuration space Clifford algebra as a hyper-finite factor of type $II_1$

The naive expectation is that the trace of the unit matrix associated with the Clifford algebra spanned by configuration space sigma matrices is infinite and thus defines an excellent candidate for a source of divergences in perturbation theory. This potential source of infinities remained un-noticed until it became clear that there is a connection with von Neumann algebras [A41] . In fact, for a separable Hilbert space defines a standard representation for so called [A51] . This guarantees that the trace of the unit matrix equals to unity and there is no danger about divergences.

### Philosophical ideas behind von Neumann algebras

The goal of von Neumann was to generalize the algebra of quantum mechanical observables. The basic ideas behind the von Neumann algebra are dictated by physics. The algebra elements allow Hermitian conjugation $^*$ and observables correspond to Hermitian operators. Any measurable function $f(A)$ of operator $A$ belongs to the algebra and one can say that non-commutative measure theory is in question.

The predictions of quantum theory are expressible in terms of traces of observables. Density matrix defining expectations of observables in ensemble is the basic example. The highly non-trivial requirement of von Neumann was that identical a priori probabilities for a detection of states of infinite state system must make sense. Since quantum mechanical expectation values are expressible in terms of operator traces, this requires that unit operator has unit trace: $tr(Id) = 1$.

In the finite-dimensional case it is easy to build observables out of minimal projections to 1-dimensional eigen spaces of observables. For infinite-dimensional case the probably of projection to 1-dimensional sub-space vanishes if each state is equally probable. The notion of observable must thus be modified by excluding 1-dimensional minimal projections, and allow only projections for which the trace would be infinite using the straightforward generalization of the matrix algebra trace as the dimension of the projection.

The non-trivial implication of the fact that traces of projections are never larger than one is that the eigen spaces of the density matrix must be infinite-dimensional for non-vanishing projection probabilities. Quantum measurements can lead with a finite probability only to mixed states with a density matrix which is projection operator to infinite-dimensional subspace. The simple von Neumann algebras for which unit operator has unit trace are known as factors of type $II_1$ [A51] .

The definitions of adopted by von Neumann allow however more general algebras. Type $I_n$ algebras correspond to finite-dimensional matrix algebras with finite traces whereas $I_\infty$ associated with a separable infinite-dimensional Hilbert space does not allow bounded traces. For algebras of type $III$ non-trivial traces are always infinite and the notion of trace becomes useless.

### von Neumann, Dirac, and Feynman

The association of algebras of type $I$ with the standard quantum mechanics allowed to unify matrix mechanism with wave mechanics. Note however that the assumption about continuous momentum state basis is in conflict with separability but the particle-in-box idealization allows to circumvent this problem (the notion of space-time sheet brings the box in physics as something completely real).

Because of the finiteness of traces von Neumann regarded the factors of type $II_1$ as fundamental and factors of type III as pathological. The highly pragmatic and successful approach of Dirac based on the notion of delta function, plus the emergence of Feynman graphs, the possibility to formulate the notion of delta function rigorously in terms of distributions, and the emergence of path integral approach meant that von Neumann approach was forgotten by particle physicists.

Algebras of type $II_1$ have emerged only much later in conformal and topological quantum field theories [A66, A72] allowing to deduce invariants of knots, links and 3-manifolds. Also algebraic structures known as bi-algebras, Hopf algebras, and ribbon algebras [A52, A38] relate closely to type $II_1$ factors. In topological quantum computation [B31] based on braid groups [A39] modular S-matrices they play an especially important role.

### Clifford algebra of configuration space as von Neumann algebra

The Clifford algebra of the configuration space provides a school example of a hyper-finite factor of type $II_1$, which means that fermionic sector does not produce divergence problems. Super-symmetry means that also "orbital" degrees of freedom corresponding to the deformations of 3-surface define similar factor. The general theory of hyper-finite factors of type $II_1$ is very rich and leads to rather detailed understanding of the general structure of S-matrix in TGD framework. For instance, there is a unitary evolution operator intrinsic to the von Neumann algebra defining in a natural manner single particle time evolution. Also a connection with 3-dimensional topological quantum field theories and knot theory, conformal field theories, braid groups, quantum groups, and quantum counterparts of quaternionic and octonionic division algebras emerges naturally. These aspects are discussed in detail in [K62] .

## 4.3 Hierarchy of Planck constants and the generalization of the notion of imbedding space

In the following the recent view about structure of imbedding space forced by the quantization of Planck constant is summarized. The question is whether it might be possible in some sense to replace $H$ or its Cartesian factors by their necessarily singular multiple coverings and factor spaces. One can consider two options: either $M^4$ or the causal diamond $CD$. The latter one is the more plausible option from the point of view of WCW geometry.

### 4.3.1 The evolution of physical ideas about hierarchy of Planck constants

The evolution of the physical ideas related to the hierarchy of Planck constants and dark matter as a hierarchy of phases of matter with non-standard value of Planck constants was much faster than the evolution of mathematical ideas and quite a number of applications have been developed during last five years.

1. The starting point was the proposal of Nottale [E1] that the orbits of inner planets correspond to Bohr orbits with Planck constant $\hbar_{gr} = GMm/v_0$ and outer planets with Planck constant $\hbar_{gr} = 5GMm/v_0$, $v_0/c \simeq 2^{-11}$. The basic proposal [K47, K38] was that ordinary matter condenses around dark matter which is a phase of matter characterized by a non-standard value of Planck constant whose value is gigantic for the space-time sheets mediating gravitational interaction. The interpretation of these space-time sheets could be as magnetic flux quanta or as massless extremals assignable to gravitons.

2. Ordinary particles possibly residing at these space-time sheet have enormous value of Compton length meaning that the density of matter at these space-time sheets must be very slowly varying. The string tension of string like objects implies effective negative pressure characterizing dark energy so that the interpretation in terms of dark energy might make sense [K48] . TGD predicted a one-parameter family of Robertson-Walker cosmologies with critical or over-critical mass density and the "pressure" associated with these cosmologies is negative.

3. The quantization of Planck constant does not make sense unless one modifies the view about standard space-time is. Particles with different Planck constant must belong to different worlds in the sense local interactions of particles with different values of $\hbar$ are not possible. This inspires the idea about the book like structure of the imbedding space obtained by gluing almost copies of $H$ together along common "back" and partially labeled by different values of Planck constant.

4. Darkness is a relative notion in this framework and due to the fact that particles at different pages of the book like structure cannot appear in the same vertex of the generalized Feynman diagram. The phase transitions in which partonic 2-surface $X^2$ during its travel along $X_l^3$ leaks to another page of book are however possible and change Planck constant. Particle (say photon -) exchanges of this kind allow particles at different pages to interact. The interactions are strongly constrained by charge fractionization and are essentially phase transitions involving many particles. Classical interactions are also possible. It might be that we are actually observing dark matter via classical fields all the time and perhaps have even photographed it [K56] .

5. The realization that non-standard values of Planck constant give rise to charge and spin fractionization and anyonization led to the precise identification of the prerequisites of anyonic phase. If the partonic 2-surface, which can have even astrophysical size, surrounds the tip of $CD$, the matter at the surface is anyonic and particles are confined at this surface. Dark matter could be confined inside this kind of light-like 3-surfaces around which ordinary matter condenses. If the radii of the basic pieces of these nearly spherical anyonic surfaces - glued to a connected structure by flux tubes mediating gravitational interaction - are given by Bohr rules, the findings of Nottale [E1] can be understood. Dark matter would resemble to a high degree matter in black holes replaced in TGD framework by light-like partonic 2-surfaces with a minimum size of order Schwartschild radius $r_S$ of order scaled up Planck length $l_{Pl} = \sqrt{\hbar_{gr}G} = GM$. Black hole entropy is inversely proportional to $\hbar$ and predicted to be of order unity so that dramatic modification of the picture about black holes is implied.

6. Perhaps the most fascinating applications are in biology. The anomalous behavior ionic currents through cell membrane (low dissipation, quantal character, no change when the membrane is replaced with artificial one) has a natural explanation in terms of dark supra currents. This leads to a vision about how dark matter and phase transitions changing the value of Planck constant could relate to the basic functions of cell, functioning of DNA and aminoacids, and to the mysteries of bio-catalysis. This leads also a model for EEG interpreted as a communication and control tool of magnetic body containing dark matter and using biological body as motor instrument and sensory receptor. One especially amazing outcome is the emergence of genetic code of vertebrates from the model of dark nuclei as nuclear strings [L2, K56] , [L2] .

### 4.3.2 The most general option for the generalized imbedding space

Simple physical arguments pose constraints on the choice of the most general form of the imbedding space.

1. The fundamental group of the space for which one constructs a non-singular covering space or factor space should be non-trivial. This is certainly not possible for $M^4$, $CD$, $CP_2$, or $H$. One can however construct singular covering spaces. The fixing of the quantization axes implies a selection of the sub-space $H_4 = M^2 \times S^2 \subset M^4 \times CP_2$, where $S^2$ is geodesic sphere of $CP_2$. $\hat{M}^4 = M^4 \backslash M^2$ and $\hat{CP}_2 = CP_2 \backslash S^2$ have fundamental group $Z$ since the codimension of the excluded sub-manifold is equal to two and homotopically the situation is like that for a punctured plane. The exclusion of these sub-manifolds defined by the choice of quantization axes could naturally give rise to the desired situation.

2. $CP_2$ allows two geodesic spheres which left invariant by $U(2$ *resp.* $SO(3)$. The first one is homologically non-trivial. For homologically non-trivial geodesic sphere $H_4 = M^2 \times S^2$ represents a straight cosmic string which is non-vacuum extremal of Kähler action (not necessarily preferred extremal). One can argue that the many-valuedness of $\hbar$ is un-acceptable for non-vacuum extremals so that only homologically trivial geodesic sphere $S^2$ would be acceptable. One could go even further. If the extremals in $M^2 \times CP_2$ can be preferred non-vacuum extremals, the singular coverings of $M^4$ are not possible. Therefore only the singular coverings and factor spaces of $CP_2$ over the homologically trivial geodesic sphere $S^2$ would be possible. This however looks a non-physical outcome.

   (a) The situation changes if the extremals of type $M^2 \times Y^2$, $Y^2$ a holomorphic surface of $CP_3$, fail to be hyperquaternionic. The tangent space $M^2$ represents hypercomplex sub-space and the product of the modified gamma matrices associated with the tangent spaces of $Y^2$ should belong to $M^2$ algebra. This need not be the case in general.

   (b) The situation changes also if one reinterprets the gluing procedure by introducing scaled up coordinates for $M^4$ so that metric is continuous at $M^2 \times CP_2$ but $CD$s with different size have different sizes differing by the ratio of Planck constants and would thus have only piece of lower or upper boundary in common.

3. For the more general option one would have four different options corresponding to the Cartesian products of singular coverings and factor spaces. These options can be denoted by $C-C$, $C-F$, $F-C$, and $F-F$, where $C$ $(F)$ signifies for covering (factor space) and first (second) letter signifies for $CD$ $(CP_2)$ and correspond to the spaces $(\hat{CD}\hat{\times}G_a) \times (\hat{CP}_2\hat{\times}G_b)$, $(\hat{CD}\hat{\times}G_a) \times \hat{CP}_2/G_b$, $\hat{CD}/G_a \times (\hat{CP}_2\hat{\times}G_b)$, and $\hat{CD}/G_a \times \hat{CP}_2/G_b$.

4. The groups $G_i$ could correspond to cyclic groups $Z_n$. One can also consider an extension by replacing $M^2$ and $S^2$ with its orbit under more general group $G$ (say tedrahedral, octahedral, or icosahedral group). One expects that the discrete subgroups of $SU(2)$ emerge naturally in this framework if one allows the action of these groups on the singular sub-manifolds $M^2$ or $S^2$. This would replace the singular manifold with a set of its rotated copies in the case that the subgroups have genuinely 3-dimensional action (the subgroups which corresponds to exceptional groups in the ADE correspondence). For instance, in the case of $M^2$ the quantization axes for angular momentum would be replaced by the set of quantization axes going through the vertices of tedrahedron, octahedron, or icosahedron. This would bring non-commutative homotopy groups into the picture in a natural manner.

### 4.3.3 About the phase transitions changing Planck constant

There are several non-trivial questions related to the details of the gluing procedure and phase transition as motion of partonic 2-surface from one sector of the imbedding space to another one.

1. How the gluing of copies of imbedding space at $M^2 \times CP_2$ takes place? It would seem that the covariant metric of $CD$ factor proportional to $\hbar^2$ must be discontinuous at the singular manifold since only in this manner the idea about different scaling factor of $CD$ metric can make sense. On the other hand, one can always scale the $M^4$ coordinates so that the metric is continuous but the sizes of $CDs$ with different Planck constants differ by the ratio of the Planck constants.

2. One might worry whether the phase transition changing Planck constant means an instantaneous change of the size of partonic 2-surface in $M^4$ degrees of freedom. This is not the case. Light-likeness in $M^2 \times S^2$ makes sense only for surfaces $X^1 \times D^2 \subset M^2 \times S^2$, where $X^1$ is light-like geodesic. The requirement that the partonic 2-surface $X^2$ moving from one sector of $H$ to another one is light-like at $M^2 \times S^2$ irrespective of the value of Planck constant requires that $X^2$ has single point of $M^2$ as $M^2$ projection. Hence no sudden change of the size $X^2$ occurs.

3. A natural question is whether the phase transition changing the value of Planck constant can occur purely classically or whether it is analogous to quantum tunneling. Classical non-vacuum extremals of Chern-Simons action have two-dimensional $CP_2$ projection to homologically non-trivial geodesic sphere $S^2_I$. The deformation of the entire $S^2_I$ to homologically trivial geodesic sphere $S^2_{II}$ is not possible so that only combinations of partonic 2-surfaces with vanishing total homology charge (Kähler magnetic charge) can in principle move from sector to another one, and this process involves fusion of these 2-surfaces such that $CP_2$ projection becomes single homologically trivial 2-surface. A piece of a non-trivial geodesic sphere $S^2_I$ of $CP_2$ can be deformed to that of $S^2_{II}$ using 2-dimensional homotopy flattening the piece of $S^2$ to curve. If this homotopy cannot be chosen to be light-like, the phase transitions changing Planck constant take place only via quantum tunnelling. Obviously the notions of light-like homotopies (cobordisms) are very relevant for the understanding of phase transitions changing Planck constant.

### 4.3.4 How one could fix the spectrum of Planck constants?

The question how the observed Planck constant relates to the integers $n_a$ and $n_b$ defining the covering and factors spaces, is far from trivial and I have considered several options. The basic physical inputs are the condition that scaling of Planck constant must correspond to the scaling of the metric of $CD$ (that is Compton lengths) on one hand and the scaling of the gauge coupling strength $g^2/4\pi\hbar$ on the other hand.

1. One can assign to Planck constant to both $CD$ and $CP_2$ by assuming that it appears in the commutation relations of corresponding symmetry algebras. Algebraist would argue that Planck constants $\hbar(CD)$ and $\hbar(CP_2)$ must define a homomorphism respecting multiplication and division (when possible) by $G_i$. This requires $r(X) = \hbar(X)\hbar_0 = n$ for covering and $r(X) = 1/n$ for factor space or vice versa.

2. If one assumes that $\hbar^2(X)$, $X = M^4$, $CP_2$ corresponds to the scaling of the covariant metric tensor $g_{ij}$ and performs an over-all scaling of $H$-metric allowed by the Weyl invariance of Kähler action by dividing metric with $\hbar^2(CP_2)$, one obtains the scaling of $M^4$ covariant metric by $r^2 \equiv \hbar^2/\hbar_0^2 = \hbar^2(M^4)/\hbar^2(CP_2)$ whereas $CP_2$ metric is not scaled at all.

3. The condition that $\hbar$ scales as $n_a$ is guaranteed if one has $\hbar(CD) = n_a\hbar_0$. This does not fix the dependence of $\hbar(CP_2)$ on $n_b$ and one could have $\hbar(CP_2) = n_b\hbar_0$ or $\hbar(CP_2) = \hbar_0/n_b$. The intuitive picture is that $n_b$- fold covering gives in good approximation rise to $n_an_b$ sheets and multiplies YM action action by $n_an_b$ which is equivalent with the $\hbar = n_an_b\hbar_0$ if one effectively compresses the covering to $CD \times CP_2$. One would have $\hbar(CP_2) = \hbar_0/n_b$ and $\hbar = n_an_b\hbar_0$. Note that the descriptions using ordinary Planck constant and coverings and scaled Planck constant but contracting the covering would be alternative descriptions.

This gives the following formulas $r \equiv \hbar/\hbar_0 = r(M^4)/r(CP_2)$ in various cases.

| | $C-C$ | $F-C$ | $C-F$ | $F-F$ |
|---|---|---|---|---|
| $r$ | $n_an_b$ | $\frac{n_a}{n_b}$ | $\frac{n_b}{n_a}$ | $\frac{1}{n_an_b}$ |

### 4.3.5 Preferred values of Planck constants

Number theoretic considerations favor the hypothesis that the integers corresponding to Fermat polygons constructible using only ruler and compass and given as products $n_F = 2^k \prod_s F_s$, where $F_s = 2^{2^s} + 1$ are distinct Fermat primes, are favored. The reason would be that quantum phase $q = exp(i\pi/n)$ is in this case expressible using only iterated square root operation by starting from rationals. The known Fermat primes correspond to $s = 0, 1, 2, 3, 4$ so that the hypothesis is very strong and predicts that p-adic length scales have satellite length scales given as multiples of $n_F$ of fundamental p-adic length scale. $n_F = 2^{11}$ corresponds in TGD framework to a fundamental constant expressible as a combination of Kähler coupling strength, $CP_2$ radius and Planck length appearing in the expression for the tension of cosmic strings, and the powers of $2^{11}$ seem to be especially favored as values of $n_a$ in living matter [K16] .

### 4.3.6 How Planck constants are visible in Kähler action?

$\hbar(M^4)$ and $\hbar(CP_2)$ appear in the commutation and anticommutation relations of various superconformal algebras. Only the ratio of $M^4$ and $CP_2$ Planck constants appears in Kähler action and is due to the fact that the $M^4$ and $CP_2$ metrics of the imbedding space sector with given values of Planck constants are proportional to the corresponding Planck constants. This implies that Kähler function codes for radiative corrections to the classical action, which makes possible to consider the possibility that higher order radiative corrections to functional integral vanish as one might expect at quantum criticality. For a given p-adic length scale space-time sheets with all allowed values of Planck constants are possible. Hence the spectrum of quantum critical fluctuations could in the ideal case correspond to the spectrum of $\hbar$ coding for the scaled up values of Compton lengths and other quantal lengths and times. If so, large $\hbar$ phases could be crucial for understanding of quantum critical superconductors, in particular high $T_c$ superconductors.

### 4.3.7 Updated view about the hierarchy of Planck constants

The original hypothesis was that the hierarchy of Planck constants is real. In this formulation the imbedding space was replaced with its covering space assumed to decompose to a Cartesian product of singular finite-sheeted coverings of $M^4$ and $CP_2$.

Few years ago came the realization that it could be only effective but have same practical implications. The basic observation was that the effective hierarchy need not be postulated separately but follows as a prediction from the vacuum degeneracy of Kähler action. In this formulation Planck constant at fundamental level has its standard value and its effective values come as its integer multiples so that one should write $\hbar_{eff} = n\hbar$ rather than $\hbar = n\hbar_0$ as I have done. For most practical purposes the states in question would behave as if Planck constant were an integer multiple of the ordinary one. In this formulation the singular covering of the imbedding space became only a convenient auxiliary tool. It is no more necessary to assume that the covering reduces to a Cartesian product of singular coverings of $M^4$ and $CP_2$ but for some reason I kept this assumption.

The formulation based on multi-furcations of space-time surfaces to $N$ branches. For some reason I assumed that they are simultaneously present. This is too restrictive an assumption. The $N$ branches are very much analogous to single particle states and second quantization allowing all $0 < n \leq N$-particle states for given $N$ rather than only $N$-particle states looks very natural. As a matter fact, this interpretation was the original one, and led to the very speculative and fuzzy notion of $N$-atom, which I later more or less gave up. Quantum multi-furcation could be the root concept implying the effective hierarchy of Planck constants, anyons and fractional charges, and related notions- even the notions of $N$-nuclei, $N$-atoms, and $N$-molecules.

### Basic physical ideas

The basic phenomenological rules are simple and there is no need to modify them.

1. The phases with non-standard values of effective Planck constant are identified as dark matter. The motivation comes from the natural assumption that only the particles with the same value of effective Planck can appear in the same vertex. One can illustrate the situation in terms of the book metaphor. Imbedding spaces with different values of Planck constant form a book like structure and matter can be transferred between different pages only through the back of the book where the pages are glued together. One important implication is that light exotic charged particles lighter than weak bosons are possible if they have non-standard value of Planck constant. The standard argument excluding them is based on decay widths of weak bosons and has led to a neglect of large number of particle physics anomalies [K57].

2. Large effective or real value of Planck constant scales up Compton length - or at least de Broglie wave length - and its geometric correlate at space-time level identified as size scale of the space-time sheet assignable to the particle. This could correspond to the Kähler magnetic flux tube for the particle forming consisting of two flux tubes at parallel space-time sheets and short flux tubes at ends with length of order $CP_2$ size.

   This rule has far reaching implications in quantum biology and neuroscience since macroscopic quantum phases become possible as the basic criterion stating that macroscopic quantum phase becomes possible if the density of particles is so high that particles as Compton length sized objects overlap. Dark matter therefore forms macroscopic quantum phases. One implication is the explanation of mysterious looking quantal effects of ELF radiation in EEG frequency range on vertebrate brain: $E = hf$ implies that the energies for the ordinary value of Planck constant are much below the thermal threshold but large value of Planck constant changes the situation. Also the phase transitions modifying the value of Planck constant and changing the lengths of flux tubes (by quantum classical correspondence) are crucial as also reconnections of the flux tubes.

   The hierarchy of Planck constants suggests also a new interpretation for FQHE (fractional quantum Hall effect) [K40] in terms of anyonic phases with non-standard value of effective Planck constant realized in terms of the effective multi-sheeted covering of imbedding space: multi-sheeted space-time is to be distinguished from many-sheeted space-time.

3. In astrophysics and cosmology the implications are even more dramatic if one believes that also $\hbar_{gr}$ correponds to effective Planck constant interpreted as number of sheets of multi-furcation. It was Nottale [E1] who first introduced the notion of gravitational Planck constant as $\hbar_{gr} = GMm/v_0$, $v_0 < 1$ has interpretation as velocity light parameter in units $c = 1$. This would be true for $GMm/v_0 \geq 1$. The interpretation of $\hbar_{gr}$ in TGD framework is as an effective Planck constant associated with space-time sheets mediating gravitational interaction between masses $M$ and $m$. The huge value of $\hbar_{gr}$ means that the integer $\hbar_{gr}/\hbar_0$ interpreted as the number of sheets of covering is gigantic and that Universe possesses gravitational quantum coherence in super-astronomical scales for masses which are large. This would suggest that gravitational radiation is emitted as dark gravitons which decay to pulses of ordinary gravitons replacing continuous flow of gravitational radiation.

   It must be however emphasized that the interpretation of $\hbar_{gr}$ could be different, and it will be found that one can develop an argument demonstrating how $\hbar_{gr}$ with a correct order of magnitude emerges from the effective space-time metric defined by the anticommutators appearing in the modified Dirac equation.

4. Why Nature would like to have large effective value of Planck constant? A possible answer relies on the observation that in perturbation theory the expansion takes in powers of gauge couplings strengths $\alpha = g^2/4\pi\hbar$. If the effective value of $\hbar$ replaces its real value as one might expect to happen for multi-sheeted particles behaving like single particle, $\alpha$ is scaled down and perturbative expansion converges for the new particles. One could say that Mother Nature loves theoreticians and comes in rescue in their attempts to calculate. In quantum gravitation the problem is especially acute since the dimensionless parameter $GMm/\hbar$ has gigantic value. Replacing $\hbar$ with $\hbar_{gr} = GMm/v_0$ the coupling strength becomes $v_0 < 1$.

### Space-time correlates for the hierarchy of Planck constants

The hierarchy of Planck constants was introduced to TGD originally as an additional postulate and formulated as the existence of a hierarchy of imbedding spaces defined as Cartesian products of singular coverings of $M^4$ and $CP_2$ with numbers of sheets given by integers $n_a$ and $n_b$ and $\hbar = n\hbar_0$. $n = n_a n_b$.

With the advent of zero energy ontology, it became clear that the notion of singular covering space of the imbedding space could be only a convenient auxiliary notion. Singular means that the sheets fuse together at the boundary of multi-sheeted region. The effective covering space emerges naturally from the vacuum degeneracy of Kähler action meaning that all deformations of canonically imbedded $M^4$ in $M^4 \times CP_2$ have vanishing action up to fourth order in small perturbation. This is clear from the fact that the induced Kähler form is quadratic in the gradients of $CP_2$ coordinates and Kähler action is essentially Maxwell action for the induced Kähler form. The vacuum degeneracy implies that the correspondence between canonical momentum currents $\partial L_K/\partial(\partial_\alpha h^k)$ defining the modified gamma matrices [K67] and gradients $\partial_\alpha h^k$ is not one-to-one. Same canonical momentum current corresponds to several values of gradients of imbedding space coordinates. At the partonic 2-surfaces at the light-like boundaries of $CD$ carrying the elementary particle quantum numbers this implies that the two normal derivatives of $h^k$ are many-valued functions of canonical momentum currents in normal directions.

Multi-furcation is in question and multi-furcations are indeed generic in highly non-linear systems and Kähler action is an extreme example about non-linear system. What multi-furcation means in quantum theory? The branches of multi-furcation are obviously analogous to single particle states. In quantum theory second quantization means that one constructs not only single particle states but also the many particle states formed from them. At space-time level single particle states would correspond to $N$ branches $b_i$ of multi-furcation carrying fermion number. Two-particle states would correspond to 2-fold covering consisting of 2 branches $b_i$ and $b_j$ of multi-furcation. $N-$particle state would correspond to $N$-sheeted covering with all branches present and carrying elementary particle quantum numbers. The branches co-incide at the partonic 2-surface but since their normal space data are different they correspond to different tensor product factors of state space. Also now the factorization $N = n_a n_b$ occurs but now $n_a$ and $n_b$ would relate to branching in the direction of space-like 3-surface and light-like 3-surface rather than $M^4$ and $CP_2$ as in the original hypothesis.

In light of this the working hypothesis adopted during last years has been too limited: for some reason I ended up to propose that only $N$-sheeted covering corresponding to a situation in which all $N$ branches are present is possible. Before that I quite correctly considered more general option based on intuition that one has many-particle states in the multi-sheeted space. The erratic form of the working hypothesis has not been used in applications.

Multi-furcations relate closely to the quantum criticality of Kähler action. Feigenbaum bifurcations represent a toy example of a system which via successive bifurcations approaches chaos. Now more general multi-furcations in which each branch of given multi-furcation can multi-furcate further, are possible unless on poses any additional conditions. This allows to identify additional aspect of the geometric arrow of time. Either the positive or negative energy part of the zero energy state is "prepared" meaning that single $n$-sub-furcations of $N$-furcation is selected. The most general state of this kind involves superposition of various $n$-sub-furcations.

### Basic phenomenological rules of thumb in the new framework

It is important to check whether or not the refreshed view about dark matter is consistent with existent rules of thumb.

1. The interpretation of quantized multi-furcations as WCW anyons explains also why the effective hierarchy of Planck constants defines a hierarchy of phases which are dark relative to each other. This is trivially true since the phases with different number of branches in multi-furcation correspond to disjoint regions of WCW so that the particles with different effective value of Planck constant cannot appear in the same vertex.

2. The phase transitions changing the value of Planck constant are just the multi-furcations and can be induced by changing the values of the external parameters controlling the properties of preferred extremals. Situation is very much the same as in any non-linear system.

3. In the case of massless particles the scaling of wavelength in the effective scaling of $\hbar$ can be understood if dark $n$-photons consist of $n$ photons with energy $E/n$ and wavelength $n\lambda$.

4. For massive particle it has been assumed that masses for particles and they dark counterparts are same and Compton wavelength is scaled up. In the new picture this need not be true. Rather, it would seem that wave length are same as for ordinary electron.

   On the other hand, p-adic thermodynamics predicts that massive elemenetary particles are massless most of the time. ZEO predicts that even virtual wormhole throats are massless. Could this mean that the picture applying on massless particle should apply to them at least at relativistic limit at which mass is negligible. This might be the case for bosons but for fermions also fermion number should be fractionalized and this is not possible in the recent picture. If one assumes that the $n$-electron has same mass as electron, the mass for dark single electron state would be scaled down by $1/n$. This does not look sensible unless the p-adic length defined by prime is scaled down by this fact in good approximation.

   This suggests that for fermions the basic scaling rule does not hold true for Compton length $\lambda_c = \hbar_m$. Could it however hold for de-Broglie lengths $\lambda = \hbar/p$ defined in terms of 3-momentum? The basic overlap rule for the formation of macroscopic quantum states is indeed formulated for de Broglie wave length. One could argue that an $1/N$-fold reduction of density that takes place in the delocalization of the single particle states to the $N$ branches of the cover, implies that the volume per particle increases by a factor $N$ and single particle wave function is delocalized in a larger region of 3-space. If the particles reside at effectively one-dimensional 3-surfaces - say magnetic flux tubes - this would increase their de Broglie wave length in the direction of the flux tube and also the length of the flux tube. This seems to be enough for various applications.

One important notion in TGD inspired quantum biology is dark cyclotron state.

1. The scaling $\hbar \rightarrow k\hbar$ in the formula $E_n = (n + 1/2)\hbar eB/m$ implies that cyclotron energies are scaled up for dark cyclotron states. What this means microscopically has not been obvious but the recent picture gives a rather clearcut answer. One would have $k$-particle state formed from cyclotron states in $N$-fold branched cover of space-time surface. Each branch would carry magnetic field $B$ and ion or electron. This would give a total cyclotron energy equal to $kE_n$. These cyclotron states would be excited by $k$-photons with total energy $E = khf$ and for large enough value of $k$ the energies involved would be above thermal threshold. In the case of $Ca^{++}$ one has $f = 15$ Hz in the field $B_{end} = .2$ Gauss. This means that the value of $\hbar$ is at least the ratio of thermal energy at room temperature to $E = hf$. The thermal frequency is of order $10^{12}$ Hz so that one would have $k \simeq 10^{11}$. The number branches would be therefore rather high.

2. It seems that this kinds of states which I have called cyclotron Bose-Einstein condensates could make sense also for fermions. The dark photons involved would be Bose-Einstein condensates of $k$ photons and wall of them would be simultaneously absorbed. The biological meaning of this would be that a simultaneous excitation of large number of atoms or molecules can take place if they are localized at the branches of $N$-furcation. This would make possible coherent macroscopic changes. Note that also Cooper pairs of electrons could be $n = 2$-particle states associated with $N$-furcation.

There are experimental findings suggesting that photosynthesis involves delocalized excitations of electrons and it is interesting so see whether this could be understood in this framework.

1. The TGD based model relies on the assumption that cyclotron states are involved and that dark photons with the energy of visible photons but with much longer wavelength are involved. Single electron excitations (or single particle excitations of Cooper pairs) would generate negentropic entanglement automatically.

2. If cyclotron excitations are the primary ones, it would seem that they could be induced by dark $n$-photons exciting all $n$ electrons simultaneously. $n$-photon should have energy of a visible photon. The number of cyclotron excited electrons should be rather large if the total excitation energy is to be above thermal threshold. In this case one could not speak about cyclotron excitation however. This would require that solar photons are transformed to $n$-photons in $N$-furcation in biosphere.

3. Second - more realistic looking - possibility is that the incoming photons have energy of visible photon and are therefore $n = 1$ dark photons delocalized to the branches of the $N$-furcation. They would induce delocalized single electron excitation in WCW rather than 3-space.

### Charge fractionalization and anyons

It is easy to see how the effective value of Planck constant as an integer multiple of its standard value emerges for multi-sheeted states in second quantization. At the level of Kähler action one can assume that in the first approximation the value of Kähler action for each branch is same so that the total Kähler action is multiplied by $n$. This corresponds effectively to the scaling $\alpha_K \to \alpha_K/n$ induced by the scaling $\hbar_0 \to n\hbar_0$.

Also effective charge fractionalization and anyons emerge naturally in this framework.

1. In the ordinary charge fractionalization the wave function decomposes into sharply localized pieces around different points of 3-space carrying fractional charges summing up to integer charge. Now the same happens at at the level of WCW ("world of classical worlds") rather than 3-space meaning that wave functions in $E^3$ are replaced with wave functions in the space-time of 3-surfaces (4-surfaces by holography implied by General Coordinate Invariance) replacing point-like particles. Single particle wave function in WCW is a sum of $N$ sharply localized contributions: localization takes place around one particular branch of the multi-sheeted space time surface. Each branch carries a fractional charge $q/N$ for teh analogs of plane waves.

   Therefore all quantum numbers are additive and fractionalization is only effective and observable in a localization of wave function to single branch occurring with probability $p = 1/N$ from which one can deduce that charge is $q/N$.

2. The is consistent with the proposed interpretation of dark photons/gravitons since they could carry large spin and this kind of situation could decay to bunches of ordinary photons/gravitons. It is also consistent with electromagnetic charge fractionalization and fractionalization of spin.

3. The original - and it seems wrong - argument suggested what might be interpreted as a genuine fractionalization for orbital angular momentum and also of color quantum numbers, which are analogous to orbital angular momentum in TGD framework. The observation was that a rotation through $2\pi$ at space-time level moving the point along space-time surface leads to a new branch of multi-furcation and $N + 1$:th branch corresponds to the original one. This suggests that angular momentum fractionalization should take place for $M^4$ angle coordinate $\phi$ because for it $2\pi$ rotation could lead to a different sheet of the effective covering.

   The orbital angular momentum eigenstates would correspond to waves $exp(i\phi m/N)$, $m = 0, 2, ..., N-1$ and the maximum orbital angular momentum would correspond the sum $\sum_{m=0}^{N-1} m/N = (N-1)/2$. The sum of spin and orbital angular momentum be therefore fractional.

   The different prediction is due to the fact that rotations are now interpreted as flows rotating the points of 3-surface along 3-surface rather than rotations of the entire partonic surface in imbedding space. In the latter interpretation the rotation by $2\pi$ does nothing for the 3-surface. Hence fractionalization for the total charge of the single particle states does not take place unless one adopts the flow interpretation. This view about fractionalization however leads to problems with fractionalization of electromagnetic charge and spin for which there is evidence from fractional quantum Hall effect.

### What about the relationship of gravitational Planck constant to ordinary Planck constant?

Gravitational Planck constant is given by the expression $\hbar_{gr} = GMm/v_0$, where $v_0 < 1$ has interpretation as velocity parameter in the units $c = 1$. Can one interpret also $\hbar_{gr}$ as effective value of Planck constant so that its values would correspond to multifurcation with a gigantic number of sheets. This does not look reasonable.

Could one imagine any other interpretation for $\hbar_{gr}$? Could the two Planck constants correspond to inertial and gravitational dichotomy for four-momenta making sense also for angular momentum identified as a four-vector? Could gravitational angular momentum and the momentum associated with the flux tubes mediating gravitational interaction be quantized in units of $\hbar_{gr}$ naturally?

1. Gravitational four-momentum can be defined as a projection of the $M^4$-four-momentum to space-time surface. It's length can be naturally defined by the effective metric $g_{eff}^{\alpha\beta}$ defined by the anticommutators of the modified gamma matrices. Gravitational four-momentum appears as a measurement interaction term in the modified Dirac action and can be restricted to the space-like boundaries of the space-time surface at the ends of $CD$ and to the light-like orbits of the wormhole throats and which induced 4- metric is effectively 3-dimensional.

2. At the string world sheets and partonic 2-surfaces the effective metric degenerates to 2-D one. At the ends of braid strands representing their intersection, the metric is effectively 4-D. Just for definiteness assume that the effective metric is proportional to the $M^4$ metric or rather - to its $M^2$ projection: $g_{eff}^{kl} = K^2 m^{kl}$.

   One can express the length squared for momentum at the flux tubes mediating the gravitational interaction between massive objects with masses $M$ and $m$ as

$$g_{eff}^{\alpha\beta} p_\alpha p_\beta = g_{eff}^{\alpha\beta} \partial_\alpha h^k \partial_\beta h^l p_k p_l \equiv g_{eff}^{kl} p_k p_l = n^2 \frac{\hbar^2}{L^2} \quad . \tag{4.3.1}$$

   Here $L$ would correspond to the length of the flux tube mediating gravitational interaction and $p_k$ would be the momentum flowing in that flux tube. $g_{eff}^{kl} = K^2 m^{kl}$ would give

$$p^2 = \frac{n^2 \hbar^2}{K^2 L^2} \quad .$$

   $\hbar_{gr}$ could be identifed in this simplified situation as $\hbar_{gr} = \hbar/K$.

3. Nottale's proposal requires $K = GMm/v_0$ for the space-time sheets mediating gravitational interacting between massive objects with masses $M$ and $m$. This gives the estimate

$$p_{gr} = \frac{GMm}{v_0} \frac{1}{L} \quad . \tag{4.3.2}$$

   For $v_0 = 1$ this is of the same order of magnitude as the exchanged momentum if gravitational potential gives estimate for its magnitude. $v_0$ is of same order of magnitude as the rotation velocity of planet around Sun so that the reduction of $v_0$ to $v_0 \simeq 2^{-11}$ in the case of inner planets does not mean that the propagation velocity of gravitons is reduced.

4. Nottale's formula requires that the order of magnitude for the components of the energy momentum tensor at the ends of braid strands at partonic 2-surface should have value $GMm/v_0$. Einstein's equations $T = \kappa G + \Lambda g$ give a further constraint. For the vacuum solutions of Einstein's equations with a vanishing cosmological constant the value of $h_{gr}$ approaches infinity. At the flux tubes mediating gravitational interaction one expects $T$ to be proportional to the factor $GMm$ simply because they mediate the gravitational interaction.

5. One can consider similar equation for gravitational angular momentum:

$$g_{eff}^{\alpha\beta} L_\alpha L_\beta = g_{eff}^{kl} L_k L_l = l(l+1)\hbar^2 \quad . \tag{4.3.3}$$

   This would give under the same simplifying assumptions

$$L^2 = l(l+1) \frac{\hbar^2}{K^2} \quad . \tag{4.3.4}$$

   This would justify the Bohr quantization rule for the angular momentum used in the Bohr quantization of planetary orbits.

Maybe the proposed connection might make sense in some more refined formulation. In particular the proportionality between $m^{kl}_{eff} = Km^{kl}$ could make sense as a quantum average. Also the fact, that the constant $v_0$ varies, could be understood from the dynamical character of $m^{kl}_{eff}$.

**Summary**

The hierarchy of Planck constants reduces to second quantization of multi-furcations in TGD framework and the hierarchy is only effective. Anyonic physics and effective charge fractionalization are consequences of second quantized multi-furcations. This framework also provides quantum version for the transition to chaos via quantum multi-furcations and living matter represents the basic application. The key element of dynamics of TGD is vacuum degeneracy of Kähler action making possible quantum criticality having the hierarchy of multi-furcations as basic aspect. The potential problems relate to the question whether the effective scaling of Planck constant involves scaling of ordinary wavelength or not. For particles confined inside linear structures such as magnetic flux tubes this seems to be the case.

There is also an intriguing connection with the vision about physics as generalized number theory. The conjecture that the preferred extremals of Kähler action consist of quaternionic or co-quaternionic regions led to a construction of them using iteration and also led to the hierarchy of multi-furcations [K67]. Therefore it seems that the dynamics of preferred extremals might indeed reduce to associativity/co-associativity condition at space-time level , to commutativity/co-commutativity condition at the level of string world sheets and partonic 2-surfaces, and to reality at the level of stringy curves (conformal invariance makes stringy curves causal determinants [K63] so that conformal dynamics represents conformal evolution) [K54].

## 4.4 Number theoretic compactification and $M^8 - H$ duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to number theory. In strong form $M^8 - H$ duality boils down to the assumption that space-time surfaces can be regarded either as surfaces of $H$ or as surfaces of $M^8$ composed of hyper-quaternionic and co-hyper-quaternionic regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric.

### 4.4.1 Basic idea behind $M^8 - M^4 \times CP_2$ duality

The hopes of giving $M^4 \times CP_2$ hyper-octonionic structure are meager. This circumstance forces to ask whether four-surfaces $X^4 \subset M^8$ could under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly so that the spontaneous compactification of super string models would correspond in TGD to two different manners to interpret the space-time surface. The following arguments suggest that this is indeed the case.

The hard mathematical fact behind number theoretical compactification is that the quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space) are parameterized by $CP_2$ just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of $CP_2$, as the automorphism sub-group of octonions, and as color group.

1. The space of complex structures of the octonion space is parameterized by $S^6$. The subgroup $SU(3)$ of the full automorphism group $G_2$ respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it $e_1$. Hyper-quaternions can be identified as $U(2)$ Lie-algebra but it is obvious that hyper-octonions do not allow an identification as $SU(3)$ Lie algebra. Rather, octonions decompose as $1 \oplus 1 \oplus 3 \oplus \overline{3}$ to the irreducible representations of $SU(3)$.

2. Geometrically the choice of a preferred complex (quaternionic) structure means fixing of complex (quaternionic) sub-space of octonions. The fixing of a hyper-quaternionic structure of hyper-octonionic $M^8$ means a selection of a fixed hyper-quaternionic sub-space $M^4 \subset M^8$ implying the decomposition $M^8 = M^4 \times E^4$. If $M^8$ is identified as the tangent space of $H = M^4 \times CP_2$,

this decomposition results naturally. It is also possible to select a fixed hyper-complex structure, which means a further decomposition $M^4 = M^2 \times E^2$.

3. The basic result behind number theoretic compactification and $M^8 - H$ duality is that hyper-quaternionic sub-spaces $M^4 \subset M^8$ containing a fixed hyper-complex sub-space $M^2 \subset M^4$ or its light-like line $M_\pm$ are parameterized by $CP_2$. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of $e_1$) are labeled by $U(2) \subset SU(3)$. The choice of $e_2$ and $e_3$ amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of $e_1$ and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having $e_2$ and $e_3$ components. Hence all possible completions of $1, e_1$ by adding $e_2, e_3$ doublet are labeled by $SU(3)/U(2) = CP_2$.

4. Space-time surface $X^4 \subset M^8$ is by the standard definition hyper-quaternionic if the tangent spaces of $X^4$ are hyper-quaternionic planes. Co-hyper-quaternionicity means the same for normal spaces. The presence of fixed hyper-complex structure means at space-time level that the tangent space of $X^4$ contains fixed $M^2$ at each point. Under this assumption one can map the points $(m, e) \in M^8$ to points $(m, s) \in H$ by assigning to the point $(m, e)$ of $X^4$ the point $(m, s)$, where $s \in CP_2$ characterize $T(X^4)$ as hyper-quaternionic plane. This definition is not the only one and even the appropriate one in TGD context the replacement of the tangent plane with the 4-D plane spanned by modified gamma matrices defined by Kähler action is a more natural choice. This plane is not parallel to tangent plane in general. In the sequel $T(X^4)$ denotes the preferred 4-plane which co-incides with tangent plane of $X^4$ only if the action defining modified gamma matrices is 4-volume.

5. The choice of $M^2$ can be made also local in the sense that one has $T(X^4) \supset M^2(x) \subset M^4 \subset H$. It turns out that strong form of number theoretic compactification requires this kind of generalization. In this case one must be able to fix the convention how the point of $CP_2$ is assigned to a hyper-quaternionic plane so that it applies to all possible choices of $M^2 \subset M^4$. Since $SO(3)$ hyper-quaternionic rotation relates the hyper-quaternionic planes to each other, the natural assumption is hyper-quaternionic planes related by $SO(3)$ rotation correspond to the same point of $CP_2$. Under this assumption it is possible to map hyper-quaternionic surfaces of $M^8$ for which $M^2 \subset M^4$ depends on point of $X^4$ to $H$.

### 4.4.2 Hyper-octonionic Pauli "matrices" and modified definition of hyper-quaternionicity

Hyper-octonionic Pauli matrices suggest an interesting possibility to define precisely what hyper-quaternionicity means at space-time level (for background see [K61] ).

1. According to the standard definition space-time surface $X^4$ is hyper-quaternionic if the tangent space at each point of $X^4$ in $X^4 \subset M^8$ picture is hyper-quaternionic. What raises worries is that this definition involves in no manner the action principle so that it is far from obvious that this identification is consistent with the vacuum degeneracy of Kähler action. It also unclear how one should formulate hyper-quaternionicity condition in $X^4 \subset M^4 \times CP_2$ picture.

2. The idea is to map the modified gamma matrices $\Gamma^\alpha = \frac{\partial L_K}{\partial h^k_\alpha}\Gamma^k$, $\Gamma_k = e^A_k\gamma_A$, to hyper-octonionic Pauli matrices $\sigma^\alpha$ by replacing $\gamma_A$ with hyper-octonion unit. Hyper-quaternionicity would state that the hyper-octonionic Pauli matrices $\sigma^\alpha$ obtained in this manner span complexified quaternion sub-algebra at each point of space-time. These conditions would provide a number theoretic manner to select preferred extremals of Kähler action. Remarkably, this definition applies both in case of $M^8$ and $M^4 \times CP_2$.

3. Modified Pauli matrices span the tangent space of $X^4$ if the action is four-volume because one has $\frac{\partial L_K}{\partial h^k_\alpha} = \sqrt{g}g^{\alpha\beta}\partial h^l_\beta h_{kl}$. Modified gamma matrices reduce to ordinary induced gamma matrices in this case: 4-volume indeed defines a super-conformally symmetric action for ordinary gamma matrices since the mass term of the Dirac action given by the trace of the second fundamental form vanishes for minimal surfaces.

4. For Kähler action the hyper-quaternionic sub-space does not coincide with the tangent space since $\frac{\partial L_K}{\partial h^k_\alpha}$ contains besides the gravitational contribution coming from the induced metric also the "Maxwell contribution" from the induced Kähler form not parallel to space-time surface. Modified gamma matrices are required by super conformal symmetry for the extremals of Kähler action and they also guarantee that vacuum extremals defined by surfaces in $M^4 \times Y^2$, $Y^2$ a Lagrange sub-manifold of $CP_2$, are trivially hyper-quaternionic surfaces. The modified definition of hyper-quaternionicity does not affect in any manner $M^8 \leftrightarrow M^4 \times CP_2$ duality allowing purely number theoretic interpretation of standard model symmetries.

A side comment not strictly related to hyper-quaternionicity is in order. The anticommutators of the modified gamma matrices define an effective Riemann metric and one can assign to it the counterparts of Riemann connection, curvature tensor, geodesic line, volume, etc... One would have two different metrics associated with the space-time surface. Only if the action defining space-time surface is identified as the volume in the ordinary metric, these metrics are equivalent. The index raising for the effective metric could be defined also by the induced metric and it is not clear whether one can define Riemann connection also in this case. Could this effective metric have concrete physical significance and play a deeper role in quantum TGD? For instance, AdS-CFT duality leads to ask whether interactions be coded in terms of the gravitation associated with the effective metric.

### 4.4.3 Minimal form of $M^8 - H$ duality

The basic problem in the construction of quantum TGD has been the identification of the preferred extremals of Kähler action playing a key role in the definition of the theory. The most elegant manner to do this is by fixing the 4-D tangent space $T(X^4(X^3_l))$ of $X^4(X^3_l)$ at each point of $X^3_l$ so that the boundary value problem is well defined. What I called number theoretical compactification allows to achieve just this although I did not fully realize this in the original vision. The minimal picture is following.

1. The basic observations are following. Let $M^8$ be endowed with hyper-octonionic structure. For hyper-quaternionic space-time surfaces in $M^8$ tangent spaces are by definition hyper-quaternionic. If they contain a preferred plane $M^2 \subset M^4 \subset M^8$ in their tangent space, they can be mapped to 4-surfaces in $M^4 \times CP_2$. The reason is that the hyper-quaternionic planes containing preferred the hyper-complex plane $M^2$ of $M_\pm \subset M^2$ are parameterized by points of $CP_2$. The map is simply $(m, e) \to (m, s(m, e))$, where $m$ is point of $M^4$, $e$ is point of $E^4$, and $s(m, 2)$ is point of $CP_2$ representing the hyperquaternionic plane. The inverse map assigns to each point $(m, s)$ in $M^4 \times CP_2$ point $m$ of $M^4$, undetermined point $e$ of $E^4$ and 4-D plane. The requirement that the distribution of planes containing the preferred $M^2$ or $M_\pm$ corresponds to a distribution of planes for 4-D surface is expected to fix the points $e$. The physical interpretation of $M^2$ is in terms of plane of non-physical polarizations so that gauge conditions have purely number theoretical interpretation.

2. In principle, the condition that $T(X^4)$ contains $M^2$ can be replaced with a weaker condition that either of the two light-like vectors of $M^2$ is contained in it since already this condition assigns to $T(X^4)$ $M^2$ and the map $H \to M^8$ becomes possible. Only this weaker form applies in the case of massless extremals [K5] as will be found.

3. The original idea was that hyper-quaternionic 4-surfaces in $M^8$ containing $M^2 \subset M^4$ in their tangent space could correspond to preferred extremals of Kähler action. This condition does not seem to be consistent with what is known about the extremals of Kähler action. The weaker form of the hypothesis is that hyper-quaternionicity holds only for 4-D tangent spaces of $X^3_l \subset H = M^4 \times CP_2$ identified as wormhole throats or boundary components lifted to 3-surfaces in 8-D tangent space $M^8$ of $H$. The minimal hypothesis would be that only $T(X^4(X^3_l))$ at $X^3_l$ is associative that is hyper-quaternionic for fixed $M^2$. $X^3_l \subset M^8$ and $T(X^4(X^3_l))$ at $X^3_l$ can be mapped to $X^3_l \subset H$ if tangent space contains also $M_\pm \subset M^2$ or $M^2 \subset M^4 \subset M^8$ itself having interpretation as preferred hyper-complex plane. This condition is not satisfied by all surfaces $X^3_l$ as is clear from the fact that the inverse map involves local $E^4$ translation. The requirements that the distribution of hyper-quaternionic planes containing $M^2$ corresponds to a distribution of 4-D tangent planes should fix the $E^4$ translation to a high degree.

4. A natural requirement is that the image of $X_l^3 \subset H$ in $M^8$ is light-like. The condition that the determinant of induced metric vanishes gives an additional condition reducing the number of free parameters by one. This condition cannot be formulated as a condition on $CP_2$ coordinate characterizing the hyper-quaternionic plane. Since $M^4$ projections are same for the two representations, this condition is satisfied if the contributions from $CP_2$ and $E^4$ and projections to the induced metric are identical: $s_{kl}\partial_\alpha s^k \partial_\beta s^l = e_{kl}\partial_\alpha e^k \partial_\beta e^l$. This condition means that only a subset of light-like surfaces of $M^8$ are realized physically. One might argue that this is as it must be since the volume of $E^4$ is infinite and that of $CP_2$ finite: only an infinitesimal portion of all possible light-like 3-surfaces in $M^8$ can can have $H$ counterparts. The conclusion would be that number theoretical compactification is 4-D isometry between $X^4 \subset H$ and $X^4 \subset M^8$ at $X_l^3$. This unproven conjecture is unavoidable.

5. $M^2 \subset T(X^4(X_l^3))$ condition fixes $T(X^4(X_l^3))$ in the generic case by extending the tangent space of $X_l^3$, and the construction of configuration space spinor structure fixes boundary conditions completely by additional conditions necessary when $X_l^3$ corresponds to a light-like 3 surfaces defining wormhole throat at which the signature of induced metric changes. What is especially beautiful that only the data in $T(X^4(X_l^3))$ at $X_l^3$ is needed to calculate the vacuum functional of the theory as Dirac determinant: the only remaining conjecture (strictly speaking un-necessary but realistic looking) is that this determinant gives exponent of Kähler action for the preferred extremal and there are excellent hopes for this by the structure of the basic construction.

The basic criticism relates to the condition that light-like 3-surfaces are mapped to light-like 3-surfaces guaranteed by the condition that $M^8 - H$ duality is isometry at $X_l^3$.

### 4.4.4 Strong form of $M^8 - H$ duality

The proposed picture is the minimal one. One can of course ask whether the original much stronger conjecture that the preferred extrema of Kähler action correspond to hyper-quaternionic surfaces could make sense in some form. One can also wonder whether one could allow the choice of the plane $M^2$ of non-physical polarization to be local so that one would have $M^2(x) \subset M^4 \subset M^4 \times E^4$, where $M^4$ is fixed hyper-quaternionic sub-space of $M^8$ and identifiable as $M^4$ factor of $H$.

1. If $M^2$ is same for all points of $X_l^3$, the inverse map $X_l^3 \subset H \to X_l^3 \subset M^8$ is fixed apart from possible non-uniquencess related to the local translation in $E^4$ from the condition that hyper-quaternionic planes represent light-like tangent 4-planes of light-like 3-surfaces. The question is whether not only $X_l^3$ but entire four-surface $X^4(X_l^3)$ could be mapped to the tangent space of $M^8$. By selecting suitably the local $E^4$ translation one might hope of achieving the achieving this. The conjecture would be that the preferred extrema of Kähler action are those for which the distribution integrates to a distribution of tangent planes.

2. There is however a problem. What is known about extremals of Kähler action is not consistent with the assumption that fixed $M^2$ of $M_\pm \subset M^2$is contained in the tangent space of $X^4$. This suggests that one should relax the condition that $M^2 \subset M^4 \subset M^8$ is a fixed hyper-complex plane associated with the tangent space or normal space $X^4$ and allow $M^2$ to vary from point to point so that one would have $M^2 = M^2(x)$. In $M^8 \to H$ direction the justification comes from the observation (to be discussed below) that it is possible to uniquely fix the convention assigning $CP_2$ point to a hyper-quaternionic plane containing varying hyper-complex plane $M^2(x) \subset M^4$.

   Number theoretic compactification fixes naturally $M^4 \subset M^8$ so that it applies to any $M^2(x) \subset M^4$. Under this condition the selection is parameterized by an element of $SO(3)/SO(2) = S^2$. Note that $M^4$ projection of $X^4$ would be at least 2-dimensional in hyper-quaternionic case. In co-hyper-quaternionic case $E^4$ projection would be at least 2-D. $SO(2)$ would act as a number theoretic gauge symmetry and the $SO(3)$ valued chiral field would approach to constant at $X_l^3$ invariant under global $SO(2)$ in the case that one keeps the assumption that $M^2$ is fixed ad $X_l^3$.

3. This picture requires a generalization of the map assigning to hyper-quaternionic plane a point of $CP_2$ so that this map is defined for all possible choices of $M^2 \subset M^4$. Since the $SO(3)$ rotation of the hyper-quaternionic unit defining $M^2$ rotates different choices parameterized by $S^2$ to each other, a natural assumption is that the hyper-quaternionic planes related by $SO(3)$ rotation

correspond to the same point of $CP_2$. Denoting by $M^2$ the standard representative of $M^2$, this means that for the map $M^8 \to H$ one must perform $SO(3)$ rotation of hyper-quaternionic plane taking $M^2(x)$ to $M^2$ and map the rotated plane to $CP_2$ point. In $M^8 \to H$ case one must first map the point of $CP_2$ to hyper-quaternionic plane and rotate this plane by a rotation taking $M^2(x)$ to $M^2$.

4. In this framework local $M^2$ can vary also at the surfaces $X_l^3$, which considerably relaxes the boundary conditions at wormhole throats and light-like boundaries and allows much more general variety of light-like 3-surfaces since the basic requirement is that $M^4$ projection is at least 1-dimensional. The physical interpretation would be that a local choice of the plane of non-physical polarizations is possible everywhere in $X^4(X_l^3)$. This does not seem to be in any obvious conflict with physical intuition.

These observation provide support for the conjecture that (classical) $S^2 = SO(3)/SO(2)$ conformal field theory might be relevant for (classical) TGD.

1. General coordinate invariance suggests that the theory should allow a formulation using any light-like 3-surface $X^3$ inside $X^4(X_l^3)$ besides $X_l^3$ identified as union of wormhole throats and boundary components. For these surfaces the element $g(x) \in SO(3)$ would vary also at partonic 2-surfaces $X^2$ defined as intersections of $\delta CD \times CP_2$ and $X^3$ (here $CD$ denotes causal diamond defined as intersection of future and past directed light-cones). Hence one could have $S^2 = SO(3)/SO(2)$ conformal field theory at $X^2$ (regarded as quantum fluctuating so that also $g(x)$ varies) generalizing to WZW model for light-like surfaces $X^3$.

2. The presence of $E^4$ factor would extend this theory to a classical $E^4 \times S^2$ WZW model bringing in mind string model with 6-D Euclidian target space extended to a model of light-like 3-surfaces. A further extension to $X^4$ would be needed to integrate the WZW models associated with 3-surfaces to a full 4-D description. General Coordinate Invariance however suggests that $X_l^3$ description is enough for practical purposes.

3. The choices of $M^2(x)$ in the interior of $X_l^3$ is dictated by dynamics and the first optimistic conjecture is that a classical solution of $SO(3)/SO(2)$ Wess-Zumino-Witten model obtained by coupling $SO(3)$ valued field to a covariantly constant $SO(2)$ gauge potential characterizes the choice of $M^2(x)$ in the interior of $M^8 \supset X^4(X_l^3) \subset H$ and thus also partially the structure of the preferred extremal. Second optimistic conjecture is that the Kähler action involving also $E^4$ degrees of freedom allows to assign light-like 3-surface to light-like 3-surface.

4. The best that one can hope is that $M^8 - H$ duality could allow to transform the extremely non-linear classical dynamics of TGD to a generalization of WZW-type model. The basic problem is to understand how to characterize the dynamics of $CP_2$ projection at each point.

In $H$ picture there are two basic types of vacuum extremals: $CP_2$ type extremals representing elementary particles and vacuum extremals having $CP_2$ projection which is at most 2-dimensional Lagrange manifold and representing say hadron. Vacuum extremals can appear only as limiting cases of preferred extremals which are non-vacuum extremals. Since vacuum extremals have so decisive role in TGD, it is natural to requires that this notion makes sense also in $M^8$ picture. In particular, the notion of vacuum extremal makes sense in $M^8$.

This requires that Kähler form exist in $M^8$. $E^4$ indeed allows full $S^2$ of covariantly constant Kähler forms representing quaternionic imaginary units so that one can identify Kähler form and construct Kähler action. The obvious conjecture is that hyper-quaternionic space-time surface is extremal of this Kähler action and that the values of Kähler actions in $M^8$ and $H$ are identical. The elegant manner to achieve this, as well as the mapping of vacuum extremals to vacuum extremals and the mapping of light-like 3-surfaces to light-like 3-surfaces is to assume that $M^8 - H$ duality is Kähler isometry so that induced Kähler forms are identical.

This picture contains many speculative elements and some words of warning are in order.

1. Light-likeness conjecture would boil down to the hypothesis that $M^8 - H$ correspondence is Kähler isometry so that the metric and Kähler form of $X^4$ induced from $M^8$ and $H$ would be identical. This would guarantee also that Kähler actions for the preferred extremal are identical. This conjecture is beautiful but strong.

2. The slicing of $X^4(X_l^3)$ by light-like 3-surfaces is very strong condition on the classical dynamics of Kähler action and does not make sense for pieces of $CP_2$ type vacuum extremals.

#### Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative

The 8-dimensionality of $M^8$ allows to consider both associativity (hyper-quaternionicity) of the tangent space and associativity of the normal space- let us call this co-assosiativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, $k$ positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as $CP_2$ type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the $CP_2$ type extremal and $CP_2$ size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \to k$ duality.

#### Are the known extremals of Kähler action consistent with the strong form of $M^8 - H$ duality

It is interesting to check whether the known extremals of Kähler action [K5] are consistent with strong form of $M^8 - H$ duality assuming that $M^2$ or its light-like ray is contained in $T(X^4)$ or normal space.

1. $CP_2$ type vacuum extremals correspond cannot be hyper-quaternionic surfaces but co-hyper-quaternionicity is natural for them. In the same manner canonically imbedded $M^4$ can be only hyper-quaternionic.

2. String like objects are associative since tangent space obviously contains $M^2(x)$. Objects of form $M^1 \times X^3 \subset M^4 \times CP_2$ do not have $M^2$ either in their tangent space or normal space in $H$. So that the map from $H \to M^8$ is not well defined. There are no known extremals of Kähler action of this type. The replacement of $M^1$ random light-like curve however gives vacuum extremal with vanishing volume, which need not mean physical triviality since fundamental objects of the theory are light-like 3-surfaces.

3. For canonically imbedded $CP_2$ the assignment of $M^2(x)$ to normal space is possible but the choice of $M^2(x) \subset N(CP_2)$ is completely arbitrary. For a generic $CP_2$ type vacuum extremals $M^4$ projection is a random light-like curve in $M^4 = M^1 \times E^3$ and $M^2(x)$ can be defined uniquely by the normal vector $n \in E^3$ for the local plane defined by the tangent vector $dx^\mu/dt$ and acceleration vector $d^2x^\mu/dt^2$ assignable to the orbit.

4. Consider next massless extremals. Let us fix the coordinates of $X^4$ as $(t,z,x,y) = (m^0, m^2, m^1, m^2)$. For simplest massless extremals $CP_2$ coordinates are arbitrary functions of variables $u = k \cdot m = t - z$ and $v = \epsilon \cdot m = x$, where $k = (1,1,0,0)$ is light-like vector of $M^4$ and $\epsilon = (0,0,1,0)$ a polarization vector orthogonal to it. Obviously, the extremals defines a decomposition $M^4 = M^2 \times E^2$. Tangent space is spanned by the four $H$-vectors $\nabla_\alpha h^k$ with $M^4$ part given by $\nabla_\alpha m^k = \delta^k_\alpha$ and $CP_2$ part by $\nabla_\alpha s^k = \partial_u s^k k_\alpha + \partial_v s^k \epsilon_\alpha$.

   The normal space cannot contain $M^4$ vectors since the $M^4$ projection of the extremal is $M^4$. To realize hyper-quaternionic representation one should be able to from these vector two vectors of $M^2$, which means linear combinations of tangent vectors for which $CP_2$ part vanishes. The vector $\partial_t h^k - \partial_z h^k$ has vanishing $CP_2$ part and corresponds to $M^4$ vector $(1,-1,0,0)$ fix assigns to each point the plane $M^2$. To obtain $M^2$ one would need $(1,1,0,0)$ too but this is not possible. The vector $\partial_y h^k$ is $M^4$ vector orthogonal to $\epsilon$ but $M^2$ would require also $(1,0,0,0)$. The proposed generalization of massless extremals allows the light-like line $M_\pm$ to depend on point of $M^4$ [K5] , and leads to the introduction of Hamilton-Jacobi coordinates involving a local decomposition of $M^4$ to $M^2(x)$ and its orthogonal complement with light-like coordinate lines having interpretation as curved light rays. $M^2(x) \subset T(X^4)$ assumption fails fails also for vacuum extremals of form $X^1 \times X^3 \subset M^4 \times CP_2$, where $X^1$ is light-like random curve. In the latter case, vacuum property follows from the vanishing of the determinant of the induced metric.

5. The deformations of string like objects to magnetic flux quanta are basic conjectural extremals of Kähler action and the proposed picture supports this conjecture. In hyper-quaternionic case the assumption that local 4-D plane of $X^3$ defined by modified gamma matrices contains $M^2(x)$ but that $T(X^3)$ does not contain it, is very strong. It states that $T(X^4)$ at each point can be regarded as a product $M^2(x) \times T^2$, $T^2 \subset T(CP_2)$, so that hyper-quaternionic $X^4$ would be a collection of Cartesian products of infinitesimal 2-D planes $M^2(x) \subset M^4$ and $T^2(x) \subset CP_2$. The extremals in question could be seen as local variants of string like objects $X^2 \times Y^2 \subset M^4 \times CP_2$, where $X^2$ is minimal surface and $Y^2$ holomorphic surface of $CP_2$. One can say that $X^2$ is replaced by a collection of infinitesimal pieces of $M^2(x)$ and $Y^2$ with similar pieces of homologically non-trivial geodesic sphere $S^2(x)$ of $CP_2$, and the Cartesian products of these pieces are glued together to form a continuous surface defining an extremal of Kähler action. Field equations would pose conditions on how $M^2(x)$ and $S^2(x)$ can depend on $x$. This description applies to magnetic flux quanta, which are the most important must-be extremals of Kähler action.

#### Geometric interpretation of strong $M^8 - H$ duality

In the proposed framework $M^8 - H$ duality would have a purely geometric meaning and there would nothing magical in it.

1. $X^4(X^3_l) \subset H$ could be seen a curve representing the orbit of a light-like 3-surface defining a 4-D surface. The question is how to determine the notion of tangent vector for the orbit of $X^3_l$. Intuitively tangent vector is a one-dimensional arrow tangential to the curve at point $X^3_l$. The identification of the hyper-quaternionic surface $X^4(X^3_l) \subset M^8$ as tangent vector conforms with this intuition.

2. One could argue that $M^8$ representation of space-time surface is kind of chart of the real space-time surface obtained by replacing real curve by its tangent line. If so, one cannot avoid the question under which conditions this kind of chart is faithful. An alternative interpretation is that a representation making possible to realize number theoretical universality is in question.

3. An interesting question is whether $X^4(X^3_l)$ as orbit of light-like 3-surface is analogous to a geodesic line -possibly light-like- so that its tangent vector would be parallel translated in the sense that $X^4(X^3)$ for any light-like surface at the orbit is same as $X^4(X^3_l)$. This would give justification for the possibility to interpret space-time surfaces as a geodesic of configuration space: this is one of the first -and practically forgotten- speculations inspired by the construction of configuration space geometry. The light-likeness of the geodesic could correspond at the level of $X^4$ the possibility to decompose the tangent space to a direct sum of two light-like spaces and 2-D transversal space producing the foliation of $X^4$ to light-like 3-surfaces $X^3_l$ along light-like curves.

4. $M^8 - H$ duality would assign to $X^3_l$ classical orbit and its tangent vector at $X^3_l$ as a generalization of Bohr orbit. This picture differs from the wave particle duality of wave mechanics stating that once the position of particle is known its momentum is completely unknown. The outcome is however the same: for $X^3_l$ corresponding to wormhole throats and light-like boundaries of $X^4$, canonical momentum densities in the normal direction vanish identically by conservation laws and one can say that the the analog of $(q,p)$ phase space as the space carrying wave functions is replaced with the analog of subspace consisting of points $(q,0)$. The dual description in $M^8$ would not be analogous to wave functions in momentum space space but to those in the space of unique tangents of curves at their initial points.

#### The Kähler and spinor structures of $M^8$

If one introduces $M^8$ as dual of $H$, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in $H$ are also extremals of $M^8$ Kähler action with same value of Kähler action. As found, this leads to the conclusion that the$M^8 - H$ duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in $H$ should have full $M^8$ dual.

There are strong physical constraints on $M^8$ dual and they could kill the hypothesis. The basic constraint to the spinor structure of $M^8$ is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different $H$-chiralities and parity breaking.

1. By the flatness of the metric of $E^4$ its spinor connection is trivial. $E^4$ however allows full $S^2$ of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units.

2. One should be able to distinguish between quarks and leptons also in $M^8$, which suggests that one introduce spinor structure and Kähler structure in $E^4$. The Kähler structure of $E^4$ is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of $S^2$ representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of $H$.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and $Z^0$ contains both axial and vector parts. The free Kähler forms could thus allow to produce $M^8$ counterparts of these gauge potentials possessing same couplings as their $H$ counterparts. This picture would produce parity breaking in $M^8$ picture correctly.

4. Only the charged parts of classical electro-weak gauge fields would be absent. This would conform with the standard thinking that charged classical fields are not important. The predicted classical $W$ fields is one of the basic distinctions between TGD and standard model and in this framework. A further prediction is that this distinction becomes visible only in situations, where $H$ picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of $E^4$ partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

5. Also super-symmetries of quantum TGD crucial for the construction of configuration space geometry force this picture. In the absence of coupling to Kähler gauge potential all constant spinor fields and their conjugates would generate super-symmetries so that $M^8$ would allow $N =$ 8 super-symmetry. The introduction of the coupling to Kähler gauge potential in turn means that all covariantly constant spinor fields are lost. Only the representation of all three neutral parts of electro-weak gauge potentials in terms of three independent Kähler gauge potentials allows right-handed neutrino as the only super-symmetry generator as in the case of $H$.

6. The $SO(3)$ element characterizing $M^2(x)$ is fixed apart from a local $SO(2)$ transformation, which suggests an additional $U(1)$ gauge field associated with $SO(2)$ gauge invariance and representable as Kähler form corresponding to a quaternionic unit of $E^4$. A possible identification of this gauge field would be as a part of electro-weak gauge field.

### $M^8$ dual of configuration space geometry and spinor structure?

If one introduces $M^8$ spinor structure and preferred extremals of $M^8$ Kähler action, one cannot avoid the question whether it is possible or useful to formulate the notion of configuration space geometry and spinor structure for light-like 3-surfaces in $M^8$ using the exponent of Kähler action as vacuum functional.

1. The isometries of the configuration space in $M^8$ and $H$ formulations would correspond to symplectic transformation of $\delta M^4_\pm \times E^4$ and $\delta M^4_\pm \times CP_2$ and the Hamiltonians involved would belong to the representations of $SO(4)$ and $SU(3)$ with 2-dimensional Cartan sub-algebras. In $H$ picture color group would be the familiar $SU(3)$ but in $M^8$ picture it would be $SO(4)$.

Color confinement in both $SU(3)$ and $SO(4)$ sense could allow these two pictures without any inconsistency.

2. For $M^4 \times CP_2$ the two spin states of covariantly constant right handed neutrino and antineutrino spinors generate super-symmetries. This super-symmetry plays an important role in the proposed construction of configuration space geometry. As found, this symmetry would be present also in $M^8$ formulation so that the construction of $M^8$ geometry should reduce more or less to the replacement of $CP_2$ Hamiltonians in representations of $SU(3)$ with $E^4$ Hamiltonians in representations of $SO(4)$. These Hamiltonians can be taken to be proportional to functions of $E^4$ radius which is $SO(4)$ invariant and these functions bring in additional degree of freedom.

3. The construction of Dirac determinant identified as a vacuum functional can be done also in $M^8$ picture and the conjecture is that the result is same as in the case of $H$. In this framework the construction is much simpler due to the flatness of $E^4$. In particular, the generalized eigen modes of the Dirac operator $D_K(Y_l^3)$ restricted to the $X_l^3$ correspond to a situation in which one has fermion in induced Maxwell field mimicking the neutral part of electro-weak gauge field in $H$ as far as couplings are considered. Induced Kähler field would be same as in $H$. Eigen modes are localized to regions inside which the Kähler magnetic field is non-vanishing and apart from the fact that the metric is the effective metric defined in terms of canonical momentum densities via the formula $\hat{\Gamma}^\alpha = \partial L_K/\partial h^k_\alpha \Gamma_k$ for effective gamma matrices. This in fact, forces the localization of modes implying that their number is finite so that Dirac determinant is a product over finite number eigenvalues. It is clear that $M^8$ picture could dramatically simplify the construction of configuration space geometry.

4. The eigenvalue spectra of the transversal parts of $D_K$ operators in $M^8$ and $H$ should identical. This motivates the question whether it is possible to achieve a complete correspondence between $H$ and $M^8$ pictures also at the level of spinor fields at $X^3$ by performing a gauge transformation eliminating the classical $W$ gauge boson field altogether at $X_l^3$ and whether this allows to transform the modified Dirac equation in $H$ to that in $M^8$ when restricted to $X_l^3$. That something like this might be achieved is supported by the fact that in Coulombic gauge the component of gauge potential in the light-like direction vanishes so that the situation is effectively 2-dimensional and holonomy group is Abelian.

### Why $M^8 - H$ duality is useful?

Skeptic could of course argue that $M^8 - H$ duality produces only an inflation of unproven conjectures. There are however strong reasons for $M^8 - H$ duality: both theoretical and physical.

1. The map of $X_l^3 \subset H \to X_l^3 \subset M^8$ and corresponding map of space-time surfaces would allow to realize number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as natural coordinates in which one can say what it means that the point of imbedding space is rational/algebraic. The point of $X^4 \subset H$ is algebraic if it is mapped to an algebraic point of $M^8$ in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not have practical meaning. Number theoretical compactication could in fact be motivated by the number theoretical universality.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action since the Kähler form in $E^4$ has constant components. If the spinor connection in $E^4$ is combination of the three Kähler forms mimicking neutral part of electro-weak gauge potential, the eigenvalue spectrum for the modified Dirac operator would correspond to that for a fermion in $U(1)$ magnetic field defined by an Abelian magnetic field whereas in $M^4 \times CP_2$ picture $U(2)_{ew}$ magnetic fields would be present.

3. $M^8 - H$ duality provides insights to low energy hadron physics. $M^8$ description might work when $H$-description fails. For instance, perturbative QCD which corresponds to $H$-description fails at low energies whereas $M^8$ description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely

also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8-H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of $E^4$ spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in $CP_2$. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

### 4.4.5 $M^8 - H$ duality and low energy hadron physics

The description of $M^8 - H$ at the configuration space level can be applied to gain a view about color confinement and its dual for electro-weak interactions at short distance limit. The basic idea is that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quark color using $E^4$ and $CP_2$ partial waves and low energy hadron physics corresponds to a situation in which $M^8$ picture provides the perturbative approach whereas $H$ picture works at high energies. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since configuration space degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the $E^4$ Hamiltonians in $M^8$ picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of $E^4$ valued vector field or equivalently collection of four $E^4$ Hamiltonians corresponding to spherical $E^4$ coordinates. Pion corresponds to $S^3$ valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the $E^4$ radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks $E^4$ partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be be important whereas at higher energies higher partial waves would be excited and the description based on $CP_2$ partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [K35] .

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

### 4.4.6 The notion of number theoretical braid

Braids -not necessary number theoretical- provide a realization discretization as a space-time correlate for the finite measurement resolution. The notion of braid was inspired by the idea about quantum TGD as almost topological quantum field theory. Although the original form of this idea has been buried, the notion of braid has survived: in the decomposition of space-time sheets to string world sheets, the ends of strings define representatives for braid strands at light-like 3-surfaces.

The notion of number theoretic universality inspired the much more restrictive notion of number theoretic braid requiring that the points in the intersection of the braid with the partonic 2-surface correspond to rational or at most algebraic points of $H$ in preferred coordinates fixed by symmetry considerations. The challenge has been to find a unique identification of the number theoretic braid or at least of the end points of the braid. The following consideration suggest that the number theoretic braids are not a useful notion in the generic case but make sense and are needed in the intersection of real and p-adic worlds which is in crucial role in TGD based vision about living matter [K31] .

It is only the braiding that matters in topological quantum field theories used to classify braids. Hence braid should require only the fixing of the end points of the braids at the intersection of the braid at the light-like boundaries of $CD$s and the braiding equivalence class of the braid itself. Therefore it is enough is to specify the topology of the braid and the end points of the braid in accordance with the attribute "number theoretic". Of course, the condition that all points of the strand of the number theoretic braid are algebraic is impossible to satisfy.

The situation in which the equations defining $X^2$ make sense both in real sense and p-adic sense using appropriate algebraic extension of p-adic number field is central in the TGD based vision about living matter [K31] . The reason is that in this case the notion of number entanglement theoretic entropy having negative values makes sense and entanglement becomes information carrying. This motivates the identification of life as something in the intersection of real and p-adic worlds. In this situation the identification of the ends of the number theoretic braid as points belonging to the intersection of real and p-adic worlds is natural. These points -call them briefly algebraic points- belong to the algebraic extension of rationals needed to define the algebraic extension of p-adic numbers. This definition however makes sense also when the equations defining the partonic 2-surfaces fail to make sense in both real and p-adic sense. In the generic case the set of points satisfying the conditions is discrete. For instance, according to Fermat's theorem the set of rational points satisfying $X^n + Y^n = Z^n$ reduces to the point $(0, 0, 0)$ for $n = 3, 4, ....$ Hence the constraint might be quite enough in the intersection of real and p-adic worlds where the choice of the algebraic extension is unique.

One can however criticize this proposal.

1. One must fix the the number of points of the braid and outside the intersection and the non-uniquencess of the algebraic extension makes the situation problematic. Physical intuition suggests that the points of braid define carriers of quantum numbers assignable to second quantized induced spinor fields so that the total number of fermions antifermions would define the number of braids. In the intersection the highly non-trivial implication is that this number cannot exceed the number of algebraic points.

2. In the generic case one expects that even the smallest deformation of the partonic 2-surface can change the number of algebraic points and also the character of the algebraic extension of rational numbers needed. The restriction to rational points is not expected to help in the generic case. If the notion of number theoretical braid is meant to be practical, must be able to decompose WCW to open sets inside which the numbers of algebraic points of braid at its ends are constant. For real topology this is expected to be impossible and it does not make sense to use p-adic topology for WCW whose points do not allow interpretation as p-adic partonic surfaces.

3. In the intersection of real and p-adic worlds which corresponds to a discrete subset of WCW, the situation is different. Since the coefficients of polynomials involved with the definition of the partonic 2-surface must be rational or at most algebraic, continuous deformations are not possible so that one avoids the problem.

4. This forces to ask the reason why for the number theoretic braids. In the generic case they seem to produce only troubles. In the intersection of real and p-adic worlds they could however allow the construction of the elements of $M$-matrix describing quantum transitions changing p-adic to real surfaces and vice versa as realizations of intentions and generation of cognitions. In this the case it is natural that only the data from the intersection of the two worlds are used. In [K31] I have sketched the idea about number theoretic quantum field theory as a description of intentional action and cognition.

There is also the the problem of fixing the interior points of the braid modulo deformations not affecting the topology of the braid.

1. Infinite number of non-equivalent braidings are possible. Should one allow all possible braidings for a fixed light-like 3-surface and say that their existence is what makes the dynamics essentially three-dimensional even in the topological sense? In this case there would be no problems with the condition that the points at both ends of braid are algebraic.

2. Or should one try to characterize the braiding uniquely for a given partonic 2-surfaces and corresponding 4-D tangent space distributions? The slicing of the space-time sheet by partonic 2-surfaces and string word sheets suggests that the ends of string world sheets could define the braid strands in the generic context when there is no algebraicity condition involved. This could be taken as a very natural manner to fix the topology of braid but leave the freedom to choose the representative for the braid. In the intersection of real and p-adic worlds there is no good reason for the end points of strands in this case to be algebraic at both ends of the string world sheet. One can however start from the braid defined by the end points of string world sheets, restrict the end points to be algebraic at the end with a smaller number of algebraic points and and then perform a topologically non-trivial deformation of the braid so that also the points at the other end are algebraic? Non-trivial deformations need not be possible for all possible choices of algebraic braid points at the other end of braid and different choices of the set of algebraic points would give rise to different braidings. A further constraint is that only the algebraic points at which one has assign fermion or antifermion are used so that the number of braid points is not always maximal.

3. One can also ask whether one should perform the gauge fixing for the strands of the number theoretic braid using algebraic functions making sense both in real and p-adic context. This question does not seem terribly relevant since since it is only the topology of the braid that matters.

### 4.4.7 Connection with string model and Equivalence Principle at space-time level

Coset construction allows to generalize Equivalence Principle and understand it at quantum level. This is however not quite enough: a precise understanding of Equivalence Principle is required also at the classical level. Also the mechanism selecting via stationary phase approximation a preferred extremal of Kähler action providing a correlation between quantum numbers of the particle and geometry of the preferred extremals is still poorly understood.

#### Is stringy action principle coded by the geometry of preferred extremals?

It seems very difficult to deduce Equivalence Principle as an identity of gravitational and inertial masses identified as Noether charges associated with corresponding action principles. Since string model is an excellent theory of quantum gravitation, one can consider a less direct approach in which one tries to deduce a connection between classical TGD and string model and hope that the bridge from string model to General Relativity is easier to build. Number theoretical compactification gives good hopes that this kind of connection exists.

1. Number theoretic compactification implies that the preferred extremals of Kähler action have the property that one can assign to each point of $M^4$ projection $P_{M^4}(X^4(X^3_l))$ of the preferred extremal $M^2(x)$ identified as the plane of non-physical polarizations and also as the plane in which local massless four-momentum lies.

2. If the distribution of the planes $M^2(x)$ is integrable, one can slice $P_{M^4}(X^4(X^3_l))$ to string world-sheets. The intersection of string world sheets with $X^3 \subset \delta M^4_\pm \times CP_2$ corresponds to a light-like curve having tangent in local tangent space $M^2(x)$ at light-cone boundary. This is the first candidate for the definition of number theoretic braid. Second definition assumes $M^2$ to be fixed at $\delta CD$: in this case the slicing is parameterized by the sphere $S^2$ defined by the light rays of $\delta M^4_\pm$.

3. One can assign to the string world sheet -call it $Y^2$ - the standard area action

$$S_G(Y^2) = \int_{Y^2} T\sqrt{g_2}d^2y \ , \quad (4.4.1)$$

where $g_2$ is either the induced metric or only its $M^4$ part. The latter option looks more natural since $M^4$ projection is considered. $T$ is string tension.

4. The naivest guess would be $T = 1/\hbar G$ apart from some numerical constant but one must be very cautious here since $T = 1/L_p^2$ apart from a numerical constant is also a good candidate if one accepts the basic argument identifying $G$ in terms of p-adic length $L_p$ and Kähler action for two pieces of $CP_2$ type vacuum extremals representing propagating graviton. The formula reads $G = L_p^2 exp(-2aS_K(CP_2))$, $a \leq 1$ [K3, K18] . The interaction strength which would be $L_p^2$ without the presence of $CP_2$ type vacuum extremals is reduced by the exponential factor coming from the exponent of Kähler function of configuration space.

5. One would have string model in either $CD \times CP_2$ or $CD \subset M^4$ with the constraint that stringy world sheet belongs to $X^4(X_l^3)$. For the extremals of $S_G(Y^2)$ gravitational four-momentum defined as Noether charge is conserved. The extremal property of string world sheet need not however be consistent with the preferred extremal property. This constraint might bring in coupling of gravitons to matter. The natural guess is that graviton corresponds to a string connecting wormhole contacts. The strings could also represent formation of gravitational bound states when they connect wormhole contacts separated by a large distance. The energy of the string is roughly $E \sim \hbar TL$ and for $T = 1/\hbar G$ gives $E \sim L/G$. Macroscopic strings are not allowed except as models of black holes. The identification $T \sim 1/L_p^2$ gives $E \sim \hbar L/L_p^2$, which does not favor long strings for large values of $\hbar$. The identification $G_p = L_p^2/\hbar_0$ gives $T = 1/\hbar G_p$ and $E \sim \hbar_0 L/L_p^2$, which makes sense and allows strings with length not much longer than p-adic length scale. Quantization - that is the presence of configuration space degrees of freedom- would bring in massless gravitons as deformations of string whereas strings would carry the gravitational mass.

6. The exponent $exp(iS_G)$ can appear as a phase factor in the definition of quantum states for preferred extremals. $S_G$ is not however enough. One can assign also to the points of number theoretic braid action describing the interaction of a point like current $Qdx^\mu/ds$ with induced gauge potentials $A_\mu$. The corresponding contribution to the action is

$$S_{braid} = \int_{braid} iTr(Q\frac{dx^\mu}{ds}A_\mu)dx \ . \quad (4.4.2)$$

In stationary phase approximation subject to the additional constraint that a preferred extremal of Kähler action is in question one obtains the desired correlation between the geometry of preferred extremal and the quantum numbers of elementary particle. This interaction term carries information only about the charges of elementary particle. It is quite possible that the interaction term is more complex: for instance, it could contain spin dependent terms (Stern-Gerlach experiment).

7. The constraint coming from preferred extremal property of Kähler action can be expressed in terms of Lagrange multipliers

$$S_c = \int_{Y^2} \lambda^k D_\alpha(\frac{\partial L_K}{\partial_\alpha h^k})\sqrt{g_2}d^2y \ . \quad (4.4.3)$$

8. The action exponential reads as

$$exp(iS_G + S_{braid} + S_c) \ . \tag{4.4.4}$$

The resulting field equations couple stringy $M^4$ degrees of freedom to the second variation of Kähler action with respect to $M^4$ coordinates and involve third derivatives of $M^4$ coordinates at the right hand side. If the second variation of Kähler action with respect to $M^4$ coordinates vanishes, free string results. This is trivially the case if a vacuum extremal of Kähler action is in question.

9. An interesting question is whether the preferred extremal property boils down to the condition that the second variation of Kähler action with respect to $M^4$ coordinates or actually all coordinates vanishes so that gravitonic string is free. As a matter fact, the stronger condition is required that the Noether currents associated with the modified Dirac action are conserved. The physical interpretation would be in terms of quantum criticality which is the basic conjecture about the dynamics of quantum TGD. This is clear from the fact that in 1-D system criticality means that the potential $V(x) = ax+bx^2+..$ has $b = 0$. In field theory criticality corresponds to the vanishing of the term $m^2\phi^2/2$ so that massless situation corresponds to massless theory and criticality and long range correlations. For more than one dynamical variable there is a hierarchy of criticalities corresponding to the gradual reduction of the rank of the matrix of the matrix defined by the second derivatives of $V(x)$ and this gives rise to a classification of criticalities. Maximum criticality would correspond to the total vanishing of this matrix. In infinite-D case this hierarchy is infinite.

**What does the equality of gravitational and inertial masses mean?**

Consider next the question in what form Equivalence Principle could be realized in this framework.

1. Coset construction inspires the conjecture that gravitational and inertial four-momenta are identical. Also some milder form of it would make sense. What is clear is that the construction of preferred extremal involving the distribution of $M^2(x)$ implies that conserved four-momentum associated with Kähler action can be expressed formally as stringy four-momentum. The integral of the conserved inertial momentum current over $X^3$ indeed reduces to an integral over the curve defining string as one integrates over other two degrees of freedom. It would not be surprising if a stringy expression for four-momentum would result but with string tension depending on the point of string and possibly also on the component of four-momentum. If the dependence of string tension on the point of string and on the choice of the stringy world sheet is slow, the interpretation could be in terms of coupling constant evolution associated with the stringy coordinates. An alternative interpretation is that string tension corresponds to a scalar field. A quite reasonable option is that for given $X^3_l$ $T$ defines a scalar field and that the observed $T$ corresponds to the average value of $T$ over deformations of $X^3_l$.

2. The minimum option is that Kähler mass is equal to the sum gravitational masses assignable to strings connecting points of wormhole throat or two different wormhole throats. This hypothesis makes sense even for wormhole contacts having size of order Planck length.

3. The condition that gravitational mass equals to the inertial mass (rest energy) assigned to Kähler action is the most obvious condition that one can imagine. The breaking of Poincare invariance to Lorentz invariance with respect to the tip of $CD$ supports this form of Equivalence Principle. This would predict the value of the ratio of the parameter $R^2T$ and p-adic length scale hypothesis would allow only discrete values for this parameter. $p \simeq 2^k$ following from the quantization of the temporal distance $T(n)$ between the tips of $CD$ as $T(n) = 2^nT_0$ would suggest string tension $T_n = 2^nR^2$ apart from a numerical factor. $G_p \propto 2^nR^2/\hbar_0$ would emerge as a prediction of the theory. $G$ can be seen either as a prediction or RG invariant input parameter fixed by quantum criticality. The arguments related to p-adic coupling constant evolution suggest $R^2/\hbar_0G = 3 \times 2^{23}$ [K3, K18] .

4. The scalar field property of string tension should be consistent with the vacuum degeneracy of Kähler action. For instance, for the vacuum extremals of Kähler action stringy action is non-vanishing. The simplest possibility is that one includes the integral of the scalar $J^{\mu\nu}J_{\mu\nu}$ over the degrees transversal to $M^2$ to the stringy action so that string tension vanishes for vacuum extremals. This would be nothing but dimensional reduction of 4-D theory to a 2-D theory using the slicing of $X^4(X^3_l)$ to partonic 2-surfaces and stringy word sheets. For cosmic strings Kähler action reduces to stringy action with string tension $T \propto 1/g_K^2 R^2$ apart from a numerical constant. If one wants consistency with $T \propto 1/L_p^2$, one must have $T \propto 1/g_K^2 2^n R^2$ for the cosmic strings deformed to Kähler magnetic flux tubes. This looks rather plausible if the thickness of deformed string in $M^4$ degrees of freedom is given by p-adic length scale.

# 4.5 An attempt to understand preferred extremals of Kähler action

There are pressing motivations for understanding the preferred extremals of Kähler action. For instance, the conformal invariance of string models naturally generalizes to 4-D invariance defined by quantum Yangian of quantum affine algebra (Kac-Moody type algebra) characterized by two complex coordinates and therefore explaining naturally the effective 2-dimensionality [K63]. The problem is however how to assign a complex coordinate with the string world sheet having Minkowskian signature of metric. One can hope that the understanding of preferred extremals could allow to identify two preferred complex coordinates whose existence is also suggested by number theoretical vision giving preferred role for the rational points of partonic 2-surfaces in preferred coordinates. The best one could hope is a general solution of field equations in accordance with the hints that TGD is integrable quantum theory.

A lot is is known about properties of preferred extremals and just by trying to integrate all this understanding, one might gain new visions. The problem is that all these arguments are heuristic and rely heavily on physical intuition. The following considerations relate to the space-time regions having Minkowskian signature of the induced metric. The attempt to generalize the construction also to Euclidian regions could be very rewarding. Only a humble attempt to combine various ideas to a more coherent picture is in question.

The core observations and visions are following.

1. Hamilton-Jacobi coordinates for $M^4$ (discussed in this chapter) define natural preferred coordinates for Minkowskian space-time sheet and might allow to identify string world sheets for $X^4$ as those for $M^4$. Hamilton-Jacobi coordinates consist of light-like coordinate $m$ and its dual defining local 2-plane $M^2 \subset M^4$ and complex transversal complex coordinates $(w, \overline{w})$ for a plane $E^2_x$ orthogonal to $M^2_x$ at each point of $M^4$. Clearly, hyper-complex analyticity and complex analyticity are in question.

2. Space-time sheets allow a slicing by string world sheets (partonic 2-surfaces) labelled by partonic 2-surfaces (string world sheets).

3. The quaternionic planes of octonion space containing preferred hyper-complex plane are labelled by $CP_2$, which might be called $CP_2^{mod}$ [K54]. The identification $CP_2 = CP_2^{mod}$ motivates the notion of $M^8 - -M^4 \times CP_2$ duality [K13]. It also inspires a concrete solution ansatz assuming the equivalence of two different identifications of the quaternionic tangent space of the space-time sheet and implying that string world sheets can be regarded as strings in the 6-D coset space $G_2/SU(3)$. The group $G_2$ of octonion automorphisms has already earlier appeared in TGD framework.

4. The duality between partonic 2-surfaces and string world sheets in turn suggests that the $CP_2 = CP_2^{mod}$ conditions reduce to string model for partonic 2-surfaces in $CP_2 = SU(3)/U(2)$. String model in both cases could mean just hypercomplex/complex analyticity for the coordinates of the coset space as functions of hyper-complex/complex coordinate of string world sheet/partonic 2-surface.

The considerations of this section lead to a revival of an old very ambitious and very romantic number theoretic idea.

1. To begin with express octonions in the form $o = q_1 + Iq_2$, where $q_i$ is quaternion and $I$ is an octonionic imaginary unit in the complement of fixed a quaternionic sub-space of octonions. Map preferred coordinates of $H = M^4 \times CP_2$ to octonionic coordinate, form an arbitrary octonion analytic function having expansion with real Taylor or Laurent coefficients to avoid problems due to non-commutativity and non-associativity. Map the outcome to a point of $H$ to get a map $H \rightarrow H$. This procedure is nothing but a generalization of Wick rotation to get an 8-D generalization of analytic map.

2. Identify the preferred extremals of Kähler action as surfaces obtained by requiring the vanishing of the imaginary part of an octonion analytic function. Partonic 2-surfaces and string world sheets would correspond to commutative sub-manifolds of the space-time surface and of imbedding space and would emerge naturally. The ends of braid strands at partonic 2-surface would naturally correspond to the poles of the octonion analytic functions. This would mean a huge generalization of conformal invariance of string models to octonionic conformal invariance and an exact solution of the field equations of TGD and presumably of quantum TGD itself.

## 4.5.1 Basic ideas about preferred extremals

### The slicing of the space-time sheet by partonic 2-surfaces and string world sheets

The basic vision is that space-time sheets are sliced by partonic 2-surfaces and string world sheets. The challenge is to formulate this more precisely at the level of the preferred extremals of Kähler action.

1. Almost topological QFT property means that the Kähler action reduces to Chern-Simons terms assignable to 3-surfaces. This is guaranteed by the vanishing of the Coulomb term in the action density implied automatically if conserved Kähler current is proportional to the instanton current with proportionality coefficient some scalar function.

2. The field equations reduce to the conservation of isometry currents. An attractive ansatz is that the flow lines of these currents define global coordinates. This means that these currents are Beltrami flows [B29] so that corresponding 1-forms $J$ satisfy the condition $J \wedge dJ = 0$. These conditions are satisfied if

$$J = \Phi \nabla \Psi$$

   hold true for conserved currents. From this one obtains that $\Psi$ defines global coordinate varying along flow lines of $J$.

3. A possible interpretation is in terms of local polarization and momentum directions defined by the scalar functions involved and natural additional conditions are that the gradients of $\Psi$ and $\Phi$ are orthogonal:

$$\nabla \Phi \cdot \nabla \Psi = 0 \ ,$$

   and that the $\Psi$ satisfies massless d'Alembert equation

$$\nabla^2 \Psi = 0$$

   as a consequence of current conservation. If $\Psi$ defines a light-like vector field - in other words

$$\nabla \Psi \cdot \nabla \Psi = 0 \ ,$$

   the light-like dual of $\Phi$ -call it $\Phi_c$- defines a light-like like coordinate and $\Phi$ and $\Phi_c$ defines a light-like plane at each point of space-time sheet.

   If also $\Phi$ satisfies d'Alembert equation

$$\nabla^2 \Phi = 0 \ ,$$

also the current

$$K = \Psi \nabla \Phi$$

is conserved and its flow lines define a global coordinate in the polarization plane orthogonal to time-lik plane defined by local light-like momentum direction.

If $\Phi$ allows a contination to an analytic function of the transversal complex coordinate, one obtains a coordinatization of spacetime surface by $\Psi$ and its dual (defining hyper-complex coordinate) and $w, \overline{w}$. Complex analyticity and its hyper-complex variant would allow to provide space-time surface with four coordinates very much analogous with Hamilton-Jacobi coordinates of $M^4$.

This would mean a decomposition of the tangent space of space-time surface to orthogonal planes defined by light-like momentum and plane orthogonal to it. If the flow lines of $J$ defined Beltrami flow it seems that the distribution of momentum planes is integrable.

4. General arguments suggest that the space-time sheets allow a slicing by string world sheets parametrized by partonic 2-surfaces or vice versa. This would mean a intimate connection with the mathematics of string models. The two complex coordinates assignable to the Yangian of affine algebra would naturally relate to string world sheets and partonic 2-surfaces and the highly non-trivial challenge is to identify them appropriately.

### Hamilton-Jacobi coordinates for $M^4$

The earlier attempts to construct preferred extremals [K5] led to the realization that so called Hamilton-Jacobi coordinates $(m, w)$ for $M^4$ define its slicing by string world sheets parametrized by partonic 2-surfaces. $m$ would be pair of light-like conjugate coordinates associated with an integrable distribution of planes $M^2$ and $w$ would define a complex coordinate for the integrable distribution of 2-planes $E^2$ orthogonal to $M^2$. There is a great temptation to assume that these coordinates define prefered coorinates for $M^4$.

1. The slicing is very much analogous to that for space-time sheets and the natural question is how these slicings relate. What is of special interest is that the momentum plane $M^2$ can be defined by massless momentum. The scaling of this vector does not matter so that these planes are labelled by points $z$ of sphere $S^2$ telling the direction of the line $M^2 \cap E^3$, when one assigns rest frame and therefore $S^2$ with the preferred time coordinate defined by the line connecting the tips of $CD$. This direction vector can be mapped to a twistor consisting of a spinor and its conjugate. The complex scalings of the twistor $(u, \overline{u}) \rightarrow \lambda u, \overline{u}/\lambda)$ define the same plane. Projective twistor like entities defining $CP_1$ having only one complex component instead of three are in question. This complex number defines with certain prerequisites a local coordinate for space-time sheet and together with the complex coordinate of $E^2$ could serve as a pair of complex coordinates $(z, w)$ for space-time sheet. This brings strongly in mind the two complex coordinates appearing in the expansion of the generators of quantum Yangian of quantum affine algebra [K63].

2. The coordinate $\Psi$ appearing in Beltrami flow defines the light-like vector field defining $M^2$ distribution. Its hyper-complex conjugate would define $\Psi_c$ and conjugate light-like direction. An attractive possibility is that $\Phi$ allows analytic continuation to a holomorphic function of $w$. In this manner one would have four coordinates for $M^4$ also for space-time sheet.

3. The general vision is that at each point of space-time surface one can decompose the tangent space to $M^2(x) \subset M^4 = M^2_x \times E^2_x$ representing momentum plane and polarization plane $E^2 \subset E^2_x \times T(CP_2)$. The moduli space of planes $E^2 \subset E^6$ is 8-dimensional and parametrized by $SO(6)/SO(2) \times SO(4)$ for a given $E^2_x$. How can one achieve this selection and what conditions it must satisfy? Certainly the choice must be integrable but this is not the only condition.

### Space-time surfaces as quaternionic surfaces

The idea that number theory determines classical dynamics in terms of associativity condition means that space-time surfaces are in some sense quaternionic surfaces of an octonionic space-time. It took several trials before the recent form of this hypothesis was achieved.

1. Octonionic structure is defined in terms of the octonionic representaton of gamma matrices of the imbedding space existing only in dimension $D = 8$ since octonion units are in one-one correspondence with tangent vectors of the tangent space. Octonionic real unit corresponds to a preferred time axes (and rest frame) identified naturally as that connecting the tips of $CD$. What modified gamma matrices mean depends on variational principle for space-time surface. For volume action one would obtain induced gamma matrices. For Kähler action one obtains something different. In particular, the modified gamma matrices do not define vector basis identical with tangent vector basis of space-time surface.

2. Quaternionicity means that the modified gamma matrices defined as contractions of gamma matrices of $H$ with canonical momentum densities for Kähler action span quaternionic sub-space of the octonionic tangent space [K19]. A further condition is that each quaternionic space defined in this manner contains a preferred hyper-complex subspace of octonions.

3. The sub-space defined by the modified gamma matrices does not co-incide with the tangent space of space-time surface in general so that the interpretation of this condition is far from obvious. The canonical momentum densities need not define four independent vectors at given point. For instance, for massless extremals these densities are proportional to light-like vector so that the situation is degenerate and the space in question reduces to 2-D hyper-complex sub-space since light-like vector defines plane $M^2$.

The obvious questions are following.

1. Does the analog of tangent space defined by the octonionic modified gammas contain the local tangent space $M^2 \subset M^4$ for preferred extremals? For massless extremals [K5] this condition would be true. The orthogonal decomposition $T(X^4) = M^2 \oplus_\perp E^2$ can be defined at each point if this is true. For massless extremals also the functions $\Psi$ and $\Phi$ can be identified.

2. One should answer also the following delicate question. Can $M^2$ really depend on point $x$ of space-time? $CP_2$ as a moduli space of quaternionic planes emerges naturally if $M^2$ is *same* everywhere. It however seems that one should allow an integrable distribution of $M^2_x$ such that $M^2_x$ is same for all points of a given partonic 2-surface.

   How could one speak about fixed $CP_2$ (the imbedding space) at the entire space-time sheet even when $M^2_x$ varies?

   (a) Note first that $G_2$ defines the Lie group of octonionic automorphisms and $G_2$ action is needed to change the preferred hyper-octonionic sub-space. Various $SU(3)$ subgroups of $G_2$ are related by $G_2$ automorphism. Clearly, one must assign to each point of a string world sheet in the slicing parameterizing the partonic 2-surfaces an element of $G_2$. One would have Minkowskian string model with $G_2$ as a target space. As a matter fact, this string model is defined in the target space $G_2/SU(3)$ having dimension $D = 6$ since $SU(3)$ automorphisms leave given $SU(3)$ invariant.

   (b) This would allow to identify at each point of the string world sheet standard quaternionic basis - say in terms of complexified basis vectors consisting of two hyper-complex units and octonionic unit $q_1$ with "color isospin" $I_3 = 1/2$ and "color hypercharge" $Y = -1/3$ and its conjugate $\overline{q}_1$ with opposite color isospin and hypercharge.

   (c) The $CP_2$ point assigned with the quaternionic basis would correspond to the $SU(3)$ rotation needed to rotate the standard basis to this basis and would actually correspond to the first row of $SU(3)$ rotation matrix. Hyper-complex analyticity is the basic property of the solutions of the field equations representing Minkowskian string world sheets. Also now the same assumption is highly natural. In the case of string models in Minkowski space, the reduction of the induced metric to standard form implies Virasoro conditions and similar conditions are expected also now. There is no need to introduce action principle -just the hyper-complex analycitity is enough-since Kähler action already defines it.

3. The WZW model inspired approach to the situation would be following. The parametrization corresponds to a map $g : X^2 \to G_2$ for which $g$ defines a flat $G_2$ connection at string world sheet. WZW type action would give rise to this kind of situation. The transition $G_2 \to G_2/SU(3)$

would require that one gauges $SU(3)$ degrees of freedom by bringing in $SU(3)$ connection. Similar procedure for $CP_2 = SU(3)/U(2)$ would bring in $SU(3)$ valued chiral field and $U(2)$ gauge field. Instead of introducing these connections one can simply introduce $G_2/SU(3)$ and $SU(3)/U(2)$ valued chiral fields. What this observation suggests that this ansatz indeed predicts gluons and electroweak gauge bosons assignable to string like objects so that the mathematical picture would be consistent with physical intuition.

#### The two interpretations of $CP_2$

An old observation very relevant for what I have called $M^8 - H$ duality [K13] is that the moduli space of quaternionic sub-spaces of octonionic space (identifiable as $M^8$) containing preferred hyper-complex plane is $CP_2$. Or equivalently, the space of two planes whose addition extends hyper-complex plane to some quaternionic subspace can be parametrized by $CP_2$. This $CP_2$ can be called it $CP_2^{mod}$ to avoid confusion. In the recent case this would mean that the space $E^2(x) \subset E^2_x \times T(CP_2)$ is represented by a point of $CP_2^{mod}$. On the other hand, the imbedding of space-time surface to $H$ defines a point of "real" $CP_2$. This gives two different $CP_2$s.

1. The highly suggestive idea is that the identification $CP_2^{mod} = CP_2$ (apart from isometry) is crucial for the construction of preferred extremals. Indeed, the projection of the space-time point to $CP_2$ would fix the local polarization plane completely. This condition for $E^2(x)$ would be purely local and depend on the values of $CP_2$ coordinates only. Second condition for $E^2(x)$ would involve the gradients of imbedding space coordinates including those of $CP_2$ coordinates.

2. The conditions that the planes $M^2_x$ form an integrable distribution at space-like level and that $M^2_x$ is determined by the modified gamma matrices. The integrability of this distribution for $M^4$ could imply the integrability for $X^2$. $X^4$ would differ from $M^4$ only by a deformation in degrees of freedom transversal to the string world sheets defined by the distribution of $M^2$s.

   Does this mean that one can begin from vacuum extremal with constant values of $CP_2$ coordinates and makes them non-constant but allows to depend only on transversal degrees of freedom? This condition is too strong even for simplest massless extremals for which $CP_2$ coordinates depend on transversal coorinates defined by $\epsilon \cdot m$ and $\epsilon \cdot k$. One could however allow dependence of $CP_2$ coordinates on light-like $M^4$ coordinate since the modification of the induced metric is light-like so that light-like coordinate remains light-like coordinate in this modification of the metric.

   Therefore, if one generalizes directly what is known about massless extremals, the most general dependence of $CP_2$ points on the light-like coordinates assignable to the distribution of $M^2_x$ would be dependence on either of the light-like coordinates of Hamilton-Jacobi coordinates but not both.

### 4.5.2 What could be the construction recipe for the preferred extremals assuming $CP_2 = CP_2^{mod}$ identification?

The crucial condition is that the planes $E^2(x)$ determined by the point of $CP_2 = CP_2^{mod}$ identification and by the tangent space of $E^2_x \times CP_2$ are same. The challenge is to transform this condition to an explicit form. $CP_2 = CP_2^{mod}$ identification should be general coordinate invariant. This requires that also the representation of $E^2$ as $(e^2, e^3)$ plane is general coordinate invariant suggesting that the use of preferred $CP_2$ coordinates -presumably complex Eguchi-Hanson coordinates- could make life easy. Preferred coordinates are also suggested by number theoretical vision. A careful consideration of the situation would be required.

The modified gamma matrices define a quaternionic sub-space analogous to tangent space of $X^4$ but not in general identical with the tangent space: this would be the case only if the action were 4-volume. I will use the notation $T^m_x(X^4)$ about the modified tangent space and call the vectors of $T^m_x(X^4)$ modified tangent vectors. I hope that this would not cause confusion.

#### $CP_2 = CP_2^{mod}$ condition

Quaternionic property of the counterpart of $T^m_x(X^4)$ allows an explicit formulation using the tangent vectors of $T^m_x(X^4)$.

1. The unit vector pair $(e_2, e_3)$ should correspond to a unique tangent vector of $H$ defined by the coordinate differentials $dh^k$ in some natural coordinates used. Complex Eguchi-Hanson coordinates [L1] are a natural candidate for $CP_2$ and require complexified octonionic imaginary units. If octonionic units correspond to the tangent vector basis of $H$ uniquely, this is possible.

2. The pair $(e_2, e_3)$ as also its complexification $(q_1 = e_2 + ie_3, \overline{q}_1 = e_2 - ie_3)$ is expressible as a linear combination of octonionic units $I_2, ...I_7$ should be mapped to a point of $CP_2^{mod} = CP_2$ in canonical manner. This mapping is what should be expressed explicitly. One should express given $(e_2, e_3)$ in terms of $SU(3)$ rotation applied to a standard vector. After that one should define the corresponding $CP_2$ point by the bundle projection $SU(3) \rightarrow CP_2$.

3. The tangent vector pair

$$(\partial_w h^k, \partial_{\overline{w}} h^k)$$

defines second representation of the tangent space of $E^2(x)$. This pair should be equivalent with the pair $(q_1, \overline{q}_1)$. Here one must be however very cautious with the choice of coordinates. If the choice of $w$ is unique apart from constant the gradients should be unique. One can use also real coordinates $(x, y)$ instead of $(w = x + iy, \overline{w} = x - iy)$ and the pair $(e_2, e_3)$. One can project the tangent vector pair to the standard vielbein basis which must correspond to the octonioni basis

$$(\partial_x h^k, \partial_y h^k) \rightarrow (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A) e_A) \leftrightarrow (e_2, e_3) \ \ ,$$

where the $e_A$ denote the octonion units in 1-1 correspondence with vielbein vectors. This expression can be compared to the expression of $(e_2, e_3)$ derived from the knowledge of $CP_2$ projection.

**Formulation of quaternionicity condition in terms of octonionic structure constants**

One can consider also a formulation of the quaternionic tangent planes in terms of $(e_2, e_3)$ expressed in terms of octonionic units deducible from the condition that unit vectors obey quaternionic algebra. The expressions for octonionic *resp.* quaternionic structure constants can be found at [A19] *resp.* [A21].

1. The ansatz is

$$\begin{aligned} \{E_k\} &= \{1, I_1, E_2, E_3\} \ , \\ E_2 &= E_{2k} e^k \equiv \sum_{k=2}^{7} E_{2k} e^k \ \ , \ \ E_3 = E_{3k} e^k \equiv \sum_{k=2}^{7} E_{3k} e^k \ \ , \\ |E_2| &= 1 \ \ , \ \ |E_3| = 1 \ \ . \end{aligned} \tag{4.5.1}$$

2. The multiplication table for octonionic units expressible in terms of octonionic triangle [A19] gives

$$f^{1kl} E_{2k} = E_{3l} \ \ , \ \ f^{1kl} E_{3k} = -E_{2l} \ \ , \ \ f^{klr} E_{2k} E_{3l} = \delta_1^r \ \ . \tag{4.5.2}$$

Here the indices are raised by unit metric so that there is no difference between lower and upper indices. Summation convention is assumed. Also the contribution of the real unit is present in the structure constants of third equation but this contribution must vanish.

3. The conditions are linear and quadratic in the coefficients $E_{2k}$ and $E_{3k}$ and are expected to allow an explicit solution. The first two conditions define homogenous equations which must allow solution. The coefficient matrix acting on $(E_2, E_3)$ is of the form

$$\left( \begin{array}{cc} f_1 & 1 \\ -1 & f_1 \end{array} \right) ,$$

where 1 denotes unit matrix. The vanishing of the determinant of this matrix should be due to the highly symmetric properties of the structure constants. In fact the equations can be written as eigen conditions

$$f_1 \circ (E_2 \pm iE_3) = \mp i(E_2 \pm iE_3) \ ,$$

and one can say that the structure constants are eigenstates of the hermitian operator defined by $I_1$ analogous to color hyper charge. Both values of color hyper charged are obtained.

**Explicit expression for the $CP_2 = CP_2^{mod}$ conditions**

The symmetry under $SU(3)$ allows to construct the solutions of the above equations directly.

1. One can introduce complexified basis of octonion units transforming like $(1, 1, 3, \overline{3})$ under SU(3). Note the analogy of triplet with color triplet of quarks. One can write complexified basis as $(1, e_1, (q_1, q_2, q_3), (\overline{q}_1\overline{q}_2, \overline{q}_3))$. The expressions for complexified basis elements are

$$(q_1, q_2, q_3) = \frac{1}{\sqrt{2}}(e_2 + ie_3, e_4 + ie_5, e_6 + ie_7) \ .$$

These options can be seen to be possible by studying octonionic triangle in which all lines containing 3 units defined associative triple: any pair of octonion units at this kind o fline can be used to form pair of complexified unit and its conjugate. In the tangent space of $M^4 \times CP_2$ the basis vectors $q_1$, and $q_2$ are mixtures of $E^2_x$ and $CP_2$ tangent vectors. $q_3$ involves only $CP_2$ tangent vectors and there is a temptation to interpret it as the analog of the quark having no color isospin.

2. The quaternionic basis is real and must transform like $(1, 1, q_1, \overline{q}_1)$, where $q_1$ is any quark in the triplet and $\overline{q}_1$ its conjugate in antitriplet. Having fixed some basis one can perform SU(3) rotations to get a new basis. The action of the rotation is by $3 \times 3$ special unitary matrix. The over all phases of its rows do not matter since they induce only a rotation in $(e_2, e_3)$ plane not affecting the plane itself. The action of $SU(3)$ on $q_1$ is simply the action of its first row on $(q_1, q_2, q_3)$ triplet:

$$\begin{aligned} q_1 &\rightarrow (Uq)_1 = U_{11}q_1 + U_{12}q_2 + U_{13}q_3 \equiv z_1q_1 + z_2q_2 + z_3q_3 \\ &= z_1(e_2 + ie_3) + z_2(e_4 + ie_5) + z_3(e_6 + ie_7) \ . \end{aligned} \tag{4.5.3}$$

The triplets $(z_1, z_2, z_3)$ defining a complex unit vector and point of $S^5$. Since overall phase does not matter a point of $CP_2$ is in question. The new real octonion units are given by the formulas

$$\begin{aligned} e_2 &\rightarrow Re(z_1)e_2 + Re(z_2)e_4 + Re(z_3)e_6 - Im(z_1)e_3 - Im(z_2)e_5 - Im(z_3)e_7 \ , \\ e_3 &\rightarrow Im(z_1)e_2 + Im(z_2)e_4 + Im(z_3)e_6 + Re(z_1)e_3 + Re(z_2)e_5 + Re(z_3)e_7 \ . \end{aligned} \tag{4.5.4}$$

For instance the $CP_2$ coordinates corresponding to the coordinate patch $(z_1, z_2, z_3)$ with $z_3 \neq 0$ are obtained as $(\xi_1, \xi_2) = (z_1/z_3, z_2/z_3)$.

Using these expressions the equations expressing the conjecture $CP_2 = CP_2^{mod}$ equivalence can be expressed explicitly as first order differential equations. The conditions state the equivalence

$$(e_2, e_3) \quad \leftrightarrow \quad (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A e_A) \ , \tag{4.5.5}$$

where $e_A$ denote octonion units. The comparison of two pairs of vectors requires normalization of the tangent vectors on the right hand side to unit vectors so that one takes unit vector in the direction of the tangent vector. After this the vectors can be equated. This allows to expresses the contractions of the partial derivatives with vielbein vectors with the 6 components of $e_2$ and $e_3$. Each condition gives 6+6 first order partial differential equations which are non-linear by the presence of the overal normalization factor for the right hand side. The equations are invariant under scalings of $(x, y)$. The very special form of these equations suggests that some symmetry is involved.

It must be emphasized that these equations make sense only in preferred coordinates: ordinary Minkowski coordinates and Hamiltonin-Jacobi coordinates for $M^4$ and Eguchi-Hanson complex coordinates in which $SU(2) \times U(1)$ is represented linearly for $CP_2$. These coordinates are preferred because they carry deep physical meaning.

**Does TGD boil down to two string models?**

It is good to look what have we obtained. Besides Hamilton-Jacobi conditions, and $CP_2 = CP_2^{mod}$ conditions one has what one might call string model with 6-dimensional $G_2/SU(3)$ as targent space. The orbit of string in $G_2/SU(3)$ allows to deduce the $G_2$ rotation identifiable as a point of $G_2/SU(3)$ defining what one means with standard quaternionic plane at given point of string world sheet. The hypothesis is that hyper-complex analyticity solves these equations.

The conjectured electric-magnetic duality implies duality between string world sheet and partonic 2-surfaces central for the proposed mathematical applications of TGD [K25, K26, K52, K64]. This duality suggests that the solutions to the $CP_2 = CP_2^{mod}$ conditions could reduce to holomorphy with respect to the coordinate $w$ for partonic 2-surface plus the analogs of Virasoro conditions. The dependence on light-like coordinate would appear as a parametric dependence.

If this were the case, TGD would reduce at least partially to what might be regaded as dual string models in $G_2/SU(3)$ and $SU(3)/U(2)$ and also to string model in $M^4$ and $X^4$! In the previous arguments one ends up to string models in moduli spaces of string world sheets and partonic 2-surfaces. TGD seems to yield an inflation of string models! This not actually surprising since the slicing of space-time sheets by string world sheets and partonic 2-surfaces implies automatically various kinds of maps having interpretation in terms of string orbits.

### 4.5.3 Could octonion analyticity solve the field equations?

The interesting question is what happens in the space-time regions with Euclidian signature of induced metric. In this case it is not possible to introduce light-like plane at each point of the space-time sheet. Nothing however prevents from applying the above described procedure to construct conserved currents whose flow lines define global coordinates. In both cases analytic continuation allows to extend the coordinates to complex coordinates. Therefore one would have two complex functions satisfying Laplace equation and having orthogonal gradients.

1. When $CP_2$ projection is 4-dimensional, there is strong temptation to assume that these functions could be reduced to complex $CP_2$ coordinates analogous to the Hamilton-Jacobi coordinates for $M^4$. Complex Eguchi-Hanson coordinates transforming linearly under $U(2) \subset SU(3)$ define the simplest candidates in this respect. Laplace-equations are satisfied utomatically since holomorphic functions are in question. The gradients are also orthogonal automatically since the metric is Kähler metric. Note however that one could argue that in innner product the conjugate of the function appears. Any holomorphic map defines new coordinates of this kind. Note that the maps need not be globally holomorphic since $CP_2$ projection of space-time sheet need not cover the entire $CP_2$.

2. For string like objects $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$ with Minkowskian signature of the metric the coordinate pair would be hyper-complex coordinate in $M^4$ and complex coordinate in $CP_2$.

If $X^2$ has Euclidian signature of induced metric the coordinate in question would be complex coordinate. The proposal in the case of $CP_2$ allows all holomorphic functions of the complex coordinates.

There is an objection against this construction. There should be a symmetry between $M^4$ and $CP_2$ but this is not the case. Therefore this picture cannot be quite correct.

Could the construction of new preferred coordinates by holomorphic maps generalize as electic-magnetic duality suggests? One can imagine several options, which bring in mind old ideas that what I have christened as "romantic stuff" [K54].

1. Should one generalize the holomorphic map to a quaternion analytic map with real Taylor coefficients so that non-commutativity would not produce problems. One would map first $M^4$ coordinates to quaternions, map these coordinates to new ones by quaternion analytic map defined by a Taylor or even Laurnte expansion with real coefficients, and then map the resulting quaternion valued coordinate back to hyper-quaternion defining four coordinates as fuctions in $M^4$. This procedure would be very much analogous to Wick rotation used in quantum field theories. Similar quaternion analytic map be applied also in $CP_2$ degrees of freedom followed by the map of the quaternion to two complex numbers. This would give additional constraints on the map. This option could be seen as a quaternionic generalization of conformal invariance.

   The problem is that one decouples $M^4$ and $CP_2$ degrees of freedom completely. These degrees are however coupled in the proposed construction since the $E^2(x)$ corresponds to subspace of $E^2_x \times T(CP_2)$. Something goes still wrong.

2. This motivates to imagine even more ambitious and even more romantic option realizing the original idea about octonionic generalization of conformal invariance. Assume linear $M^4 \times CP_2$ coordinates (Eguchi-Hanson coordinates transforming linearly under $U(2)$ in the case of $CP_2$). Map these to octonionic coordinate $h$. Map the octonionic coordinate to itself by an octonionic analytic map defined by Taylor or even Laurent series with real coefficients so that non-commutativity and non-associativity do not cause troubles. Map the resulting octonion valued coordinates back to ordinary $H$-coordinates and expressible as functions of original coordinates.

   It must be emphasized that this would be nothing but a generalization of Wick rotation and its inverse used routinely in quantum field theories in order to define loop integrals.

### Could octonion real-analyticity make sense?

Suppose that one -for a fleeting moment- takes octonionic analyticity seriously. For space-time surfaces themselves one should have in some sense quaternionic variant of conformal invariance. What does this mean?

1. Could one regard space-time surfaces analogous to the curves at which the imaginary part of analytic function of complex argument vanishes so that complex analyticity reduces to real analyticity. One can indeed divide octonion to quaternion and its imaginary part to give $o = q_1 + Iq_2$: $q_1$ and $q_2$ are quaternionis and $I$ is octonionic imaginary unit in the complement of the quaternionic sub-space. This decomposition actually appears in the standard construction of octonions. Therefore 4-dimensional surfaces at which the imaginary part of octonion valued function vanishes make sense and defined in well-defined sense quaternionic 4-surfaces.

   This kind of definition would be in nice accord with the vision about physics as algebraic geometry. Now the algebraic geometry would be extended from complex realm to the octonionic realm since quaternionic surfaces/string world sheets could be regarded as associative/commutative sub-algebras of the algebra of the octionic real-analytic functions.

2. Could these surfaces correspond to quaternionic 4-surfaces defined in terms of the modified gamma matrices or induced gamma matrices? Contrary to the original expectations it will be found that only induced gamma matrices is a plausible option. This would be an enormous simplification and would mean that the theory is exactly solvable in the same sense as string models are: complex analyticity would be replaced with octonion analyticity. I have considered this option in several variants using the notion of real octonion analyticity [K54] but have not managed to build any satisfactory scenario.

3. Hyper-complex and complex conformal symmetries would result by a restriction to hyper-complex *resp.* complex sub-manifods of the imbedding space defined by string world sheets *resp.* partonic 2-surfaces. The principle forcing this restriction would be commutativity. Yangian of an affine algebra would unify these views to single coherent view [K63].

   4-D n-point functions of the theory should result from the restriction on partonic 2-surfaces or string world sheets with arguments of n-point functions identified as the ends of braid strands so that a kind of analytic continuation from 2-D to the 4-D case would be in question. The octonionic conformal invariance would be induced by the ordinary conformal invariance in accordance with strong form of General Coordinate Invariance.

4. This algebraic continuation of the ordinary conformal invariance could help to construct also the representations of Yangians of affine Kac-Moody type algebras. For the Yangian symmetry of 1+1 D integrable QFTs the charges are multilocal involving multiple integrals over ordered multiple points of 1-D space. I

   In the recent case multiple 1-D space is replaced with a space-like 3-surface at the light-like end of $CD$. The point of the 1-D space appearing in the multiple integral are replaced by a partonic 2-surface represented by a collection of punctures. There is a strong temptation to assume that the intermediate points on the line correspond to genuine physical particles and therefore to partonic 2-surfaces at which the signature of the induced metric changes. If so, the 1-D space would correspond to a closed curve connecting punctures of different partonic 2-surfaces representing physical particles and ordered along a loop. The integral over multiple points would correspond to an integral over WCW rather than over fixed back-ground space-time.

   1-D space would be replaced with a closed curve going through punctures of a subset of partonic 2-surfaces associated with a space-like 3-surface. If a given partonic surface or a given puncture can contribute only once to the multiple integral the multi-locality is bounded from above and only a finite number of Yangian generators are obtained in this manner unless one allows the number of partonic 2-surfaces and of punctures for them to vary. This variation is physically natural and would correspond to generation of particle pairs by vacuum polarization. Although only punctures would contribute, the Yangian charges would be defined in WCW rather than in fixed space-time. Integral over positions of punctures and possible numbers of them would be actually an integral over WCW. 2-D modular invariance of Yangian charges for the partonic 2-surfaces is a natural constraint.

   The question is whether some conformal fields at the punctures of the partonic 2-surfaces appearing in the multiple integral define the basic building bricks of the conserved quantum charges representing the multilocal generators of the Yangian algebra? Note that Wick rotation would be involved.

**What Wick rotation could mean?**

Second definition of quaternionicity is on more shaky basis and motivated by the solutions of 2-D Laplace equation: quaternionic space-time surfaces would be obtained as zero loci of octonion real–analytic functions. Unfortunately octonion real–analyticity does not make sense in Minkowskian signature.

One could understand octonion real-analyticity in Minkowskian signature if one could understand the deeper meaning of Wick rotation. Octonion real analyticity formulated as a condition for the vanishing of the imaginary part of octonion real-analytic function makes sense for in octonionic coordinates for $E^4 \times CP_2$ with Euclidian signature of metric. $M^4 \times CP_2$ is however only a subspace of complexified octonions and not closed with respect to multiplication so that octonion real-analytic functions do not make sense in $M^4 \times CP_2$ . Wick rotation should transform the solution candidate defined by an octonion real-analytic function to that defined in $M^4 \times CP_2$. A natural additional condition is that Wick rotation should reduce to that taking $M^2 \subset M^4$ to $E^2 \subset E^4$.

The following trivial observation made in the construction of Hamilton-Jacobi structure in $M^4$ with Minkowskian signature of the induced metric (see the appendix of [K67]) as a Wick rotation of Hermitian structure in $E^4$ might help here.

1. The components of the metric of $E^2$ in complex coordinates $(z, \overline{z})$ for $E^2$ are given by $g_{w\overline{w}} = -1$ whereas the metric of $M^2$ in light-like coordinates $(u = x+t, v = x-t)$ is given by $g_{uv} = -1$. The

metric is same and $M^2$ and $E^2$ correspond only to different interpretations for the coordinates! One could say that $M^4 \times CP_2$ and $E^4 \times CP_2$ have same metric tensor, Kähler structure, and spinor structure. Since only these appear in field equations, one could hope that the solutions of field equations in $M^4 \times CP_2$ and $E^4 \times CP_2$ are obtained by Wick rotation. This for preferred extremals at least and if the field equations reduce to purely algebraic ones.

2. If one accepts the proposed construction of preferred extremals of Kähler action discussed in [K67], the field equations indeed reduce to purely algebraic conditions satisfied if space-time surface possesses Hermitian structure in the case of Euclidian signature of the induced metric and Hamilton-Jacobi structure in the case of Minkowskian signature. Just as in the case of minimal surfaces, energy momentum tensor and second fundamental form have no common non-vanishing components. The algebraization requires as a consistency condition Einstein's equations with a cosmological term. Gravitational constant and cosmological constant follow as predictions.

3. If Wick rotation in the replacement of $E^2$ coordinates $(z, \overline{z})$ with $M^2$ coordinates $(u, v)$ makes sense, one can hope that field equations for the preferred extremals hold true also for a Wick rotated surfaces obtained by mapping $M^2 \subset M^4$ to $E^2 \subset E^4$. Also Einstein's equations should be satisfied by the Wick rotated metric with Euclidian signature.

4. Wick rotation makes sense also for the surfaces defined by the vanishing of the imaginary part (complementary to quaternionic part) of octonion real-analytic function. Therefore one can hope that this ansatz could work. Wick rotation is non-trivial geometrically. For instance, light-like lines $v = 0$ of hyper-complex plane $M^2$ are taken to $\overline{z} = 0$ defining a point of complex plane $E^2$. Note that non-invertible hyper-complex numbers correspond to the two light-like lines $u = 0$ and $v = 0$ whereas non-invertible complex numbers correspond to the origin of $E^2$.

5. If the conjecture holds true, one can apply to both factors in $E^4 = E^2 \times E^2$ and to get preferred extremals in $M^{2,2} \times CP_2$. Minkowski space $M^{2,2}$ is essential in twistor approach and the possibility to carry out Wick rotation for preferred extremals could justify Wick rotation in quantum theory.

#### What the non-triviality of the moduli space of the octonionic structures means?

The moduli space $G_2$ of the octonionic structures is essentially the Galois group defined as maps of octonions to itself respecting octonionic sum and multiplication. This raises the question whether octonion analyticity should be generalized in such a manner that the global choice of the octonionic imaginary units - in particular that of preferred commuting complex sub-space- would become local. Physically this would correspond to the choice of momentum plane $M^2_x$ for a position dependent light-iike momentum defining the plane of non-physical polarizations.

This question is inspired by the general solution ansatz based on the slicing of space-time sheets which involves the dependence of the choice of the momentum plane $M^2_x$ on the point of string world sheet. This dependence is parameterized by a point of $G_2/SU(3)$ and assumed to be constant along partonic 2-surfaces. These slicings would be naturally associated with the two complex parts $c_i$ of the quaternionic coordinate $q_1 = c_1 + Ic_2$ of the space-time sheet.

This dependence is well-defined only for the quaternionic 4-surface defining the space-time surface and can be seen as a local choice of a preferred complex imaginary unit along string world sheets. $CP_2$ would parametrize the remaining geometric degrees of freedom. Should/could one extend this dependence to entire 8-D imbedding space? This is possible if the 8-D imbedding space allows a slicing by the string world sheets. If the string world sheets correspond to the string world sheets appearing in the slicing of $M^4$ defined by Hamiton-Jacobi coordinates [K5], this slicing indeed exists.

#### Zero energy ontology and octonion analyticity

How does this picture relate to zero energy ontology and how partonic 2-surfaces and string world sheets could be identified in this framework?

1. The intersection of the quaternionic four-surfaces with the 7-D light-like boundaries of $CD$s is 3-D space-like surface. String world sheets are obtained as 2-D complex surfaces by putting $c_2 = 0$,

where $c_2$ is the imaginary part of the quaternion coordinate $q = c_1 + Ic_2$. Their intersections with $CD$ boundaries are generally 1-dimensional and represent space-like strings.

2. Partonic 2-surfaces could correspond to the intersections of $Re(c_1) = constant$ 3-surfaces with the boundaries of $CD$. The variation of $Re(c_1)$ would give a family of (possibly light-like) 3-surfaces whose intersection with the boundaries of $CD$ would be 2-dimensional. The interpretation $Re(c_1) = constant$ surfaces as (possibly light-like) orbits of partonic 2-surfaces would be natural. Wormhole throats at which the signature of the induced metric changes (by definition) would correspond to some special value of $Re(c_1)$, naturally $Re(c_1) = 0$.

   What comes first in mind is that partonic 2-surfaces assignable to wormhole throats correspond to co-complex 2-surfaces obtained by putting $c_1 = 0$ (or $c_1 = constant$) in the decomposition $q = c_1 + ic_2$. This option is consistent with the above assumption if $Im(c_1) = 0$ holds true at the boundaries of $CD$. Note that also co-quaternionic surfaces make sense and would have Euclidian signature of the induced metric: the interpretation as counterparts of lines of generalized Feynman graphs might make sense.

3. One can of course wonder whether also the poles of $c_1$ might be relevant. The most natural idea is that the value of $Re(c_1)$ varies between 0 and $\infty$ between the ends of the orbit of partonic 2-surface. This would mean that $c_1$ has a pole at the other end of $CD$ (or light-like orbit of partonic 2-surface). In light of this the earlier proposal [K52] that zero energy states might correspond to rational functions assignable to infinite primes and that the zeros/poles of these functions correspond to the positive/negative energy part of the state is interesting.

   The intersections of string world sheets and partonic 2-surfaces identifiable as the common ends of space-like and time like brand strands would correspond to the points $q = c_1 + Ic_2 = 0$ and $q = \infty + Ic_2$, where $\infty$ means real infinity. In other words, to the zeros and real poles of quaternion analytic function with real coefficients. In the number theoretic vision especially interesting situations correspond to polynomials with rational number valued coefficients and rational functions formed from these. In this kind of situations the number of zeros and therefore of braid strands is always finite.

**Do induced or modified gamma matrices define quaternionicity?**

The are two options to be considered: either induced or modified gamma matrices define quaternionicity.

1. There are several arguments supporting this view that induced gamma matrices define quaternionicity and that quaternionic planes are therefore tangent planes for space-time sheet.

   (a) $H - M^8$ correspondence is based on the observation that quaternionic sub-spaces of octonions containing preferred complex sub-space are labelled by points of $CP_2$. The integrability of the distribution of quaternionic spaces could follow from the parametrization by points of $CP_2$ ($CP_2 = CP_{mod}$ condition). Quaternionic planes would be necessarily tangent planes of space-time surface. Induced gamma matrices correspond naturally to the tangent space vectors of the space-time surface.

   Here one should however understand the role of the $M^4$ coordinates. What is the functional form of $M^4$ coordinates as functions of space-time coordinates or does this matter at all (general coordinate invariance): could one choose the space-time coordinates as $M^4$ coordinates for surfaces representable as graphs for maps $M^4 \to CP_2$? What about other cases such as cosmic strings [K14]?

   (b) Could one do entirely without gamma matrices and speak only about induced octonion structure in 8-D tangent space (raising also dimension $D = 8$ to preferred role) with reduces to quaternionic structure for quaternionic 4-surfaces. The interpretation of quaternionic plane as tangent space would be unavoidable also now. In this approach there would be no question about whether one should identify octonionic gamma matrices as induced gamma matrices or as modified octonionic gamma matrices.

   (c) If quaternion analyticity is defined in terms of modified gamma matrices defined by the volume action why it would solve the field equations for Kähler action rather than for minimal surfaces? Is the reason that quaternionic and octonionic analyticities defined as generalized differentiability are not possible. The real and imaginary parts of quaternionic real-analytic function with quaternion interpreted as bi-complex number are not analytic functions of two complex variables of either complex variable. In 4-D situation minimal surface property would be too strong a condition whereas Kähler action poses much weaker conditions. Octonionic real-analyticity however poses strong symmetries and suggests effective 2-dimensionality.

2. The following argument suggest that modified gamma matrices cannot define the notion of quaternionic plane.

   (a) Modified gamma matrices can define sub-spaces of lower dimensionality so that they do not defined a 4-plane. In this case they cannot define $CP_2$ point so that $CP_2 = CP_2^{mod}$ identity fails. Massless extremals represents the basic example about this. Hydrodynamic solutions defined in terms of Beltrami flows could represent a more general phase of this kind.

   (b) Modified gamma matrices are not in general parallel to the space-time surface. The $CP_2$ part of field equations coming from the variation of Kähler form gives the non-tangential contribution. If the distribution of the quaternionic planes is integrable it defines another space-time surface and this looks rather strange.

   (c) Integrable quaternionicity can mean only tangent space quaternionicity. For modified gamma matrices this cannot be the case. One cannot assign to the octonion analytic map modified gamma matrices in any natural manner.

The conclusion seems to be that induced gamma matrices or induced octonion structure must define quaternionicity and quaternionic planes are tangent planes of space-time surface and therefore define an integrable distribution. An open question is whether $CP_2 = CP_2^{mod}$ condition implies the integrability automatically.

**Volume action or Kähler action?**

What seems clear is that quaternionicity must be defined by the induced gamma matrices obtained as contractions of canonical momentum densities associated with volume action with imbedding space gamma matrices. Probably equivalent definition is in terms of induced octonion structure. For the believer in strings this would suggest that the volume action is the correct choice. There are however strong objections against this choice.

1. In 2-dimensional case the minimal surfaces allow conformal invariance and one can speak of complex structure in their tangent space. In particular, string world sheets can be regarded as complex 2-surfaces of quaternionic space-time surfaces. In 4-dimensional case the situation is different since quaternionic differentiability fails by non-commutativity. It is quite possible that only very few minimal surfaces (volume action) are quaternionic.

2. The possibility of Beltrami flows is a rather plausible property of quite many preferred extremals of Kähler action. Beltrami flows are also possible for a 4-D minimal surface action. In particular, $M^4$ translations would define Beltrami flows for which the 1-forms would be gradients of linear $M^4$ coordinates. If $M^4$ coordinate can be used on obtains flows in directions of all coordinate axes. Hydrodynamical picture in the strong form therefore fails whereas for Kähler action various isometry currents could be parallel (as they are for massless extremals).

3. For volume action topological QFT property fails as also fails the decomposition of solutions to massless quanta in Minkowskian regions. The same applies to criticality. The crucial vacuum degeneracy responsible for most nice features of Kähler action is absent and also the effective 2-dimensionality and almost topological QFT property are lost since the action does not reduce to 3-D term.

One can however keep Kähler action and define quaternionicity in terms of induced gamma matrices or induced octonion structure. Preferred extremals could be identified as extremals of Kähler action which are also quaternionic 4-surfaces.

1. Preferred extremal property for Kähler action could be much weaker condition than minimal surface property so that much larger set of quaternionic space-time surfaces would be extremals of the Kähler action than of volume action. The reason would be that the rank of energy momentum tensor for Maxwell action tends to be smaller than maximal. This expectation is supported by the vacuum degeneracy, the properties of massless extremals and of $CP_2$ type vacuum extremals, and by the general hydrodynamical picture.

2. There is also a long list of beautiful properties supporting Kähler action which should be also familiar: effective 2-dimensionality and slicing of space-time surface by string world sheets and partonic 2-surfaces, reduction to almost topological QFT and to abelian Chern-Simons term, weak form of electric-magnetic duality, quantum criticality, spin glass degeneracy, etc...

**Are quaternionicities defined in terms of induced gamma matrices *resp.* octonion real-analytic maps equivalent?**

Quaternionicity could be defined by induced gamma matrices or in terms of octonion real-analytic maps. Are these two definitions equivalent and how could one test the equivalence?

1. The calculation technical problem is that space-time surfaces are not defined in terms of imbedding map involving some coordinate choice but in terms of four vanishing conditions for the imaginary part of the octonion real-analytic function expressible as biquaternion valued functions.

2. Integrability to 4-D surface is achieved if there exists a 4-D closed Lie algebra defined by vector fields identifiable as tangent vector fields. This Lie algebra can be generalized to a local 4-D Lie algebra. One cannot however represent octonionic units in terms of 8-D vector fields since the commutators of the latter do not form an associative algebra. Also the representation of 7 octonionic imaginary units as 8-D vector fields is impossible since the algebra in question is non-assciative Malcev algebra [A16] which can be seen as a Lie algebra over non-associative number field (one speaks of 7-dimensional cross product [A24]). One must use instead of vector fields either octonionic units as such or octonionic gamma "matrices" to represent tangent vectors. The use of octonionic units as such would mean the introduction of the notion of octonionic tangent space structure. That the subalgebra generated by any two octonionic units is associative brings strongly in mind effective 2-dimensionality.

3. The tangent vector fields of space-time surface in the representation using octonionic units can be identified in the following manner. Map can be defined using 8-D octonionic coordinates defined by standard $M^4$ coordinates or possibly Hamilton-Jacobi coordinates and $CP_2$ complex coordinates for which U(2) is represented linearly. Gamma "matrices" for $H$ using octonionic representation are known in these coordinates. One can introduce the 8 components of the image of a given point under the octonion real-analytic map as new imbedding space coordinates. One can calculate the covariant gamma matrices of $H$ in these coordinates.

   What should check whether the octonionic gamma matrices associated with the four non-vanishing coordinates define quaternionic (and thus associative) algebra in the octonionic basis for the gamma matrices. Also the interpretation as a associative subspace of local Malcev algebra elements is possible and one should check whether if the algebra reduces to a quaternionic Lie-algebra. Local $SO(2) \times U(1)$ algebra should emerge in this manner.

4. Can one identify quaternionic imaginary units with vector fields generating SO(3) Lie algebra or its local variant? The Lie algebra of rotation generators defines algebra equivalent with that based on commutars of quaternionic units. Could the slicing of space-time sheet by time axis define local $SO(3)$ algebra? Light-like momentum direction and momentum direction and its dual define as their sum space-like vector field and together with vector fields defining transversal momentum directions they might generate a local $SO(3)$ algebra.

**Questions related to quaternion real-analyticity**

There are many poorly understood issues and and the following questions represent only some of very many such questions picked up rather randomly.

1. The above considerations are restricted to Minkowskian regions of space-time sheets. What happens in the Euclidian regions? Does the existence of light-like Beltrami field and its dual generalize to the existence of complex vector field and its dual?

2. It would be nice to find a justification for the notion of $CD$ from basic principles. The condition $q\overline{q} = 0$ implies $q = 0$ for quaternions. For hyper-quaternionic subspace of complexified quaternions obtained by Wick rotation it implies $q\overline{q} = 0$ corresponds the entire light-cone boundary. If n-point functions can be identified identified as products of quaternion valued n-point functions and their quaternionic conjugates, the outcome could be proportional to $1/q\overline{q}$ having poles at light-cone boundaries or $CD$ boundaries rather than at single point as in Euclidian realm.

3. This correspondence of points and light-cone boundaries would effectively identify the points at future and past light-like boundaries of $CD$ along light rays. Could one think that only the 2-sphere at which the upper and lower light-like boundaries of $CD$ meet remains after this identification. The structure would be homologically very much like $CP_2$ which is obtained by compactifying $E^4$ by adding a 2-sphere at infinity. Could this $CD - CP_2$ correspondence have some deep physical meaning? Do the boundaries of $CD$ somehow correspond to zeros and/or poles of quaternionic analytic functions in the Minkowskian realm? Could the light-like orbits of partonic 2-surfaces at which the signature of the induced metric changes correspond to similar counterparts of zeros or poles when the quaternion analytic variables is obtained as quaternion real analytic function of $H$ coordinates regarded as bi-quaternions?

4. Could braids correspond to zeros and poles of an octonion real-analytic function? Consider the partonic 2-surfaces at which the signature of the induced metric changes. The intersections of these surfaces with string world sheets at the ends of $CD$s. contain only complex and thus commutative points meaning that the imaginary part of bi-complex number representing quaternionic value of octonion real-analytic function vanishes. Braid ends would thus correspond to the origins of local complex coordinate patches. Finite measurement resolution would be forced by commutativity condition and correlate directly with the complexity of the partonic 2-surface measured by the minimal number of coordinate patches. Its realization would be as an upper bound on the number of braid strands. A natural expectation would be that only the values of n-point functions at these points contribute to scattering amplitudes. Number theoretic braids would be realized but in a manner different from the original guess.

**How complex analysis could generalize?**

One can make several questions related to the possible generalization of complex analysis to the quaternionic and octonionic situation.

1. Does the notion of analyticity in the sense that derivatives $df/dq$ and $df/do$ make sense hold true? The answer is "No": non-commutativity destroys all hopes about this kind of generalization. Octonion and quaternion real-analyticity has however a well-defined meaning.

2. Could the generalization of residue calculus by keeping interaction contours as 1-D curves make sense? Since residue formulas is the outcome of the fact that any analytic function $g$ can be written as $g = df/dz$ locally, the answer is "No".

3. Could one generalize of the residue calculus by replacing 1-dimensional curves with 4-D surfaces -possibly quaternionic 4-surfaces? Could one reduce the 4-D integral of quaternion analytic function to a double residue integral? This would be the case if the quaternion real-analytic function of $q = c_1 + Ic_2$ could be regarded as an analytic function of complex arguments $c_1$ and $c_2$. This is not the case. The product of two octonions decomposed to two quaternions as $o_i = q_{i1} + Iq_{i2}$ , $i = a, b$ reads as

$$o_a o_b = q_{a1}q_{b1} - \overline{q}_{a2}q_{b2} + I(\overline{q}_{a1}q_{b2} - q_{a2}q_{b1}) \ . \tag{4.5.6}$$

The conjugations result from the anticommutativity of imaginary parts and $I$. This formula gives similar formula for quaternions by restriction. As a special cas $o_a = o_b = q_1 + Iq_2$ one has

$$o^2 = q_1^2 - \overline{q}_2 q_2 + I(\overline{q}_1 q_2 - q_2 q_1)$$

From this it is clear that the real part of an octonion real-analytic function cannot be regarded as quaternion-analytic function unless one assumes that the imaginary part $q_2$ vanishes. By similar argument real part of quaternion real-analytic function $q = c_1 + Ic_2$ fails to be analytic unless one restricts the consideration to a surface at which one has $c_2 = 0$. These negative results are obviously consistent with the effective 2-dimensionality.

4. One must however notice that physicists use often what might be called analytization trick [A1] working if the non-analytic function $f(x, y) = f(z, \overline{z})$ is differentiable. The trick is to interpret $z$ and $\overline{z}$ as independent variables. In the recent case this is rather natural. Wick rotation could be used to transform the integral over the space-time sheet to integral in quaternionic domain. For 4-dimensional integrals of quaternion real-analytic function with integration measure proportional to $dc_1 d\overline{c}_1 dc_2 d\overline{c}_2$ one could formally define the integral using multiple residue integration with four complex variables. The constraint is that the poles associated with $c_i$ and $\overline{c}_i$ are conjugates of each other. Quaternion real-analyticity should guarantee this. This would of course be a *definition* of four-dimensional integral and might work for the 4-D generalization of conformal field theory.

Mandelbrot and Julia sets are fascinating fractals and already now more or less a standard piece of complex analysis. The fact that the iteration of octonion real-analytic map produces a sequence of space-time surfaces and partonic 2-surfaces encourages to ask whether these notions -and more generally, the dynamics based on iteration of analytic functions - might have a higher-dimensional generalization in the proposed framework.

1. The canonical Mandelbrot set corresponds to the set of the complex parameters $c$ in $f(z) = z^2+c$ for which iterates of $z = 0$ remain finite. In octonionic and quaternionic real-analytic case $c$ would be real so that one would obtain only the intersection of the Mandelbrot set with real axes and the outcome would be rather uninteresting. This is true quite generally.

2. Julia set corresponds to the boundary of the Fatou set in which the dynamics defined by the iteration of $f(z)$ by definition behaves in a regular manner. In Julia set the behavior is chaotic. Julia set can be defined as a set of complex plane resulting by taking inverse images of a generic point belonging to the Julia set. For polynomials Julia set is the boundary of the region in which iterates remain finite. In Julia set the dynamics defined by the iteration is chaotic.

   Julia set could be interesting also in the recent case since it could make sense for real analytic functions of both quaternions and octonions, and one might hope that the dynamics determined by the iterations of octonion real-analytic function could have a physical meaning as a space-time correlate for quantal self-organization by quantum jump in TGD framework. Single step in iteration would be indeed a very natural space-time correlate for quantum jump. The restriction of octonion analytic functions to string world sheets should produce the counterparts of the ordinary Julia sets since these surfaces are mapped to themselves under iteration and octonion real-analytic functions reduces to ordinary complex real-analytic functions at them. Therefore one might obtain the counterparts of Julia sets in 4-D sense as extensions of ordinary Julia sets. These extensions would be 3-D sets obtained as piles of ordinary Julia sets labelled by partonic 2-surfaces.

## 4.6 Does modified Dirac action define the fundamental action principle?

Although quantum criticality in principle predicts the possible values of Kähler coupling strength, one might hope that there exists even more fundamental approach involving no coupling constants and predicting even quantum criticality and realizing quantum gravitational holography. The Dirac determinant associated with the modified Dirac action is an excellent candidate in this respect.

The original working hypothesis was that Dirac determinant defines the vacuum functional of the theory having interpretation as the exponent of Kähler function of world of classical worlds (WCW) expressible and that Kähler function reduces to Kähler action for a preferred extremal of Kähler action.

### 4.6.1 What are the basic equations of quantum TGD?

A good place to start is to as what might the basic equations of quantum TGD. There are two kinds of equations at the level of space-time surfaces.

1. Purely classical equations define the dynamics of the space-time sheets as preferred extremals of Kähler action. Preferred extremals are quantum critical in the sense that second variation vanishes for critical deformations representing zero modes. This condition guarantees that corresponding fermionic currents are conserved. There is infinite hierarchy of these currents and they define fermionic counterparts for zero modes. Space-time sheets can be also regarded as hyper-quaternionic surfaces. What these statements precisely mean has become clear only during this year. A rigorous proof for the equivalence of these two identifications is still lacking.

2. The purely quantal equations are associated with the representations of various super-conformal algebras and with the modified Dirac equation. The requirement that there are deformations of the space-time surface -actually infinite number of them- giving rise to conserved fermionic charges implies quantum criticality at the level of Kähler action in the sense of critical deformations. The precise form of the modified Dirac equation is not however completely fixed without further input. Quantal equations involve also generalized Feynman rules for $M$-matrix generalizing $S$-matrix to a "complex square root" of density matrix and defined by time-like entanglement coefficients between positive and negative energy parts of zero energy states is certainly the basic goal of quantum TGD.

3. The notion of weak electric-magnetic duality generalizing the notion of electric-magnetic duality [K19] , [L9] leads to a detailed understanding of how TGD reduces to almost topological quantum field theory [K19] , [L9] . If Kähler current defines Beltrami flow [B29] it is possible to find a gauge in which Coulomb contribution to Kähler action vanishes so that it reduces to Chern-Simons term. If light-like 3-surfaces and ends of space-time surface are extremals of Chern-Simons action also effective 2-dimensionality is realized. The condition that the theory reduces to almost topological QFT and the hydrodynamical character of field equations leads to a detailed ansatz for the general solution of field equations and also for the solutions of the modified Dirac equation relying on the notion of Beltrami flow for which the flow parameter associated with the flow lines defined by a conserved current extends to a global coordinate. This makes the theory is in well-defined sense completely integrable. Direct connection with massless theories emerges: every conserved Beltrami currents corresponds to a pair of scalar functions with the first one satisfying massless d'Alembert equation in the induced metric. The orthogonality of the gradients of these functions allows interpretation in terms of polarization and momentum directions. The Beltrami flow property can be also seen as one aspect of quantum criticality since the conserved currents associated with critical deformations define this kind of pairs.

4. The hierarchy of Planck constants provides also a fresh view to the quantum criticality. The original justification for the hierarchy of Planck constants came from the indications that Planck constant could have large values in both astrophysical systems involving dark matter and also in biology. The realization of the hierarchy in terms of the singular coverings and possibly also factor spaces of $CD$ and $CP_2$ emerged from consistency conditions. It however seems that TGD actually predicts this hierarchy of covering spaces. The extreme non-linearity of the field equations defined by Kähler action means that the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is 1-to-many. This leads naturally to the introduction of the covering space of $CD \times CP_2$, where $CD$ denotes causal diamond defined as intersection of future and past directed light-cones.

At the level of WCW there is the generalization of the Dirac equation which can be regarded as a purely classical Dirac equation. The modified Dirac operators associated with quarks and leptons carry fermion number but the Dirac equations are well-defined. An orthogonal basis of solutions of these Dirac operators define in zero energy ontology a basis of zero energy states. The $M$-matrices defining entanglement between positive and negative energy parts of the zero energy state define what can be regarded as analogs of thermal S-matrices. The M-matrices associated with the solution basis of the WCW Dirac equation define by their orthogonality unitary U-matrix between zero energy states. This matrix finds the proper interpretation in TGD inspired theory of consciousness. WCW Dirac equation as the analog of super-Virasoro conditions for the "gamma fields" of superstring models defining super counterparts of Virasoro generators was the main focus during earlier period of quantum TGD but has not received so much attention lately and will not be discussed in this chapter.

### 4.6.2 Quantum criticality and modified Dirac action

The precise mathematical formulation of quantum criticality has remained one of the basic challenges of quantum TGD. The question leading to a considerable progress in the problem was simple: Under what conditions the modified Dirac action allows to assign conserved fermionic currents with the deformations of the space-time surface? The answer was equally simple: These currents exists only if these deformations correspond to vanishing second variations of Kähler action - which is what criticality is. The vacuum degeneracy of Kähler action strongly suggests that the number of critical deformations is always infinite and that these deformations define an infinite inclusion hierarchy of super-conformal algebras. This inclusion hierarchy would correspond to a fractal hierarchy of breakings of super-conformal symmetry generalizing the symmetry breaking hierarchies of gauge theories. These super-conformal inclusion hierarchies would realize the inclusion hierarchies for hyper-finite factors of type $II_1$.

#### Quantum criticality and fermionic representation of conserved charges associated with second variations of Kähler action

It is rather obvious that TGD allows a huge generalizations of conformal symmetries. The development of the understanding of conservation laws has been slow. Modified Dirac action provides excellent candidates for quantum counterparts of Noether charges. Unfortunately, the isometry charges vanish for Cartan algebras. The only manner to obtain non-trivial isometry charges is to add a direct coupling to the charges in Cartan algebra as will be found later. This addition involves Chern-Simons Dirac action so that the original intuition guided by almost TQFT idea was not wrong after all.

*1. Conservation of the fermionic current requires the vanishing of the second variation of Kähler action*

1. The modified Dirac action assigns to a deformation of the space-time surface a conserved charge expressible as bilinears of fermionic oscillator operators only if the first variation of the modified Dirac action under this deformation vanishes. The vanishing of the first variation for the modified Dirac action is equivalent with the vanishing of the second variation for the Kähler action. This can be seen by the explicit calculation of the second variation of the modified Dirac action and by performing partial integration for the terms containing derivatives of $\Psi$ and $\overline{\Psi}$ to give a total divergence representing the difference of the charge at upper and lower boundaries of the causal diamond plus a four-dimensional integral of the divergence term defined as the integral of the quantity

$$\begin{aligned} \Delta S_D &= \overline{\Psi}\Gamma^k D_\alpha J^\alpha_k \Psi \ , \\ J^\alpha_k &= \frac{\partial^2 L_K}{\partial h^k_\alpha \partial h^l_\beta}\delta h^k_\beta + \frac{\partial^2 L_K}{\partial h^k_\alpha \partial h^l}\delta h^l \ . \end{aligned} \tag{4.6.1}$$

Here $h^k_\beta$ denote partial derivative of the imbedding space coordinate with respect to space-time coordinates. This term must vanish:

$$D_\alpha J^\alpha_k = 0 \ .$$

The condition states the vanishing of the second variation of Kähler action. This can of course occur only for preferred deformations of $X^4$. One could consider the possibility that these deformations vanish at light-like 3-surfaces or at the boundaries of CD. Note that covariant divergence is in question so that $J_k^\alpha$ does not define conserved classical charge in the general case.

2. It is essential that the modified Dirac equation holds true so that the modified Dirac action vanishes: this is needed to cancel the contribution to the second variation coming from the determinant of the induced metric. The condition that the modified Dirac equation is satisfied for the deformed space-time surface requires that also $\Psi$ suffers a transformation determined by the deformation. This gives

$$\delta\Psi = -\frac{1}{D}\times\Gamma^k J_k^\alpha\Psi \quad . \tag{4.6.2}$$

Here $1/D$ is the inverse of the modified Dirac operator defining the counterpart of the fermionic propagator.

3. The fermionic conserved currents associated with the deformations are obtained from the standard conserved fermion current

$$J^\alpha = \overline{\Psi}\Gamma^\alpha\Psi \quad . \tag{4.6.3}$$

Note that this current is conserved only if the space-time surface is extremal of Kähler action: this is also needed to guarantee Hermiticity and same form for the modified Dirac equation for $\Psi$ and its conjugate as well as absence of mass term essential for super-conformal invariance [A23, A25] . Note also that ordinary divergence rather only covariant divergence of the current vanishes.

The conserved currents are expressible as sums of three terms. The first term is obtained by replacing modified gamma matrices with their increments in the deformation keeping $\Psi$ and its conjugate constant. Second term is obtained by replacing $\Psi$ with its increment $\delta\Psi$. The third term is obtained by performing same operation for $\delta\overline{\Psi}$.

$$J^\alpha = \overline{\Psi}\Gamma^k J_k^\alpha\Psi + \overline{\Psi}\hat{\Gamma}^\alpha\delta\Psi + \delta\overline{\Psi}\hat{\Gamma}^\alpha\Psi \quad . \tag{4.6.4}$$

These currents provide a representation for the algebra defined by the conserved charges analogous to a fermionic representation of Kac-Moody algebra [A13] .

4. Also conserved super charges corresponding to super-conformal invariance are obtained. The first class of super currents are obtained by replacing $\Psi$ or $\overline{\Psi}$ right-handed neutrino spinor or its conjugate in the expression for the conserved fermion current and performing the above procedure giving two terms since nothing happens to the covariantly constant right handed-neutrino spinor. Second class of conserved currents is defined by the solutions of the modified Dirac equation interpreted as c-number fields replacing $\Psi$ or $\overline{\Psi}$ and the same procedure gives three terms appearing in the super current.

5. The existence of vanishing of second variations is analogous to criticality in systems defined by a potential function for which the rank of the matrix defined by second derivatives of the potential function vanishes at criticality. Quantum criticality becomes the prerequisite for the existence of quantum theory since fermionic anti-commutation relations in principle can be fixed from the condition that the algebra in question is equivalent with the algebra formed by the vector fields defining the deformations of the space-time surface defining second variations. Quantum criticality in this sense would also select preferred extremals of Kähler action as analogs of Bohr orbits and the the spectrum of preferred extremals would be more or less equivalent with the expected existence of infinite-dimensional symmetry algebras.

*2. About the general structure of the algebra of conserved charges*

Some general comments about the structure of the algebra of conserved charges are in order.

1. Any Cartan algebra of the isometry group $P \times SU(3)$ (there are two types of them for $P$ corresponding to linear and cylindrical Minkowski coordinates) defines critical deformations (one could require that the isometries respect the geometry of $CD$). The corresponding charges are conserved but vanish since the corresponding conjugate coordinates are cyclic for the Kähler metric and Kähler form so that the conserved current is proportional to the gradient of a Killing vector field which is constant in these coordinates. Therefore one cannot represent isometry charges as fermionic bilinears. Four-momentum and color quantum numbers are defined for Kähler action as classical conserved quantities but this is probably not enough. This can be seen as a problem.

   (a) Four-momentum and color Cartan algebra emerge naturally in the representations of super-conformal algebras. In the case of color algebra the charges in the complement of the Cartan algebra can be constructed in standard manner as extension of those for the Cartan algebra using free field representation of Kac-Moody algebras. In string theories four-momentum appears linearly in bosonic Kac-Moody generators and in Sugawara construction [A68] of super Virasoro generators as bilinears of bosonic Kac-Moody generators and fermionic super Kac-Moody generators [A13] . Also now quantized transversal parts for $M^4$ coordinates could define a second quantized field having interpretation as an operator acting on spinor fields of WCW. The angle coordinates conjugate to color isospin and hyper charge take the role of $M^4$ coordinates in case of $CP_2$.

   (b) Somehow one should be able to feed the information about the super-conformal representation of the isometry charges to the modified Dirac action by adding to it a term coupling fermionic current to the Cartan charges in general coordinate invariant and isometry invariant manner. As will be shown later, this is possible. The interpretation is as measurement interaction guaranteeing also the stringy character of the fermionic propagators. The values of the couplings involved are fixed by the condition of quantum criticality assumed in the sense that Kähler function of WCW suffers only a $U(1)$ gauge transformation $K \rightarrow K + f + \overline{f}$, where $f$ is a holomorphic function of WCW coordinates depending also on zero modes.

   (c) The simplest addition involves the modified gamma matrices defined by a Chern-Simon term at the light-like wormhole throats and is sum of Chern-Simons Dirac action and corresponding coupling term linear in Cartan charges assignable to the partonic 2-surfaces at the ends of the throats. Hence the modified Dirac equation in the interior of the space-time sheet is not affected and nothing changes as far as quantum criticality in interior is considered.

2. The action defined by four-volume gives a first glimpse about what one can expect. In this case modified gamma matrices reduce to the induced gamma matrices. Second variations satisfy d'Alembert type equation in the induced metric so that the analogs of massless fields are in question. Mass term is present only if some dimensions are compact. The vanishing of excitations at light-like boundaries is a natural boundary condition and might well imply that the solution spectrum could be empty. Hence it is quite possible that four-volume action leads to a trivial theory.

3. For the vacuum extremals of Kähler action the situation is different. There exists an infinite number of second variations and the classical non-determinism suggests that deformations vanishing at the light-like boundaries exist. For the canonical imbedding of $M^4$ the equation for second variations is trivially satisfied. If the $CP_2$ projection of the vacuum extremal is one-dimensional, the second variation contains a on-vanishing term and an equation analogous to massless d'Alembert equation for the increments of $CP_2$ coordinates is obtained. Also for the vacuum extremals of Kähler action with 2-D $CP_2$ projection all terms involving induced Kähler form vanish and the field equations reduce to d'Alembert type equations for $CP_2$ coordinates. A possible interpretation is as the classical analog of Higgs field. For the deformations of non-vacuum extremals this would suggest the presence of terms analogous to mass terms: these kind

of terms indeed appear and are proportional to $\delta s^k$. $M^4$ degrees of freedom decouple completely and one obtains QFT type situation.

4. The physical expectation is that at least for the vacuum extremals the critical manifold is infinite-dimensional. The notion of finite measurement resolution suggests infinite hierarchies of inclusions of hyper-finite factors of type $II_1$ possibly having interpretation in terms of inclusions of the super conformal algebras defined by the critical deformations.

5. The properties of Kähler action give support for this expectation. The critical manifold is infinite-dimensional in the case of vacuum extremals. Canonical imbedding of $M^4$ would correspond to maximal criticality analogous to that encountered at the tip of the cusp catastrophe. The natural guess would be that as one deforms the vacuum extremal the previously critical degrees of freedom are transformed to non-critical ones. The dimension of the critical manifold could remain infinite for all preferred extremals of the Kähler action. For instance, for cosmic string like objects any complex manifold of $CP_2$ defines cosmic string like objects so that there is a huge degeneracy is expected also now. For $CP_2$ type vacuum extremals $M^4$ projection is arbitrary light-like curve so that also now infinite degeneracy is expected for the deformations.

*3. Critical super algebra and zero modes*

The relationship of the critical super-algebra to configuration space geometry is interesting.

1. The vanishing of the second variation plus the identification of Kähler function as a Kähler action for preferred extremals means that the critical variations are orthogonal to all deformations of the space-time surface with respect to the configuration space metric and thus correspond to zero modes. This conforms with the fact that configuration space metric vanishes identically for canonically imbedded $M^4$. Zero modes do not seem to correspond to gauge degrees of freedom so that the super-conformal algebra associated with the zero modes has genuine physical content.

2. Since the action of $X^4$ local Hamiltonians of $\delta M^4_\times CP_2$ corresponds to the action in quantum fluctuating degrees of freedom, critical deformations cannot correspond to this kind of Hamiltonians.

3. The notion of finite measurement resolution suggests that the degrees of freedom which are below measurement resolution correspond to vanishing gauge charges. The sub-algebras of critical super-conformal algebra for which charges annihilate physical states could correspond to this kind of gauge algebras.

4. The conserved super charges associated with the vanishing second variations cannot give configuration space metric as their anti-commutator. This would also lead to a conflict with the effective 2-dimensionality stating that the configuration space line-element is expressible as sum of contribution coming from partonic 2-surfaces as also with fermionic anti-commutation relations.

*4. Connection with quantum criticality*

The vanishing of the second variation for some deformations means that the system is critical, in the recent case quantum critical. Basic example of criticality is bifurcation diagram for cusp catastrophe. For some mysterious reason I failed to realize that quantum criticality realized as the vanishing of the second variation makes possible a more or less unique identification of preferred extremals and considered alternative identifications such as absolute minimization of Kähler action which is just the opposite of criticality. Both the super-symmetry of $D_K$ and conservation Dirac Noether currents for modified Dirac action have thus a connection with quantum criticality.

1. Finite-dimensional critical systems defined by a potential function $V(x^1, x^2, ..)$ are characterized by the matrix defined by the second derivatives of the potential function and the rank of system classifies the levels in the hierarchy of criticalities. Maximal criticality corresponds to the complete vanishing of this matrix. Thom's catastrophe theory classifies these hierarchies, when the numbers of behavior and control variables are small (smaller than 5). In the recent case the situation is infinite-dimensional and the criticality conditions give additional field equations as existence of vanishing second variations of Kähler action.

2. The vacuum degeneracy of Kähler action allows to expect that this kind infinite hierarchy of criticalities is realized. For a general vacuum extremal with at most 2-D $CP_2$ projection the matrix defined by the second variation vanishes because $J_{\alpha\beta} = 0$ vanishes and also the matrix $(J^{\alpha}_{\ k} + J_k^{\ \alpha})(J^{\beta}_{\ l} + J_l^{\ \beta})$ vanishes by the antisymmetry $J^{\alpha}_{\ k} = -J_k^{\ \alpha}$. Recall that the formulation of Equivalence Principle in string picture demonstrated that the reduction of stringy dynamics to that for free strings requires that second variation with respect to $M^4$ coordinates vanish. This condition would guarantee the conservation of fermionic Noether currents defining gravitational four-momentum and other Poincare quantum numbers but not those for gravitational color quantum numbers. Encouragingly, the action of $CP_2$ type vacuum extremals having random light-like curve as $M^4$ projection have vanishing second variation with respect to $M^4$ coordinates (this follows from the vanishing of Kähler energy momentum tensor, second fundamental form, and Kähler gauge current). In this case however the momentum is vanishing.

3. Conserved bosonic and fermionic Noether charges would characterize quantum criticality. In particular, the isometries of the imbedding space define conserved currents represented in terms of the fermionic oscillator operators if the second variations defined by the infinitesimal isometries vanish for the modified Dirac action. For vacuum extremals the dimension of the critical manifold is infinite: maybe there is hierarchy of quantum criticalities for which this dimension decreases step by step but remains always infinite. This hierarchy could closely relate to the hierarchy of inclusions of hyper-finite factors of type $II_1$. Also the conserved charges associated with Super-symplectic and Super Kac-Moody algebras would require infinite-dimensional critical manifold defined by the spectrum of second variations.

4. Phase transitions are characterized by the symmetries of the phases involved with the transitions, and it is natural to expect that dynamical symmetries characterize the hierarchy of quantum criticalities. The notion of finite quantum measurement resolution based on the hierarchy of Jones inclusions indeed suggests the existence of a hierarchy of dynamical gauge symmetries characterized by gauge groups in ADE hierarchy [K18] with degrees of freedom below the measurement resolution identified as gauge degrees of freedom.

5. A breakthrough in understanding of the criticality was the discovery that the realization that the hierarchy of singular coverings of $CD \times CP_2$ needed to realize the hierarchy of Planck constants could correspond directly to a similar hierarchy of coverings forced by the factor that classical canonical momentum densities correspond to several values of the time derivatives of the imbedding space coordinates led to a considerable progress if the understanding of the relationship between criticality and hierarchy of Planck constants [K24] , [L4] . Therefore the problem which led to the geometrization program of quantum TGD, also allowed to reduce the hierarchy of Planck constants introduced on basis of experimental evidence to the basic quantum TGD. One can say that the 3-surfaces at the ends of $CD$ *resp.* wormhole throats are critical in the sense that they are unstable against splitting to $n_b$ *resp.* $n_a$ surfaces so that one obtains space-time surfaces which can be regarded as surfaces in $n_a \times n_b$ fold covering of $CD \times CP_2$. This allows to understand why Planck constant is effectively replaced with $n_a n_b \hbar_0$ and explains charge fractionization.

#### Preferred extremal property as classical correlate for quantum criticality, holography, and quantum classical correspondence

The Noether currents assignable to the modified Dirac equation are conserved only if the first variation of the modified Dirac operator $D_K$ defined by Kähler action vanishes. This is equivalent with the vanishing of the second variation of Kähler action -at least for the variations corresponding to dynamical symmetries having interpretation as dynamical degrees of freedom which are below measurement resolution and therefore effectively gauge symmetries.

The vanishing of the second variation in interior of $X^4(X^3_l)$ is what corresponds exactly to quantum criticality so that the basic vision about quantum dynamics of quantum TGD would lead directly to a precise identification of the preferred extremals. Something which I should have noticed for more than decade ago! The question whether these extremals correspond to absolute minima remains however open.

The vanishing of second variations of preferred extremals -at least for deformations representing dynamical symmetries, suggests a generalization of catastrophe theory of Thom, where the rank of the matrix defined by the second derivatives of potential function defines a hierarchy of criticalities with the tip of bifurcation set of the catastrophe representing the complete vanishing of this matrix. In the recent case this theory would be generalized to infinite-dimensional context. There are three kind of variables now but quantum classical correspondence (holography) allows to reduce the types of variables to two.

1. The variations of $X^4(X_l^3)$ vanishing at the intersections of $X^4(X_l^3)$ wth the light-like boundaries of causal diamonds $CD$ would represent behavior variables. At least the vacuum extremals of Kähler action would represent extremals for which the second variation vanishes identically (the "tip" of the multi-furcation set).

2. The zero modes of Kähler function would define the control variables interpreted as classical degrees of freedom necessary in quantum measurement theory. By effective 2-dimensionality (or holography or quantum classical correspondence) meaning that the configuration space metric is determined by the data coming from partonic 2-surfaces $X^2$ at intersections of $X_l^3$ with boundaries of $CD$, the interiors of 3-surfaces $X^3$ at the boundaries of $CD$s in rough sense correspond to zero modes so that there is indeed huge number of them. Also the variables characterizing 2-surface, which cannot be complexified and thus cannot contribute to the Kähler metric of configuration space represent zero modes. Fixing the interior of the 3-surface would mean fixing of control variables. Extremum property would fix the 4-surface and behavior variables if boundary conditions are fixed to sufficient degree.

3. The complex variables characterizing $X^2$ would represent third kind of variables identified as quantum fluctuating degrees of freedom contributing to the configuration space metric. Quantum classical correspondence requires 1-1 correspondence between zero modes and these variables. This would be essentially holography stating that the 2-D "causal boundary" $X^2$ of $X^3(X^2)$ codes for the interior. Preferred extremal property identified as criticality condition would realize the holography by fixing the values of zero modes once $X^2$ is known and give rise to the holographic correspondence $X^2 \rightarrow X^3(X^2)$. The values of behavior variables determined by extremization would fix then the space-time surface $X^4(X_l^3)$ as a preferred extremal.

4. Clearly, the presence of zero modes would be absolutely essential element of the picture. Quantum criticality, quantum classical correspondence, holography, and preferred extremal property would all represent more or less the same thing. One must of course be very cautious since the boundary conditions at $X_l^3$ involve normal derivative and might bring in delicacies forcing to modify the simplest heuristic picture.

5. There is a possible connection with the notion of self-organized criticality [B10] introduced to explain the behavior of systems like sand piles. Self-organization in these systems tends to lead "to the edge". The challenge is to understand how system ends up to a critical state, which by definition is unstable. Mechanisms for this have been discovered and based on phase transitions occurring in a wide range of parameters so that critical point extends to a critical manifold. In TGD Universe quantum criticality suggests a universal mechanism of this kind. The criticality for the preferred extremals of Kähler action would mean that classically all systems are critical in well-defined sense and the question is only about the degree of criticality. Evolution could be seen as a process leading gradually to increasingly critical systems. One must however distinguish between the criticality associated with the preferred extremals of Kähler action and the criticality caused by the spin glass like energy landscape like structure for the space of the maxima of Kähler function.

### 4.6.3 Handful of problems with a common resolution

Theory building could be compared to pattern recognition or to a solving a crossword puzzle. It is essential to make trials, even if one is aware that they are probably wrong. When stares long enough to the letters which do not quite fit, one suddenly realizes what one particular crossword must actually be and it is soon clear what those other crosswords are. In the following I describe an example in which this analogy is rather concrete. Let us begin by listing the problems.

1. The condition that modified Dirac action allows conserved charges leads to the condition that the symmetries in question give rise to vanishing second variations of Kähler action. The interpretation is as quantum criticality and there are good arguments suggesting that the critical symmetries define an infinite-dimensional super-conformal algebra forming an inclusion hierarchy related to a sequence of symmetry breakings closely related to a hierarchy of inclusions of hyper-finite factors of types $II_1$ and $III_1$. This means an enormous generalization of the symmetry breaking patterns of gauge theories.

   There is however a problem. For the translations of $M^4$ and color hyper charge and isospin (more generally, any Cartan algebra of $P \times SU(3)$) the resulting fermionic charges vanish. The trial for the crossword in absence of nothing better would be the following argument. By the abelianity of these charges the vanishing of quantal representation of four-momentum and color Cartan charges is not a problem and that classical representation of these charges or their super-conformal representation is enough.

2. Modified Dirac equation is satisfied in the interior of space-time surface always. This means that one does not obtain off-mass shell propagation at all in 4-D sense. Effective 2-dimensionality suggests that off mass shell propagation takes place along wormhole throats. The reduction to almost topological QFT with Kähler function reducing to Chern-Simonst type action implied by the weak form of electric-magnetic duality and a proper gauge choice for the induced Kähler gauge potential implies effective 3-dimensionality at classical level. This inspires the question whether Chern-Simons type action resulting from an instanton term could define the modified gamma matrices appearing in the 3-D modified Dirac action associated with wormhole throats and the ends of the space-time sheet at the boundaries of $CD$.

   The assumption that modified Dirac equation is satisfied also at the ends and wormhole throats would realize effective 2-dimensionality as conditions on the boundary values of the 4-D Dirac equation but would would not allow off mass shell propagation. Therefore one could argue that effective 2-dimensionality in this sense holds true only for incoming and outgoing particles.

   The reduction of Kähler action to Chern-Simons term together with effective 2-dimensionality suggests that Kähler function corresponds to an extremum of this action with a constraint term due to the weak form of electric-magnetic duality. Without this term the extrema of Chern-Simons action have 2-D $CP_2$ projection not consistent with the weak form of electric-magnetic duality. The extrema are not maxima of Kähler function: they are obtained by varying with respect to tangent space data of the partonic 2-surfaces. Lagrange multiplier term induces also to the modified gamma matrices a contribution which is of the same general form as for any general coordinate invariant action.

3. Quantum classical correspondence requires that the geometry of the space-time sheet should correlate with the quantum numbers characterizing positive (negative) energy part of the quantum state. One could argue that by multiplying WCW spinor field by a suitable phase factor depending on the charges of the state, the correspondence follows from stationary phase approximation. This crossword looks unconvincing. A more precise connection between quantum and classical is required.

4. In quantum measurement theory classical macroscopic variables identified as degrees of freedom assignable to the interior of the space-time sheet correlate with quantum numbers. Stern Gerlach experiment is an excellent example of the situation. The generalization of the imbedding space concept by replacing it with a book like structure implies that imbedding space geometry at given page and for given causal diamond ($CD$) carries information about the choice of the quantization axes (preferred plane $M^2$ of $M^4$ *resp.* geodesic sphere of $CP_2$ associated with singular covering/factor space of $CD$ *resp.* $CP_2$ ). This is a big step but not enough. Modified Dirac action as such does not seem to provide any hint about how to achieve this correspondence. One could even wonder whether dissipative processes or at least the breaking of $T$ and $CP$ characterizing the outcome of quantum jump sequence should have space-time correlate. How to achieve this?

Each of these problems makes one suspect that something is lacking from the modified Dirac action: there should exist an elegant manner to feed information about quantum numbers of the state to the modified Dirac action in turn determining vacuum functional as an exponent Kähler function identified as Kähler action for the preferred extremal assumed to be dictated by by quantum criticality and equivalently by hyper-quaternionicity.

This observation leads to what might be the correct question. Could a general coordinate invariant and Poincare invariant modification of the modified Dirac action consistent with the vacuum degeneracy of Kähler action allow to achieve this information flow somehow? In the following one manner to achieve this modification is discussed. It must be however emphasized that I have considered many alternatives and the one discussed below finds its justification only from the fact that it is the simplest one found hitherto.

### The identification of the measurement interaction term

The idea is simple: add to the modified Dirac action a term which is analogous to the Dirac action in $M^4 \times CP_2$. One can consider two options according to whether the term is assigned with interior or with a 3-D light-like 3-surface and last years have been continual argumentation about which option is the correct one.

1. The additional term would be essentially the analog of the ordinary Dirac action at the imbedding space level.

$$\begin{aligned} S_{int} &= \sum_A Q_A \int \overline{\Psi} g^{AB} j_{B\alpha} \hat{\Gamma}^\alpha \Psi \sqrt{g} d^4x \ , \\ g_{AB} &= j_A^k h_{kl} j_B^l \ , \ \ g^{AB} g_{BC} = \delta_C^A \ , \\ j_{B\alpha} &= j_B^k h_{kl} \partial_\alpha h^l \ . \end{aligned} \tag{4.6.5}$$

   The sum is over isometry charges $Q_A$ interpreted as quantal charges and $j^{Ak}$ denotes the Killing vector field of the isometry. $g^{AB}$ is the inverse of the tensor $g_{AB}$ defined by the local inner products of Killing vectors fields in $M^4$ and $CP_2$. The space-time projections of the Killing vector fields $j_{B\alpha}$ have interpretation as classical color gauge potentials in the case of $SU(3)$. In $M^4$ degrees of freedom and for Cartan algebra of $SU(3)$ $j_{B\alpha}$ reduce to the gradients of linear $M^4$ coordinates in case of translations. Modified gamma matrices could be assigned to Kähler action or its instanton term or with Chern-Simons action.

2. The added term containing quantal charges must make sense in the modified Dirac equation. This requires that the physical state is an eigenstate of momentum and color charges. This allows only color hyper-charge and color isospin so that there is no hope of obtaining exactly the stringy formula for the propagator. The modified Dirac operator is given by

$$\begin{aligned} D &= D + D_{int} = \hat{\Gamma}^\alpha D_\alpha + \hat{\Gamma}^\alpha \sum_A Q_A g^{AB} j_{B\alpha} \\ &= \hat{\Gamma}^\alpha (D_\alpha + \partial_\alpha \phi) \ , \ \ \partial_\alpha \phi = \sum_A Q_A g^{AB} j_{B\alpha} \ . \end{aligned} \tag{4.6.6}$$

   The conserved fermionic isometry currents are

$$J^{A\alpha} = \sum_B Q_B \overline{\Psi} g^{BC} j_C^k h_{kl} j_A^l \hat{\Gamma}^\alpha \Psi = Q_A \overline{\Psi} \hat{\Gamma}^\alpha \Psi \ . \tag{4.6.7}$$

   Here the sum is restricted to a Cartan sub-algebra of Poincare group and color group.

3. An important restriction is that by four-dimensionality of $M^4$ and $CP_2$ the rank of $g_{AB}$ is 4 so that $g^{AB}$ exists only when one considers only four conserved charges. In the case of $M^4$ this is achieved by a restriction to translation generators $Q_A = p_A$. $g_{AB}$ reduces to Minkowski metric and Killing vector fields are constants. The Cartan sub-algebra could be however replaced by any four commuting charges in the case of Poincare algebra (second one corresponds to time translation plus translation, boost and rotation in given direction). In the case of $SU(3)$ one must restrict the consideration either to $U(2)$ sub-algebra or its complement. $CP_2 = SU(3)/SU(2)$ decomposition would suggest the complement as the correct choice. One can indeed build the generators of $U(2)$ as commutators of the charges in the complement. On the other hand, Cartan algebra is enough in free field construction of Kac-Moody algebras.

4. What is remarkable that for the Cartan algebra of $M^4 \times SU(3)$ the measurement interaction term is equivalent with the addition of gauge part $\partial_\alpha \phi$ of the induced Kähler gauge potential $A_\alpha$. This property might hold true for any measurement interaction term. This also suggests that the change in Kähler function is only the transformation $A_\alpha \to A_\alpha + \partial_\alpha \phi$, $\partial_\alpha \phi = \sum_A Q_A g^{AB} j_{B\alpha}$.

5. Recall that the $\phi$ for $U(1)$ gauge transformations respecting the vanishing of the Coulomb interaction term of Kähler action [K24] , [L4] the current $j_K^\alpha \phi$ is conserved, which implies that the change of the Kähler action is trivial. These properties characterize the gauge transformations respecting the gauge in which Coulombic interaction term of the Kähler action vanishes so that Kähler action reduces to 3-dimensional generalized Chern-Simons term if the weak form of electric-magnetic duality holds true guaranteeing among other things that the induced Kähler field is not too singular at the wormhole throats [K24] , [L4] . The scalar function assignable to the measurement interaction terms does not have this property and this is what is expected since it must change the value of the Kähler function and therefore affect the preferred extremal.

Concerning the precise form of the modified Dirac action the basic clue comes from the observation that the measurement interaction term corresponds to the addition of a gauge part to the induced $CP_2$ Kähler gauge potential $A_\alpha$. The basic question is what part of the action one assigns the measurement interaction term.

1. One could define the measurement interaction term using either the four-dimensional instanton term or its reduction to Chern-Simons terms. The part of Dirac action defined by the instanton term in the interior does not reduce to a 3-D form unless the Dirac equation defined by the instanton term is satisfied : this cannot be true. Hence Chern-Simons term is the only possibility.

   The classical field equations associated with the Chern-Simons term cannot be assumed since they would imply that the $CP_2$ projection of the wormhole throat and space-like 3-surface are 2-dimensional. This might hold true for space-like 3-surfaces at the ends of $CD$ and incoming and outgoing particles but not for off mass shell particles. This is however not a problem since $D_\alpha \hat{\Gamma}^\alpha_{C-S}$ for the modified gamma matrices for Chern-Simons action does not contain second derivatives. This is due to the topological character of this term. For Kähler action second derivatives are present and this forces extremal property of Kähler action in the modified Dirac Kähler action so that classical physics results as a consistency condition.

2. If one assigns measurement interaction term to both $D_K$ and $D_{C-S}$ the measurement interaction corresponds to a mere gauge transformation for $AS_\alpha$ and is trivial. Therefore it seems that one must choose between $D_K$ or $D_{C-S}$. At least formally the measurement interaction term associated with $D_K$ is gauge equivalent with its negative $D_{C-S}$. The addition of the measurement interaction to $D_K$ changes the basis for the 4-D induced spinors by the phase $exp(-iQK\phi)$ and therefore also the basis for the generalized eigenstates of $D_{C-S}$ and this brings in effectively the measurement interaction term affecting the Dirac determinant.

3. The definition of Dirac determinant should be in terms of Chern-Simons action induced by the instanton term and identified as a product of the generalized eigenvalues of this operator. The modified Dirac equation for $\Psi$ is consistent with that for its conjugate if the coefficient of the instanton term is real and one uses the Dirac action $\overline{\Psi}(D^{\rightarrow} - D^{\leftarrow})\Psi$ giving modified Dirac equation as

$$D_{C-S}\Psi + \frac{1}{2}(D_\alpha \hat{\Gamma}^\alpha_{C-S})\Psi = 0 \ . \quad (4.6.8)$$

As noticed, the divergence of gamma matrices does not contain second derivatives in the case of Chern-Simons action. In the case of Kähler action they occur unless field equations equivalent with the vanishing of the divergence term are satisfied.

Also the fermionic current is conserved in this case, which conforms with the idea that fermions flow along the light-like 3-surfaces. If one uses the action $\overline{\Psi}D^{\rightarrow}\Psi$, $\overline{\Psi}$ does not satisfy the Dirac equation following from the variational principle and fermion current is not conserved. Also if the Chern-Simons term is imaginary - as a naive idea about dissipation would suggest- the Dirac equation fails to be consistent with the conjugation.

4. Off mass shell states appear in the lines of the generalized Feynman diagrams and for these $D_{C-S}$ cannot annihilate the spinor field. The generalized eigen modes lf $D_{C-S}$ should be such that one obtains the counterpart of Dirac propagator which is purely algebraic and does not therefore depend on the coordinates of the throat. This is satisfied if the generalized eigenvalues are expressible in terms of covariantly constant combinations of gamma matrices and here only $M^4$ gamma matrices are possible. Therefore the eigenvalue equation reqards as

$$D\Psi = \lambda^k\gamma_k\Psi \ , \ D = D_{C-S} + D_\alpha\hat{\Gamma}^\alpha_{C-S} \ , \ D_{C-S} = \hat{\Gamma}^\alpha_{C-S}D_\alpha \ . \quad (4.6.9)$$

Here the covariant derivatives $D_\alpha$ contain the measurement interaction term as an apparent gauge term. Covariant constancy allows to take the square of this equation and one has

$$(D^2 + \left[D, \lambda^k\gamma_k\right])\Psi + = \lambda^k\lambda_k\Psi \ . \quad (4.6.10)$$

The commutator term is analogous to magnetic moment interaction. The generalized eigenvalues correspond to $\lambda = \sqrt{\lambda^k\lambda_k}$ and Dirac determinant is defined as a product of the eigenvalues. $\lambda$ is completely analogous to mass. For incoming lines this mass would vanish so that all incoming particles irrespective their actual quantum numbers would be massless in this sense and the propagator is indeed that for a massless particle. Note that the eigen modes define the boundary values for the solutions of $D_K\Psi = 0$ so that the values of $\lambda$ indeed define the counterpart of the momentum space.

This transmutation of massive particles to effectively massless ones might make possible the application of the twistor formalism as such in TGD framework [K61] . $N = 4$ SUSY is one of the very few gauge theory which might be UV finite but it is definitely unphysical due to the masslessness of the basic quanta. Could the resolution of the interpretational problems be that the four-momenta appearing in this theory do not directly correspond to the observed four-momenta?

### Objections

The alert reader has probably raised several critical questions. Doesn't the need to solve $\lambda_k$ as functions of incoming quantum numbers plus the need to construct the measurement interactions makes the practical application of the theory hopelessly difficult? Could the resulting pseudo-momentum $\lambda_k$ correspond to the actual four-momentum? Could one drop the measurement interaction term altogether and assume that the quantum classical correspondence is through the identification of the eigenvalues as the four-momenta of the on mass shell particles propagating at the wormhole throats? Could one indeed assume that the momenta have a continuous spectrum and thus do not depend on the boundary conditions at all? Usually the thinking is just the opposite and in the general case would lead to to singular eigen modes.

1. Only the information about four-momentum would be fed into the space-time geometry. TGD however allows much more general measurement interaction terms and it would be very strange if the space-time geometry would not correlate also with the other quantum numbers. Mass formulas would of course contain information also about other quantum numbers so that this claim is not quite justified.

2. Number theoretic considerations and also the construction of octonionic variant of Dirac equation [K52] , [L7] force the conclusion that the spectrum of pseudo four-momentum is restricted to a preferred plane $M^2$ of $M^4$ and this excludes the interpretation of $\lambda^k$ as a genuine four-momentum. It also improves the hopes that the sum over pseudo-momenta does not imply divergences.

3. Dirac determinant would depend on the mass spectrum only and could not be identified as exponent of Kähler function. Note that the original guideline was the dream about stringy propagators. This is achieved for $\lambda_A\lambda^A = n$ in suitable units. This spectrum would of course also imply that Dirac determinant defined in terms of $\zeta$ function regularization is independent of the space-time surface and could not be identified with the exponent of Kähler function. One must of course take the identification of exponent of Kähler function as Dirac determinant as an additional conjecture which is not necessary for the calculation of Kähler function if the weak form of electric-magnetic duality is accepted.

4. All particles would behave as massless particles and this would not be consistent with the proposed Feynman diagrammatics inspired by zero energy ontology. Since wormhole throats carry on mass shell particles with positive or negative energy so that the net momentum can be also space-like propagators diverge for massless particles. One might overcome this problem by assuming small thermal mass (from p-adic thermodynamics [K34] ) and this is indeed assumed to reduce the number of generalized Feynman diagrams contributing to a given reaction to finite number.

Second objection of the skeptic reader relates to the delicacies of $U(1)$ gauge invariance. The modified Dirac action seems to break gauge symmetries and this breaking of gauge symmetry is absolutely essential for the dependence of the Dirac determinant on the quantum numbers. It however seems that this breaking of gauge invariance is only apparent.

1. One must distinguish between genuine $U(1)$ gauge transformations carried out for the induced Kähler gauge potential $A_\alpha$ and apparent gauge transformations of the Kähler gauge potential $A_k$ of $S^2 \times CP_2$ induced by symplectic transformations deforming the space-time surface and affect also induced metric. This delicacy of $U(1)$ gauge symmetry explains also the apparent breaking of $U(1)$ gauge symmetry of Chern-Simons Dirac action due to the presence of explicit terms $A_k$ and $A_\alpha$.

2. $CP_2$ Kähler gauge potential is obtained in complex coordinates from Kähler function as $(K_{\xi^i}, K_{\overline{\xi^i}}) = (\partial_{\xi^i}K, -\partial_{\overline{\xi^i}}K)$. Gauge transformations correspond to the additions $K \rightarrow K + f + \overline{f}$, where $f$ is a holomorphic function. Kähler gauge potential has a unique gauge in which the Kähler function of $CP_2$ is $U(2)$ invariant and contains no holomorphic part. Hence $A_k$ is defined in a preferred gauge and is a gauge invariant quantity in this sense. Same applies to $S^2$ part of the Kähler potential if present.

3. $A_\alpha$ should be also gauge invariant under gauge transformation respecting the vanishing of Coulombic interaction energy. The allowed gauge transformations $A_\alpha \rightarrow A_\alpha + \partial_\alpha\phi$ must satisfy $D_\alpha(j^\alpha_K\phi) = 0$. If the scalar function $\phi$ reduces to constant at the wormhole throats and at the ends of the space-time surface $D_{C-S}$ is gauge invariant. The gauge transformations for which $\phi$ does not satisfy this condition are identified as representations of critical deformations of space-time surface so that the change of $A_\alpha$ would code for this kind of deformation and indeed affect the modified Dirac operator and Kähler function (the change would be due to the change of zero modes).

### Some details about the modified Dirac equation defined by Chern-Simons action

First some general comments about $D_{C-S}$ are in order.

1. Quite generally, there is vacuum avoidance in the sense that $\Psi$ must vanish in the regions where the modified gamma matrices vanish. A physical analogy for the system consider is a charged particle in an external magnetic field. The effective metric defined by the anti-commutators of the modified gamma matrices so that standard intuitions might not help much. What one would naively expect would be analogs of bound states in magnetic field localized into regions inside which the magnetic field is non-vanishing.

2. If only $CP_2$ Kähler form appears in the Kähler action, the modified Dirac action defined by the Chern-Simons term is non-vanishing only when the dimension of the $CP_2$ projection of the 3-surface is $D(CP_2) \geq 2$ and the induced Kähler field is non-vanishing. This conforms with the properties of Kähler action. The solutions of the modified Dirac equation with a vanishing eigenvalue $\lambda$ would naturally correspond to incoming and outgoing particles.

3. $D(CP_2) \leq 2$ is apparently inconsistent with the weak form of electric-magnetic duality requiring $D(CP_2) = 3$. The conclusion is wrong: the variations of Chern-Simons action are subject to the constraint that electric-magnetic duality holds true expressible in terms of Lagrange multiplier term

$$\int \Lambda_\alpha (J^{n\alpha} - K\epsilon^{n\alpha\beta\gamma} J_{\beta\gamma})\sqrt{g_4} d^3x \ . \qquad (4.6.11)$$

   This gives a constraint force to the field equations and also a dependence on the induced 4-metric so that one has only almost topological QFT. This term also guarantees the $M^4$ part of WCW Kähler metric is non-trivial. The condition that the ends of space-time sheet and wormhole throats are extrema of Chern-Simons action subject to the electric-magnetic duality constraint is strongly suggested by the effective 2-dimensionality.

4. Electric-magnetic duality constraint gives an additional term to the Dirac action determined by the Lagrange multiplier term. This term gives an additional contribution to the modified gamma matrices having the same general form as coming from Kähler action and Chern-Simons action. In the following this term will not be considered. For the extremals it only affects the modified gamma matrices and leaves the general form of solutions unchanged.

In absence of the constraint from the weak form of electric-magnetic duality the explicit expression of $D_{C-S}$ is given by

$$\begin{aligned}
D &= \hat{\Gamma}^\mu D_\mu + \frac{1}{2} D_\mu \hat{\Gamma}^\mu \ , \\
\hat{\Gamma}^\mu &= \frac{\partial L_{C-S}}{\partial_\mu h^k}\Gamma_k = \epsilon^{\mu\alpha\beta}\left[2J_{kl}\partial_\alpha h^l A_\beta + J_{\alpha\beta}A_k\right]\Gamma^k D_\mu \ , \\
D_\mu \hat{\Gamma}^\mu &= B^\alpha_K (J_{k\alpha} + \partial_\alpha A_k) \ , \\
B^\alpha_K &= \epsilon^{\alpha\beta\gamma} J_{\beta\gamma} \ , \ J_{k\alpha} = J_{kl}\partial_\alpha s^l \ , \ \hat{\epsilon}^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma}\sqrt{g_3} \ .
\end{aligned} \qquad (4.6.12)$$

Note $\hat{\epsilon}^{\alpha\beta\gamma} =$ does not depend on the induced metric.

The extremals of Chern-Simons action without constraint term satisfy

$$B^\alpha_K (J_{kl} + \partial_l A_k)\partial_\alpha h^l = 0 \ , \ B^\alpha_K = \epsilon^{\alpha\beta\gamma} J_{\beta\gamma} \ . \qquad (4.6.13)$$

For a non-vanishing Kähler magnetic field $B^\alpha$ these equations hold true when $CP_2$ projection is 2-dimensional. This implies a vanishing of Chern-Simons action in absence of the constraint term realizing electric-magnetic duality, which is therefore absolutely essential in order for having a non-vanishing WCW metric.

Consider now the situation in more detail.

1. Suppose that one can assign a global coordinate to the flow lines of the Kähler magnetic field. In this case one might hope that ordinary intuitions about motion in constant magnetic field might be helpful. The repetition of the discussion of [K24] , [L4] leads to the condition $B \wedge dB = 0$ implying that a Beltrami flow for which current flows along the field lines and Lorentz forces vanishes is in question. This need not be the generic case.

2. With this assumption the modified Dirac operator reduces to a one-dimensional Dirac operator

$$D = \hat{\epsilon}^{r\alpha\beta}\left[2J_{kl}\partial_\alpha h^l A_\beta + J_{\alpha\beta}A_k\right]\Gamma^k D_r \ . \qquad (4.6.14)$$

3. The general solutions of the modified Dirac equation is covariantly constant with respect to the coordinate $r$:

$$D_r\Psi = 0 \ . \qquad (4.6.15)$$

   The solution to this condition can be written immediately in terms of a non-integrable phase factor $Pexp(i\int A_r dr)$, where integration is along curve with constant transversal coordinates. If $\hat{\Gamma^v}$ is light-like vector field also $\hat{\Gamma^v}\Psi_0$ defines a solution of $D_{C-S}$. This solution corresponds to a zero mode for $D_{C-S}$ and does not contribute to the Dirac determinant. Note that the dependence of these solutions on transversal coordinates of $X_l^3$ is arbitrary.

4. The formal solution associated with a general eigenvalue can be constructed by integrating the eigenvalue equation separately along all coordinate curves. This makes sense if $r$ indeed assigned to light-like curves indeed defines a global coordinate. What is strange that there is no correlation between the behaviors with respect longitudinal coordinate and transversal coordinates. System would be like a collection of totally uncorrelated point like particles reflecting the flow of the current along flux lines. It is difficult to say anything about the spectrum of the generalized eigenvalues in this case: it might be that the boundary conditions at the ends of the flow lines fix the allowed values of $\lambda$. Clearly, the Beltrami flow property is what makes this case very special.

**A connection with quantum measurement theory**

It is encouraging that isometry charges and also other charges could make themselves visible in the geometry of space-time surface as they should by quantum classical correspondence. This suggests an interpretation in terms of quantum measurement theory.

1. The interpretation resolves the problem caused by the fact that the choice of the commuting isometry charges is not unique. Cartan algebra corresponds naturally to the measured observables. For instance, one could choose the Cartan algebra of Poincare group to consist of energy and momentum, angular momentum and boost (velocity) in particular direction as generators of the Cartan algebra of Poincare group. In fact, the choices of a preferred plane $M^2 \subset M^4$ and geodesic sphere $S^2 \subset CP_2$ allowing to fix the measurement sub-algebra to a high degree are implied by the replacement of the imbedding space with a book like structure forced by the hierarchy of Planck constants. Therefore the hierarchy of Planck constants seems to be required by quantum measurement theory. One cannot overemphasize the importance of this connection.

2. One can add similar couplings of the net values of the measured observables to the currents whose existence and conservation is guaranteed by quantum criticality. It is essential that one maps the observables to Cartan algebra coupled to critical current characterizing the observable in question. The coupling should have interpretation as a replacement of the induced Kähler gauge potential with its gauge transform. Quantum classical correspondence encourages the identification of the classical charges associated with Kähler action with quantal Cartan charges. This would support the interpretation in terms of a measurement interaction feeding information to classical space-time physics about the eigenvalues of the observables of the measured system. The resulting field equations remain second order partial differential equations since the second order partial derivatives appear only linearly in the added terms.

3. What about the space-time correlates of electro-weak charges? The earlier proposal explains this correlation in terms of the properties of quantum states: the coupling of electro-weak charges to Chern-Simons term could give the correlation in stationary phase approximation. It would be however very strange if the coupling of electro-weak charges with the geometry of the space-time sheet would not have the same universal description based on quantum measurement theory as isometry charges have.

   (a) The hint as how this description could be achieved comes from a long standing un-answered question motivated by the fact that electro-weak gauge group identifiable as the holonomy group of $CP_2$ can be identified as $U(2)$ subgroup of color group. Could the electro-weak charges be identified as classical color charges? This might make sense since the color charges have also identification as fermionic charges implied by quantum criticality. Or could electro-weak charges be only represented as classical color charges by mapping them to classical color currents in the measurement interaction term in the modified Dirac action? At least this question might make sense.

   (b) It does not make sense to couple both electro-weak and color charges to the same fermion current. There are also other fundamental fermion currents which are conserved. All the following currents are conserved.

$$\begin{aligned} J^\alpha &= \overline{\Psi} O \hat{\Gamma}^\alpha \Psi \\ O &\in \{1 \ , \ J \equiv J_{kl}\Sigma^{kl} \ , \ \Sigma_{AB} \ , \ \Sigma_{AB}J\} \ . \end{aligned} \tag{4.6.16}$$

   Here $J_{kl}$ is the covariantly constant $CP_2$ Kähler form and $\Sigma_{AB}$ is the (also covariantly) constant sigma matrix of $M^4$ (flatness is absolutely essential).

   (c) Electromagnetic charge can be expressed as a linear combination of currents corresponding to $O = 1$ and $O = J$ and vectorial isospin current corresponds to $J$. It is natural to couple of electromagnetic charge to the the projection of Killing vector field of color hyper charge and coupling it to the current defined by $O_{em} = a+bJ$. This allows to interpret the puzzling finding that electromagnetic charge can be identified as anomalous color hyper-charge for induced spinor fields made already during the first years of TGD. There exist no conserved axial isospin currents in accordance with CVC and PCAC hypothesis which belong to the basic stuff of the hadron physics of old days.

   (d) Color charges would couple naturally to lepton and quark number current and the $U(1)$ part of electro-weak charges to the $n = 1$ multiple of quark current and $n = 3$ multiple of the lepton current (note that leptons *resp.* quarks correspond to $t = 0$ *resp.* $t = \pm 1$ color partial waves). If electro-weak *resp.* couplings to $H$-chirality are proportional to 1 *resp.* $\Gamma_9$, the fermionic currents assigned to color and electro-weak charges can be regarded as independent. This explains why the possibility of both vectorial and axial couplings in 8-D sense does not imply the doubling of gauge bosons.

   (e) There is also an infinite variety of conserved currents obtained as the quantum critical deformations of the basic fermion currents identified above. This would allow in principle to couple an arbitrary number of observables to the geometry of the space-time sheet by mapping them to Cartan algebras of Poincare and color group for a particular conserved quantum critical current. Quantum criticality would therefore make possible classical space-time correlates of observables necessary for quantum measurement theory.

   (f) The coupling constants associated with the deformations would appear in the couplings. Quantum criticality ($K \rightarrow K + f + \overline{f}$ condition) should predict the spectrum of these couplings. In the case of momentum the coupling would be proportional to $\sqrt{G/\hbar_0}= kR/\hbar_0$ and $k \sim 2^{11}$ should follow from quantum criticality. p-Adic coupling constant evolution should follow from the dependence on the scale of $CD$ coming as powers of 2.

4. Quantum criticality implies fluctuations in long length and time scales and it is not surprising that quantum criticality is needed to produce a correlation between quantal degrees of freedom and macroscopic degrees of freedom. Note that quantum classical correspondence can be regarded as an abstract form of entanglement induced by the entanglement between quantum charges $Q_A$ and fermion number type charges assignable to zero modes.

5. Space-time sheets can have an arbitrary number of wormhole contacts so that the interpretation in terms of measurement theory coupling short and long length scales suggests that the measurement interaction terms are localizable at the wormhole throats. This would favor Chern-Simons term or possibly instanton term if reducible to Chern-Simons terms. The breaking of CP and T might relate to the fact that state function reductions performed in quantum measurements indeed induce dissipation and breaking of time reversal invariance.

6. The experimental arrangement quite concretely splits the quantum state to a quantum superposition of space-time sheets such that each eigenstate of the measured observables in the superposition corresponds to different space-time sheet already before the realization of state function reduction. This relates interestingly to the question whether state function reduction really occurs or whether only a branching of wave function defined by WCW spinor field takes place as in multiverse interpretation in which different branches correspond to different observers. TGD inspired theory consciousness requires that state function reduction takes place. Maybe multiversalist might be able to find from this picture support for his own beliefs.

7. One can argue that "free will" appears not only at the level of quantum jumps but also as the possibility to select the observables appearing in the modified Dirac action dictating in turn the Kähler function defining the Kähler metric of WCW representing the "laws of physics". This need not to be the case. The choice of $CD$ fixes $M^2$ and the geodesic sphere $S^2$: this does not fix completely the choice of the quantization axis but by isometry invariance rotations and color rotations do not affect Kähler function for given $CD$ and for a given type of Cartan algebra. In $M^4$ degrees of freedom the possibility to select the observables in two manners corresponding to linear and cylindrical Minkowski coordinates could imply that the resulting Kähler functions are different. The corresponding Kähler metrics do not differ if the real parts of the Kähler functions associated with the two choices differ by a term $f(Z)+\overline{f(Z)}$, where $Z$ denotes complex coordinates of WCW, the Kähler metric remains the same. The function $f$ can depend also on zero modes. If this is the case then one can allow in given CD superpositions of WCW spinor fields for which the measurement interactions are different. This condition is expected to pose non-trivial constraints on the measurement action and quantize coupling parameters appearing in it.

#### New view about gravitational mass and matter antimatter asymmetry

The physical interpretation of the additional term in the modified Dirac action might force quite a radical revision of the ideas about matter and antimatter.

1. The term $p_A\partial_\alpha m^A$ contracted with the fermion current is analogous to a gauge potential coupling to fermion number. Since the additional terms in the modified Dirac operator induce stringy propagation, a natural interpretation of the coupling to the induced spinor fields is in terms of gravitation. One might perhaps say that the measurement of four momentum induces gravitational interaction. Besides momentum components also color charges take the role of gravitational charges. As a matter fact, any observable takes this role via coupling to the projections of Killing vector fields of Cartan algebra. The analogy of color interactions with gravitational interactions is indeed one of the oldest ideas in TGD.

2. The coupling to four-momentum is through fermion number (both quark number and lepton number). For states with a vanishing fermion number isometry charges therefore vanish. In this framework matter antimatter asymmetry would be due to the fact that matter (antimatter) corresponds to positive (negative) energy parts of zero energy states for massive systems so that the contributions to the net gravitational four-momentum are of same sign. Could antimatter be unobservable to us because it resides at negative energy space-time sheets? As a matter fact, I proposed already years ago that gravitational mass is essentially the magnitude of the inertial mass but gave up this idea.

3. Bosons do not couple at all to gravitation if they are purely local bound states of fermion and anti-fermion at the same space-time sheet (say represented by generators of super Kac-Moody algebra). Therefore the only possible identification of gauge bosons is as wormhole contacts. If the fermion and anti-fermion at the opposite throats of the contact correspond to positive

and negative energy states the net gravitational energy receives a positive contribution from both sheets. If both correspond to positive (negative) energy the contributions to the net four-momentum have opposite signs. It is not yet clear which identification is the correct one.

### 4.6.4 Generalized eigenvalues of $D_{C-S}$ and General Coordinate Invariance

The fixing of light-like 3-surface to be the wormhole throat at which the signature of induced metric changes from Minkowskian to Euclidian corresponds to a convenient fixing of gauge. General Coordinate Invariance however requires that any light-like surface $Y_l^3$ parallel to $X_l^3$ in the slicing is equally good choice. In particular, it should give rise to same Kähler metric but not necessarily the same exponent of Kähler function identified as the product of the generalized eigenvalues of $D_{C,S}$ at $Y_l^3$.

General Coordinate Invariance requires that the components of Kähler metric of configuration space defined in terms of Kähler function as

$$G_{k\bar{l}} = \partial_k \partial_{\bar{l}} K = \sum_i \partial_k \partial_{\bar{l}} \lambda_i$$

remain invariant under this flow. Here complex coordinate are of course associated with the configuration space. This is the case if the flow corresponds to the addition of sum of holomorphic function $f(z)$ and its conjugate $\overline{f}(z))$ which is anti-holomorphic function to $K$. This boils down to the scaling of eigenvalues $\lambda_i$ by

$$\lambda_i \rightarrow exp(f_i(z) + \overline{f_i(z)})\lambda_i \quad . \tag{4.6.17}$$

If the eigenvalues are interpreted as vacuum conformal weights, general coordinate transformations correspond to a spectral flow scaling the eigenvalues in this manner. This in turn would induce spectral flow of ground state conformal weights if the squares of $\lambda_i$ correspond to ground state conformal weights.

## 4.7 Representations for the configuration space gamma matrices in terms of super-symplectic charges at light cone boundary

During years I have considered several variants for the representation of WCW gamma matrices and each of these proposals has had some weakness.

1. One question has been whether the Noether currents assignable to WCW Hamiltonians should play any role in the construction or whether one can use only the generalization of flux Hamiltonians. Magnetic flux Hamiltonians do not refer to the space-time dynamics implying genuine 2-dimensionality, which is a catastrophe. If the sum of the magnetic and electric flux Hamiltonians and the weak form of self duality is assumed effective 2-dimensionality is achieved. The challenge is to identify the super-partners of the flux Hamiltonians and postulate correct anti-commutation relations for the induced spinor fields to achieve anti-commutation to flux Hamiltonians.

2. In the original proposal for WCW gamma matrices the covariantly constant right handed spinors played a key role. This led to interpretational problems with quarks. Are they needed at all or do leptons and quarks define somehow equivalent representations? I discovered only recently a brutally simple but deadly objection against this approach: the resulting WCW gamma matrices do not generate all WCW spinors from Fock vacuum. Therefore all modes of the induced spinor fields must be used.

The latter objection forced to realize that nothing is changed if one replaces the covariantly constant right handed neutrino with the collection of quark spinor modes $q_n$ *resp.* leptonic spinor modes $L_n$ multiplied by the contractions $J_{A+} = j^{Ak}\Gamma_k$ *resp.* its conjugate $J_{A-} = j^{A\bar{k}}\Gamma_{\bar{k}}$. It is essential that only of these contractions is used for a given $H$-chirality.

1. If the anti-commutator of the spinor fields is or form $J = J_{\alpha\beta}\epsilon^{\alpha\beta}\delta^2(x,y)$ at $X^2$ for magnetic flux Hamiltonians and appropriate generalization of this fro the sum of magnetic and electric flux Hamiltonians, the "half-Poisson bracket" $\partial_k H_A J^{k\bar{l}} \partial_{\bar{l}} H_B$ from the quark spinor field and its conjugate as anti-commutator from the leptonic spinor field can combine to the full Poisson bracket if the remaining factors are identical.

2. This happens if the quark modes and lepton-like modes are in 1-1 correspondence and the contractions of the eigenmodes resulting in the contraction satisfy $\overline{q}_m\gamma^0 q_n = \overline{L}_m\gamma^0 L_n = \Phi_{mn}$. The resulting Hamiltonians define an $X^2$-local algebra: that this extension is needed became obvious already earlier. A stronger condition is that the spinors can be expressed in terms of scalar function bases $\{\Phi_m\}$ so that one would have $q_{m,i} = \{\Phi_m\} q_i$ and $L_{m,i} = \{\Phi_m\} L_i$ so that one would assign to the super-currents the local Hamiltonians $\Phi_m H_A$.

3. One could of course still argue that it is questionable to use sum of quark and lepton gamma matrices since this the resulting objects to not have a well defined fermion number and cannot be used to generate physical states from vacuum. How seriously this argument should be taken is not clear to me at this moment. One could of course consider also a scenario in which one divides leptonic (or quark) modes to two classes analogous to quark and lepton modes and uses $J_{A_+}$ *resp.* $J_{A_-}$ for these two classes.

In any case, the recent view is that all modes of the induced spinor fields must be used, that lepton-quark degeneracy is absolutely essential for the construction of WCW geometry, and that the original super-symmetrization of the flux Hamiltonians combined with weak electric-magnetic duality is the correct approach. There are also fermionic Noether charges and their super counterparts implied by the criticality but these can be assigned with zero modes.

This section represents both the earlier version of the construction of configuration gamma matrices and the construction introducing explicitly the notion of finite measurement resolution. The motivation for the latter option is that if the number the generalized eigen modes of modified Dirac operator is finite, strictly local anti-commutation relations fail unless one restricts the set of points included to that corresponding to number theoretic braid. In the following integral expressions for configuration space Hamiltonians and their super-counterparts are derived first. After that the motivations for replacing integrals with sums are discussed and the expressions for Hamiltonians and super Hamiltonians are derived.

### 4.7.1 Magnetic flux representation of the super-symplectic algebra

In order to derive representation of the configuration space gamma matrices and super charges it is good to restate the basic facts about the magnetic flux representation of the configuration space gamma matrices using the original approach based on 2-dimensional integrals.

### 4.7.2 Quantization of the modified Dirac action and configuration space geometry

The quantization of the modified Dirac action involves a fusion of various number theoretical ideas. The naive approach would be based on standard canonical quantization of induced spinor fields by posing anti-commutation relations between $\Psi$ and canonical momentum density $\partial L/\partial(\partial_t\Psi)$.

#### Generalized magnetic and electric fluxes

Isometry invariants are just a special case of fluxes defining natural coordinate variables for the configuration space. Canonical transformations of $CP_2$ act as $U(1)$ gauge transformations on the Kähler potential of $CP_2$ (similar conclusion holds at the level of $\delta M^4_+ \times CP_2$).

One can generalize these transformations to local symplectic transformations by allowing the Hamiltonians to be products of the $CP_2$ Hamiltonians with the real and imaginary parts of the functions $f_{s,n,k}$ defining the Lorentz covariant function basis $H_A$, $A \equiv (a,s,n,k)$ at the light cone boundary: $H_A = H_a \times f(s,n,k)$, where $a$ labels the Hamiltonians of $CP_2$.

One can associate to any Hamiltonian $H^A$ of this kind magnetic or electric flux via the following formulas:

$$Q_{m/e}(H_A|X^2) \;=\; \int_{X^2} H_A J_{m/e} \ . \tag{4.7.1}$$

Here the magnetic (electric) flux $J_m$ ($J_e$) denotes the flux associated with induced Kähler field and its dual which is well-defined since $X^2$ is part of 4-D space-time surface.

The flux Hamiltonians

$$Q_i(H_A|X^2) \;=\; Q_i(H_A|X^2) \ , \ A \equiv (a,s,n,k) \tag{4.7.2}$$

provide a representation of WCW Hamiltonians as far as the "kinetic" part of Kähler form is considered.

### Anti-commutation relations between oscillator operators associated with same partonic 2-surface

The construction of WCW gamma matrices leads to the anti-commutation relations given by

$$\begin{aligned} \{\overline{\Psi}(x)\gamma^0,\Psi(x)\} &= [J_e + J_m)\delta^2_{x,y} \ , \\ J_e &= \int J^{03}\sqrt{g_4} \ . \end{aligned} \tag{4.7.3}$$

Kähler magnetic flux $J_m = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$ has no dependence on the induced metric.

If the weak- form of the electric-magnetic duality holds true, Kähler electric flux relates to it via the formula

$$J^{03}\sqrt{g_4} = KJ_{12} \ ,$$

where $K$ is symplectic invariant and identifiable in terms of Kähler coupling strength from classical charge quantization condition for Kähler electric flux. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge $g_K$ gives the condition $K = g_K^2/\hbar = 4\pi\alpha_K$, where $g_K$ is Kähler coupling constant. Within experimental uncertainties one has $\alpha_K = g_K^2/4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where $\alpha_{em}$ is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant. The arguments leading to the identification $\epsilon \pm 1$ at the opposite boundaries of $CD$ are discussed in [K24] , [L4] . An alternative identification is as $\epsilon = 0$ but predicts that WCW is trivial in $M^4$ degrees of freedom if Kähler function reduces to Chern-Simons terms.

The general form of the anti-commutation relations is therefore

$$\{\overline{\Psi}(x)\gamma^0,\Psi(x)\} \;=\; (1+K)J\delta^2_{x,y} \ . \tag{4.7.4}$$

What is nice that at the limit of vacuum extremals the right hand side vanishes when both $J$ and $J^1$ vanish so that spinor fields become non-dynamical. One can criticize the non-vanishing of the anti-commutator for vacuum extremals of Kähler action.

For the latter option the fermionic counterparts of local flux Hamiltonians can be written in the form

$$\begin{aligned} H_{A,\pm,n} &= \epsilon_q(A,\mp,n)H_{A,\pm,q,n} + \epsilon_L(A,\pm)H_{A,\mp,L,n} \ , \\ H_{A,+,q,n} &= \oint \overline{\Psi}J^A_+ q_n d^2x \ , \\ H_{A,-,q,n} &= \oint \overline{q}_n J^A_- \Psi d^2x \ , \\ H_{A,-,L,n} &= \oint \overline{\Psi}J^A_+ L_n d^2x \ , \\ H_{A,+,L,n} &= \oint \overline{L}_n J^A_- \Psi d^2x \ , \\ J^A_+ &= j^{Ak}\Gamma_k \ , \ J^A_- = j^{A\overline{k}}\Gamma_{\overline{k}} \ . \end{aligned} \tag{4.7.5}$$

The commutative parameters $\epsilon_q(A,\pm,n)$ *resp.* $\epsilon_L(A,\pm,n)$ are assumed to carry quark *resp.* lepton number opposite to that of $H_{A,\mp,q,n}$ *resp.* $H_{A,\mp,L,n}$ and satisfy $\epsilon_i(A,+,n)\epsilon_i(A,-,n)=1$. One encounters a hierarchy discrete algebras satisfying this condition in the construction of a symplectic analog of conformal quantum field theory required by the construction of quantum TGD [K44] . Associativity condition fixes uniquely the commutative multiplication of these units and analogs of plane waves with discrete momentum are in question.

Suppose that there is a one-one correspondence between quark modes and leptonic modes is satisfied and the label $n$ decomposes as $n=(m,i)$, where $n$ labels a scalar function basis and $i$ labels spinor components. This would give

$$\begin{aligned} q_n = q_{m,i} &= \Phi_m q_i \ , \\ L_n = L_{m,i} &= \Phi_m L_i \ , \\ \overline{q}_i\gamma^0 q_j &= \overline{L}_i\gamma^0 L_j = g_{ij} \ . \end{aligned} \tag{4.7.6}$$

Suppose that the inner products $g_{ij}$ are constant. The simplest possibility is $g_{ij}=\delta_{ij}$ Under these assumptions the anti-commutators of the super-symmetric flux Hamiltonians give flux Hamiltonians.

$$\{H_{A,+,n},H_{A,-,n}\} = g_{ij}\oint \overline{\Phi}_m\Phi_n H_A J d^2x \ . \tag{4.7.7}$$

The product of scalar functions can be expressed as

$$\overline{\Phi}_m\Phi_n = c_{mn}^{\ \ k}\Phi_k \ . \tag{4.7.8}$$

Note that the notion of symplectic QFT [K12] led to a scalar function algebra of similar kind consisting of phase factors and there excellent reasons to consider the possibility that there is a deep connection with this approach.

One expects that the symplectic algebra is restricted to a direct sum of symplectic algebras localized to the regions where the induced Kähler form is non-vanishing implying that the algebras associated with different region form to a direct sum. Also the contributions to configuration space metric are direct sums. The symplectic algebras associated with different region can be truncated to finite-dimensional spaces of symplectic algebras associated with the regions in question. As far as coordinatization of the reduced configuration space is considered, these symplectic sub-spaces are enough. These truncated algebras naturally correspond to the hyper-finite factor property of the Clifford algebra of configuration space.

### Generalization of WCW Hamiltonians and anti-commutation relations between flux Hamiltonians belonging to different ends of $CD$

This picture requires a generalization of the view about configuration space Hamiltonians since also the interaction term between the ends of the line is present not taken into account in the previous approach.

1. The proposed representation of WCW Hamiltonians as flux Hamiltonians [K10, K9] , [L5]

$$Q(H_A) = \int H_A J d^2x \ . \tag{4.7.9}$$

   works for the kinetic terms only since $J$ is not expectred to be the same at the ends of the line.

   The assumption that Poisson bracket of WCW Hamiltonians reduces to the level of imbedding space - in other words $\{Q(H_A),Q(H_B\}=Q(\{H_A,H_B\})$ - can be justified. One starts from the representation in terms of say flux Hamiltonians $Q(H_A)$ and *defines* $J_{A,B}$ as $J_{A,B}\equiv Q(\{H_A,H_B\})$. One has $\partial H_A/\partial t_B=\{H_B,H_A\}$, where $t_B$ is the parameter associated with the exponentiation of $H_B$. The inverse $J^{AB}$ of $J_{A,B}=\partial H_B/\partial t_A$ is expressible as $J^{A,B}=\partial t_A/\partial H_B$. From these formulas one can deduce by using chain rule that the bracket $\{Q(H_A),Q(H_B\}=\partial t_C Q(H_A)J^{CD}\partial t_D Q(H_B)$ of flux Hamiltonians equals to the flux Hamiltonian $Q(\{H_A,H_B\})$.

2. One should be able to assign to WCW Hamiltonians also a part corresponding to the interaction term. The symplectic conjugation associated with the interaction term permutes the WCW coordinates assignable to the ends of the line. One should reduce this apparently non-local symplectic conjugation (if one thinks the ends of line as separate objects) to a non-local symplectic conjugation for $\delta CD \times CP_2$ by identifying the points of lower and upper end of $CD$ related by time reflection and assuming that conjugation corresponds to time reflection. Formally this gives a well defined generalization of the local Poisson brackets between time reflected points at the boundaries of $CD$. The connection of Hermitian conjugation and time reflection in quantum field theories is is in accordance with this picture.

3. Perhaps the only manner to proceed is to assign to the flux Hamiltonian also a part obtained by the replacement of the flux integral over $X^2$ with an integral over the projection of $X^2$ to a sphere $S^2$ assignable to the light-cone boundary or to a geodesic sphere of $CP_2$, which come as two varieties corresponding to homologically trivial and non-trivial spheres. The projection is defined as by the geodesic line orthogonal to $S^2$ and going through the point of $X^2$. The hierarchy of Planck constants assigns to $CD$ a preferred geodesic sphere of $CP_2$ as well as a unique sphere $S^2$ as a sphere for which the radial coordinate $r_M$ or the light-cone boundary defined uniquely is constant: this radial coordinate corresponds to spherical coordinate in the rest system defined by the time-like vector connecting the tips of $CD$. Either spheres or possibly both of them could be relevant.

   Recall that also the construction of number theoretic braids and symplectic QFT [K12] led to the proposal that braid diagrams and symplectic triangulations could be defined in terms of projections of braid strands to one of these spheres. One could also consider a weakening for the condition that the points of the number theoretic braid are algebraic by requiring only that the $S^2$ coordinates of the projection are algebraic and that these coordinates correspond to the discretization of $S^2$ in terms of the phase angles associated with $\theta$ and $\phi$.

   This gives for the corresponding contribution of the WCW Hamiltonian the expression

$$Q(H_A)_{int} = (1+K)\int_{S^2_\pm} H_A X\delta^2(s_+,s_-)d^2s_\pm = (1+K)\int_{P(X^2_+)\cap P(X^2_-)} \frac{\partial(s^1,s^2)}{\partial(x^1_\pm,x^2_\pm)}d^2x_\pm \quad (4.7.10)$$

   Here the Poisson brackets between ends of the line using the rules involve delta function $\delta^2(s_+,s_-)$ at $S^2$ and the resulting Hamiltonians can be expressed as a similar integral of $H_{[A,B]}$ over the upper or lower end since the integral is over the intersection of $S^2$ projections.

   The expression must vanish when the induced Kähler form vanishes for either end. This is achieved by identifying the scalar $X$ in the following manner:

$$\begin{aligned} X &= J^+_{kl} + J^-_{kl} \ , \\ J^{kl}_\pm &= \partial_\alpha s^k \partial_\beta s^l J^{\alpha\beta}_\pm \ . \end{aligned} \quad (4.7.11)$$

   The tensors are lifts of the induced Kähler form of $X^2_\pm$ to $S^2$ (not $CP_2$).

4. One could of course ask why these Hamiltonians could not contribute also to the kinetic terms and why the brackets with flux Hamiltonians should vanish. This relate to how one *defines* the Kähler form. It was shown above that in case of flux Hamiltonians the definition of Kähler form as brackets gives the basic formula $\{Q(H_A),Q(H_B)\} = Q(\{H_A,H_B\}$ and same should hold true now. In the recent case $J_{A,B}$ would contain an interaction term defined in terms of flux Hamiltonians and the previous argument should go through also now by identifying Hamiltonians as sums of two contributions and by introducing the doubling of the coordinates $t_A$.

5. The quantization of the modified Dirac operator must be reconsidered. It would seem that one must add to the super-Hamiltonian completely analogous term obtained by replacing $J$ with $X\partial(s^1,s^2)/\partial(x^1_\pm,x^2_\pm)$. Besides the anti-commutation relations defining correct anti-commutators

to flux Hamiltonians, one should pose anti-commutation relations consistent with the anti-commutation relations of super Hamiltonians. In these anti-commutation relations $J\delta^2(x,y)$ would be replaced with $X\delta^2(s^+,s^-)$. This would guarantee that the oscillator operators at the ends of the line are not independent and that the resulting Hamiltonian reduces to integral over either end for $H_{[A,B]}$.

### 4.7.3 Expressions for configuration space super-symplectic generators in finite measurement resolution

The expressions of configuration space Hamiltonians and their super counterparts just discussed were based on 2-dimensional integrals. This is problematic for several reasons.

1. In p-adic context integrals do not makes sense so that this representation fails in p-adic context (for pe-adic numbers see [A36] ). Sums would be more appropriate if one wants number theoretic universality at the level of basic formulas.

2. The use of sums would also conform with the notion of finite measurement resolution having discretization in terms of intersections of $X^2$ with number theoretic braids as a space-time correlate.

3. Number theoretic duality suggests a unique realization of the discretization in the sense that only the points of partonic 2-surface $X^2$ whose $\delta M^4_\pm$ projections commute in hyper-octonionic sense and thus belong to the intersections of the projection $P_{M^4}(X^2)$ with radial light-like geodesics $M_\pm$ representing intersections of $M^2 \subset M^4 \subset M^8$ with $\delta M^4_\pm \times CP_2$ contribute to the configuration space Hamiltonians and super Hamiltonians and therefore to the configuration space metric.

Clearly, finite measurement resolution seems to be an unavoidable aspect of the geometrization of the configuration space as one can expect on basis of the fact that configuration space Clifford algebra provides representation for hyper-finite factors of type $II_1$ whose inclusions provide a representation for the finite measurement resolution. This means that the infinite-dimensional configuration space can be represented as a finite-dimensional space in arbitrary precise approximation so that also also configuration Clifford algebra and configuration space spinor fields becomes finite-dimensional.

The modification of anti-commutation relations to this case is

$$\{\overline{\Psi}(x_m)\gamma^0, \Psi(x_n)\} = (1+K)J\delta_{x_m,x_n} \ . \tag{4.7.12}$$

Note that the constancy of $\gamma^0$ implies a complete symmetry between the two points. The number of points must be the maximal one consistent with the Kronecker delta type anti-commutation relations so that information is not lost.

The question arises about the choice of the points $x_m$. This choice should general coordinate invariant. The number theoretic vision leads to the notion of number theoretic braid defined as the set of points common to real and p-adic variant of $X^2$. The points of the number theoretic braid are excellent candidates for points $x_n$. The p-adic variant exists only if $X^2$ is defined by rational functions with coefficients which are possibly algebraic and thus make sense both in real and p-adic sense. These points belong to the algebraic extension of rational numbers appearing in the representation of $X^2$ as an algebraic surface but one can consider quite generally the possibility that the points of the number theoretic braid are rational or in a finite algebraic extension of rationals. What is important that if one restricts the consideration to rational points this criterion makes sense even if $X^2$ is not algebraic. In the generic case one can expect that the number of these points is finite.

### 4.7.4 Configuration space geometry and hierarchy of inclusions of hyper-finite factors of $II_1$

The configuration space metric defined as anti-commutators of the configuration space gamma matrices is extremely degenerate since it effectively corresponds to a quadratic form in $N$-dimensional space, where $N_m$ is the total number of the eigenmodes of $D_K$. Since two Hamiltonians whose values and corresponding Killing vector fields co-incide at the points of $B$ are equivalent for given ray $M_{\pm}$, it is natural to pose a cutoff in the number of Hamiltonians used for the representation of reduced configuration space in given region inside which induced Kähler form is non-vanishing. The natural manner to pose this cutoff is by ordering the representations with respect to dimension and eigenvalue of Casimir operator for the irreducible representations of $SO(3) \times SO(4)$ in case of $M^8$ and for the representations of $SO(3) \times SU(3)$ in case of $H$.

This boils down to a hierarchy of approximate representations of the configuration space as Kähler manifold with spinor structure with a truncation of the Clifford algebra to a finite dimensional Clifford algebra. This is in spirit with the proposed interpretation of the inclusion sequence of hyper-finite factors of type $\text{II}_1$ and with the very notion of hyper-finiteness. A surprisingly concrete connection of the configuration space geometry with generalized eigenvalue spectrum of $D_K(X^3)$ and basic quantum physics results. For instance, from the general expression of Kähler metric in terms of Kähler function

$$G_{k\bar{l}} = \partial_k \partial_{\bar{l}} K = \frac{\partial_k \partial_{\bar{l}} exp(K)}{exp(K)} - \frac{\partial_k exp(K)}{exp(K)} \frac{\partial_{\bar{l}} exp(K)}{exp(K)} \ , \tag{4.7.13}$$

and from the expression of $exp(K) = \prod_i \lambda_i$ as the product of of finite number of eigenvalues of $D_K(X^3)$, the expression

$$G_{k\bar{l}} = \sum_i \frac{\partial_k \partial_{\bar{l}} \lambda_i}{\lambda_i} - \frac{\partial_k \lambda_i}{\lambda_i} \frac{\partial_{\bar{l}} \lambda_i}{\lambda_i} \tag{4.7.14}$$

for the configuration space metric follows. Here complex coordinates refer to the complex coordinates of configuration space.

A good candidate for these complex coordinates are the complex coordinates of $S^2 \times S$, $S = CP_2$ or $E^4$, for the points of $B$ so that a close connection with the geometry of imbedding space is obtained. Once these coordinates have been specified $G$ can be contracted with the Killing vector fields of configuration space isometries defining the coordinates for the truncated configuration space. By studying the behavior of eigenvalue spectrum under small deformations of $X_l^3$ by symplectic transformations of $\delta CD \times S$ the components of $G$ can be estimated.

## 4.8 Super-conformal symmetries at space-time and configuration space level

The physical interpretation and detailed mathematical understanding of super-conformal symmetries has developed rather slowly and has involved several side tracks. In the following I try to summarize the basic picture with minimal amount of formulas with the understanding that the statement "Noether charge associated with geometrically realized Kac-Moody symmetry" is enough for the reader to write down the needed formula explicitly.

### 4.8.1 Configuration space as a union of symmetric spaces

In finite-dimensional context globally symmetric spaces are of form $G/H$ and connection and curvature are independent of the metric, provided it is left invariant under $G$. The hope is that same holds true in infinite-dimensional context. The most one can hope of obtaining is the decomposition $C(H) = \cup_i G/H_i$ over orbits of $G$. One could allow also symmetry breaking in the sense that $G$ and $H$ depend on the orbit: $C(H) = \cup_i G_i/H_i$ but it seems that $G$ can be chosen to be same for all orbits. What is essential is that these groups are infinite-dimensional. The basic properties of the coset space decomposition give very strong constraints on the group $H$, which certainly contains the subgroup of $G$, whose action reduces to diffeomorphisms of $X^3$.

#### Consequences of the decomposition

If the decomposition to a union of coset spaces indeed occurs, the consequences for the calculability of the theory are enormous since it suffices to find metric and curvature tensor for single representative

3-surface on a given orbit (contravariant form of metric gives propagator in perturbative calculation of matrix elements as functional integrals over the configuration space). The representative surface can be chosen to correspond to the maximum of Kähler function on a given orbit and one obtains perturbation theory around this maximum (Kähler function is not isometry invariant).

The task is to identify the infinite-dimensional groups $G$ and $H$ and to understand the zero mode structure of the configuration space. Almost twenty (seven according to long held belief!) years after the discovery of the candidate for the Kähler function defining the metric, it became finally clear that these identifications follow quite nicely from $Diff^4$ invariance and $Diff^4$ degeneracy as well as special properties of the Kähler action.

The guess (not the first one!) would be following. $G$ corresponds to the symplectic transformations of $\delta M^4_\pm \times CP_2$ leaving the induced Kähler form invariant. If $G$ acts as isometries the values of Kähler form at partonic 2-surfaces (remember effective 2-dimensionality) are zero modes and configuration space allows slicing to symplectic orbits of the partonic 2-surface with fixed induced Kähler form. Quantum fluctuating degrees of freedom would correspond to symplectic group and to the fluctuations of the induced metric. The group $H$ dividing $G$ would in turn correspond to the Kac-Moody symmetries respecting light-likeness of $X^3_l$ and acting in $X^3_l$ but trivially at the partonic 2-surface $X^2$. This coset structure was originally discovered via coset construction for super Virasoro algebras of super-symplectic and super Kac-Moody algebras and realizes Equivalence Principle at quantum level.

**Configuration space isometries as a subgroup of $Diff(\delta M^4_+ \times CP_2)$**

The reduction to light cone boundary leads to the identification of the isometry group as some subgroup of for the group $G$ for the diffeomorphisms of $\delta M^4_+ \times CP_2$. These diffeomorphisms indeed act in a natural manner in $\delta CH$, the the space of 3-surfaces in $\delta M^4_+ \times CP_2$. Configuration space is expected to decompose to a union of the coset spaces $G/H_i$, where $H_i$ corresponds to some subgroup of $G$ containing the transformations of $G$ acting as diffeomorphisms for given $X^3$. Geometrically the vector fields acting as diffeomorphisms of $X^3$ are tangential to the 3-surface. $H_i$ could depend on the topology of $X^3$ and since $G$ does not change the topology of 3-surface each 3-topology defines separate orbit of $G$. Therefore, the union involves sum over all topologies of $X^3$ plus possibly other 'zero modes'. Different topologies are naturally glued together since singular 3-surfaces intermediate between two 3-topologies correspond to points common to the two sectors with different topologies.

### 4.8.2 Isometries of configuration space geometry as symplectic transformations of $\delta M^4_+ \times CP_2$

During last decade I have considered several candidates for the group $G$ of isometries of the configuration space as the sub-algebra of the subalgebra of $Diff(\delta M^4_+ \times CP_2)$. To begin with let us write the general decomposition of $diff(\delta M^4_+ \times CP_2)$:

$$diff(\delta M^4_+ \times CP_2) \quad = \quad S(CP_2) \times diff(\delta M^4_+) \oplus S(\delta M^4_+) \times diff(CP_2) \ . \tag{4.8.1}$$

Here $S(X)$ denotes the scalar function basis of space $X$. This Lie-algebra is the direct sum of light cone diffeomorphisms made local with respect to $CP_2$ and $CP_2$ diffeomorphisms made local with respect to light cone boundary.

The idea that entire diffeomorphism group would act as isometries looks unrealistic since the theory should be more or less equivalent with topological field theory in this case. Consider now the various candidates for $G$.

1. The fact that symplectic transformations of $CP_2$ and $M^4_+$ diffeomorphisms are dynamical symmetries of the vacuum extremals suggests the possibility that the diffeomorphisms of the light cone boundary and symplectic transformations of $CP_2$ could leave Kähler function invariant and thus correspond to zero modes. The symplectic transformations of $CP_2$ localized with respect to light cone boundary acting as symplectic transformations of $CP_2$ have interpretation as local color transformations and are a good candidate for the isometries. The fact that local color transformations are not even approximate symmetries of Kähler action is not a problem: if they were exact symmetries, Kähler function would be invariant and zero modes would be in question.

2. $CP_2$ local conformal transformations of the light cone boundary act as isometries of $\delta M^4_+$. Besides this there is a huge group of the symplectic symmetries of $\delta M^4_+ \times CP_2$ if light cone boundary is provided with the symplectic structure. Both groups must be considered as candidates for groups of isometries. $\delta M^4_+ \times CP_2$ option exploits fully the special properties of $\delta M^4_+ \times CP_2$, and one can develop simple argument demonstrating that $\delta M^4_+ \times CP_2$ symplectic invariance is the correct option. Also the construction of configuration space gamma matrices as super-symplectic charges supports $\delta M^4_+ \times CP_2$ option.

This picture remained same for a long time. The discovery that Kac-Moody algebra consisting of $X^2$ local symmetries generated by Hamiltonians of isometry sub-algebra of symplectic algebra forced to challenge this picture and ask whether also $X^2$-local transformations of symplectic group could be involved.

1. The basic condition is that the $X^2$ local transformation acts leaves induced Kähler form invariant apart from diffeomorphism. Denote the infinitesimal generator of $X^2$ local symplecto morphism by $\Phi_A(x)j^{Ak}$, where $A$ labels Hamiltonians in the sum and by $j^\alpha$ the generator of $X^2$ diffeomorphism.

2. The invariance of $J = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$ modulo diffeomorphism under the infinitesimal symplectic transformation gives

$$\{H^A, \Phi_A\} \equiv \partial_\alpha H^A \epsilon^{\alpha\beta}\partial_\beta \Phi_A = \partial_\alpha J j^\alpha \ . \tag{4.8.2}$$

3. Note that here the Poisson bracket is not defined by $J^\alpha\beta$ but $\epsilon^{\alpha\beta}$ defined by the induced metric. Left hand side reflects the failure of symplectomorphism property due to the dependence of $\Phi_A(x)$ on $X^2$ coordinate which and comes from the gradients of $\delta M^4 \times CP_2$ coordinates in the expression of the induced Kähler form. Right hand side corresponds to the action of infinitesimal diffeomorphism.

4. Let us assume that one can restrict the consideration to single Hamiltonian so that the transformation is generated by $\Phi(x)H_A$ and that to each $\Phi(x)$ there corresponds a diffeomorphism of $X^2$, which is a symplectic transformation of $X^2$ with respect to symplectic form $\epsilon^{\alpha\beta}$ and generated by Hamiltonian $\Psi(x)$. This transforms the invariance condition to

$$\{H^A, \Phi\} \equiv \partial_\alpha H^A \epsilon^{\alpha\beta}\partial_\beta \Phi = \partial_\alpha J \epsilon^{\alpha\beta}\partial_\beta \Psi_A = \{J, \Psi_A\} \ . \tag{4.8.3}$$

This condition can be solved identically by assuming that $\Phi_A$ and $\Psi$ are proportional to arbitrary smooth function of $J$:

$$\Phi = f(J) \ , \quad \Psi_A = -f(J)H_A \ . \tag{4.8.4}$$

Therefore the $X^2$ local symplectomorphisms of $H$ reduce to symplectic transformations of $X^2$ with Hamiltonians depending on single coordinate $J$ of $X^2$. The analogy with conformal invariance for which transformations depend on single coordinate $z$ is obvious. As far as the anti-commutation relations for induced spinor fields are considered this means that $J = consant$ curves behave as points points. For extrema of $J$ appearing as candidates for points of number theoretic braids $J = constant$ curves reduce to points.

5. From the structure of the conditions it is easy to see that the transformations generate a Lie-algebra. For the transformations $\Phi^1_A H^A$ $\Phi^2_A H^A$ the commutator is

$$\Phi_A^{[1,2]} = f_A^{\ BC}\Phi_B\Phi_C \ , \tag{4.8.5}$$

where ${f_A}^{BC}$ are the structure constants for the symplectic algebra of $\delta M^4_\pm \times CP_2$. From this form it is easy to check that Jacobi identifies are satisfied. The commutator has same form as the commutator of gauge algebra generators. BRST gauge symmetry is perhaps the nearest analog of this symmetry. In the case of isometries these transforms realized local color gauge symmetry in TGD sense.

6. If space-time surface allows a slicing to light-like 3-surfaces $Y^3_l$ parallel to $X^3_l$, these conditions make sense also for the partonic 2-surfaces defined by the intersections of $Y^3_l$ with $\delta M^4_\pm \times CP_2$ and "parallel" to $X^2$. The local symplectic transformations also generalize to their local variants in $X^3_l$. Light-likeness of $X^3_l$ means effective metric 2-dimensionality so that 2-D Kähler metric and symplectic form as well as the invariant $J = \epsilon^{\alpha\beta}J_{\alpha\beta}$ exist. A straightforward calculation shows that the the notion of local symplectic transformation makes sense also now and formulas are exactly the same as above.

### 4.8.3 SUSY algebra defined by the anticommutation relations of fermionic oscillator operators and WCW local Clifford algebra elements as chiral super-fields

Whether TGD allows space-time supersymmetry has been a long-standing question. Majorana spinors appear in $N = 1$ super-symmetric QFTs- in particular minimally super-symmetric standard model (MSSM). Majorana-Weyl spinors appear in M-theory and super string models. An undesirable consequence is chiral anomaly in the case that the numbers of left and right handed spinors are not same. For $D = 11$ and $D = 10$ these anomalies cancel which led to the breakthrough of string models and later to M-theory. The probable reason for considering these dimensions is that standard model does not predict right-handed neutrino (although neutrino mass suggests that right handed neutrino exists) so that the numbers of left and right handed Weyl-spinors are not the same.

In TGD framework the situation is different. Covariantly constant right-handed neutrino spinor acts as a super-symmetry in $CP_2$. One might think that right-handed neutrino in a well-defined sense disappears from the spectrum as a zero mode so that the number of right and left handed chiralities in $M^4 \times CP_2$ would not be same. For light-like 3-surfaces covariantly constant right-handed neutrino does not however solve the counterpart of Dirac equation for a non-vanishing four-momentum and color quantum numbers of the physical state. Therefore it does not disappear from the spectrum anymore and one expects the same number of right and left handed chiralities.

In TGD framework the separate conservation of baryon and lepton numbers excludes Majorana spinors and also the the Minkowski signature of $M^4 \times CP_2$ makes them impossible. The conclusion that TGD does not allow super-symmetry is however wrong. For $\mathcal{N} = 2N$ Weyl spinors are indeed possible and if the number of right and left handed Weyl spinors is same super-symmetry is possible. In 8-D context right and left-handed fermions correspond to quarks and leptons and since color in TGD framework corresponds to $CP_2$ partial waves rather than spin like quantum number, also the numbers of quark and lepton-like spinors are same.

The physical picture suggest a new kind of approach to super-symmetry in the sense that the anticommutations of fermionic oscillator operators associated with the modes of the induced spinor fields define a structure analogous to SUSY algebra. This means that $\mathcal{N} = 2N$ SUSY with large $N$ is in question allowing spins higher than two and also large fermion numbers. Recall that $\mathcal{N} \leq 32$ is implied by the absence of spins higher than two and the number of real spinor components is $N = 32$ also in TGD. The situation clearly differs from that encountered in super-string models and SUSYs and the large value of $N$ allows to expect very powerful constraints on dynamics irrespective of the fact that SUSY is broken. Right handed neutrino modes define a sub-algebra for which the SUSY is only slightly broken by the absence of weak interactions and one could also consider a theory containing a large number of $\mathcal{N} = 2$ super-multiplets corresponding to the addition of right-handed neutrinos and antineutrinos at the wormhole throat.

Masslessness condition is essential for super-symmetry and at the fundamental level it could be formulated in terms of modified gamma matrices using octonionic representation and assuming that they span local quaternionic sub-algebra at each point of the space-time sheet. SUSY algebra has standard interpretation with respect to spin and isospin indices only at the partonic 2-surfaces so that the basic algebra should be formulated at these surfaces. Effective 2-dimensionality would require that partonic 2-surfaces can be taken to be ends of any light-like 3-surface $Y_l^3$ in the slicing of the region surrounding a given wormhole throat.

### Super-algebra associated with the modified gamma matrices

Anti-commutation relations for fermionic oscillator operators associated with the induced spinor fields are naturally formulated in terms of the modified gamma matrices. Super-conformal symmetry suggests that the anti-commutation relations for the fermionic oscillator operators at light-like 3-surfaces or at their ends are most naturally formulated as anti-commutation relations for SUSY algebra. The resulting anti-commutation relations would fix the quantum TGD.

$$\begin{aligned} \{a^{\dagger}_{n\alpha}, a_{n\beta}\} &= D_{mn} D_{\alpha\beta} \ , \\ D &= (p^{\mu} + \sum_a Q^{\mu}_a)\hat{\sigma}^{\mu} \ . \end{aligned} \tag{4.8.6}$$

Here $p^{\mu}$ and $Q^{\mu}_a$ are space-time projections of momentum and color charges in Cartan algebra. Their action is purely algebraic. The anti-commutations are nothing but a generalization of the ordinary equal-time anticommutation relations for fermionic oscillator operators to a manifestly covariant form. The matrix $D_{m,n}$ is expected to reduce to a diagonal form with a proper normalization of the oscillator operators. The experience with extended SUSY algebra suggest that the anti-commutators could contain additional central term proportional to $\delta_{\alpha\beta}$.

One can consider basically two different options concerning the definition of the super-algebra.

1. If the super-algebra is defined at the 3-D ends of the intersection of $X^4$ with the boundaries of $CD$, the modified gamma matrices appearing in the operator $D$ appearing in the anti-commutator are associated with Kähler action. If the generalized masslessness condition $D^2 = 0$ holds true -as suggested already earlier- one can hope that no explicit breaking of super-symmetry takes place and elegant description of massive states as effectively massless states making also possible generalization of twistor is possible. One must however notice that also massive representatives of SUSY exist.

2. SUSY algebra could be also defined at 2-D ends of light-like 3-surfaces.

According to considerations of [K19] these options are equivalent for a large class of space-time sheets. If the effective 3-dimensionality realized in the sense that the effective metric defined by the modified gamma matrices is degenerate, propagation takes place along 3-D light-like 3-surfaces. This condition definitely fails for string like objects.

One can realize the local Clifford algebra also by introducing theta parameters in the standard manner and the expressing a collection of local Clifford algebra element with varying values of fermion numbers (function of $CD$ and $CP_2$ coordinates) as a chiral super-field. The definition of a chiral super field requires the introduction of super-covariant derivatives. Standard form for the anti-commutators of super-covariant derivatives $D_{\alpha}$ make sense only if they do not affect the modified gamma matrices. This is achieved if $p_k$ acts on the position of the tip of $CD$ (rather than internal coordinates of the space-time sheet). $Q_a$ in turn must act on $CP_2$ coordinates of the tip.

### Super-fields associated with WCW Clifford algebra

WCW local Clifford algebra elements possess definite fermion numbers and it is not physically sensible to super-pose local Clifford algebra elements with different fermion numbers. The extremely elegant formulation of super-symmetric theories in terms of super-fields encourages to ask whether the local Clifford algebra elements could allow expansion in terms of complex theta parameters assigned to various fermionic oscillator operator in order to obtain formal superposition of elements with different fermion numbers. One can also ask whether the notion of chiral super field might make sense.

The obvious question is whether it makes sense to assign super-fields with the modified gamma matrices.

1. Modified gamma matrices are not covariantly constant but this is not a problem since the action of momentum generators and color generators is purely algebraic space-time coordinates.

2. One can define the notion of chiral super-field also at the fundamental level. Chiral super-field would be continuation of the local Clifford algebra of associated with $CD$ to a local Clifford algebra element associated with the union of $CD$s. This would allow elegant description of cm degrees of freedom, which are the most interesting as far as QFT limit is considered.

3. Kähler function of WCW as a function of complex coordinates could be extended to a chiral super-field defined in quantum fluctuation degrees of freedom. It would depend on zero modes too. Does also the latter dependence allow super-space continuation? Coefficients of powers of theta would correspond to fermionic oscillator operators. Does this function define the propagators of various states associated with light-like 3-surface? Configuration space complex coordinates would correspond to the modes of induced spinor field so that super-symmetry would be realized very concretely.

### 4.8.4 Identification of Kac-Moody symmetries

The Kac-Moody algebra of symmetries acting as symmetries respecting the light-likeness of 3-surfaces plays a crucial role in the identification of quantum fluctuating configuration space degrees of freedom contributing to the metric.

#### Identification of Kac-Moody algebra

The generators of bosonic super Kac-Moody algebra leave the light-likeness condition $\sqrt{g_3}=0$ invariant. This gives the condition

$$\delta g_{\alpha\beta}Cof(g^{\alpha\beta}) = 0 \ , \tag{4.8.7}$$

Here $Cof$ refers to matrix cofactor of $g_{\alpha\beta}$ and summation over indices is understood. The conditions can be satisfied if the symmetries act as combinations of infinitesimal diffeomorphisms $x^\mu \rightarrow x^\mu+\xi^\mu$ of $X^3$ and of infinitesimal conformal symmetries of the induced metric

$$\delta g_{\alpha\beta} = \lambda(x)g_{\alpha\beta}+\partial_\mu g_{\alpha\beta}\xi^\mu+g_{\mu\beta}\partial_\alpha\xi^\mu+g_{\alpha\mu}\partial_\beta\xi^\mu \ . \tag{4.8.8}$$

#### Ansatz as an $X^3$-local conformal transformation of imbedding space

Write $\delta h^k$ as a super-position of $X^3$-local infinitesimal diffeomorphisms of the imbedding space generated by vector fields $J^A=j^{A,k}\partial_k$:

$$\delta h^k = c_A(x)j^{A,k} \ . \tag{4.8.9}$$

This gives

$$\begin{aligned} c_A(x) & \left[D_kj_l^A+D_lj_k^A\right]\partial_\alpha h^k\partial_\beta h^l+2\partial_\alpha c_A h_{kl}j^{A,k}\partial_\beta h^l \\ = & \lambda(x)g_{\alpha\beta}+\partial_\mu g_{\alpha\beta}\xi^\mu+g_{\mu\beta}\partial_\alpha\xi^\mu+g_{\alpha\mu}\partial_\beta\xi^\mu \ . \end{aligned} \tag{4.8.10}$$

If an $X^3$-local variant of a conformal transformation of the imbedding space is in question, the first term is proportional to the metric since one has

$$D_kj_l^A+D_lj_k^A = 2h_{kl} \ . \tag{4.8.11}$$

The transformations in question includes conformal transformations of $H_\pm$ and isometries of the imbedding space $H$.

The contribution of the second term must correspond to an infinitesimal diffeomorphism of $X^3$ reducible to infinitesimal conformal transformation $\psi^\mu$:

$$2\partial_\alpha c_A h_{kl}j^{A,k}\partial_\beta h^l = \xi^\mu\partial_\mu g_{\alpha\beta}+g_{\mu\beta}\partial_\alpha\xi^\mu+g_{\alpha\mu}\partial_\beta\xi^\mu \ . \tag{4.8.12}$$

### A rough analysis of the conditions

One could consider a strategy of fixing $c_A$ and solving solving $\xi^\mu$ from the differential equations. In order to simplify the situation one could assume that $g_{ir} = g_{rr} = 0$. The possibility to cast the metric in this form is plausible since generic 3-manifold allows coordinates in which the metric is diagonal.

1. The equation for $g_{rr}$ gives

$$\partial_r c_A h_{kl} j^{Ak} \partial_r h^k \quad = \quad 0 \ . \tag{4.8.13}$$

   The radial derivative of the transformation is orthogonal to $X^3$. No condition on $\xi^\alpha$ results. If $c_A$ has common multiplicative dependence on $c_A = f(r) d_A$ by a one obtains

$$d_A h_{kl} j^{Ak} \partial_r h^k \quad = \quad 0 \ . \tag{4.8.14}$$

   so that $J^A$ is orthogonal to the light-like tangent vector $\partial_r h^k$ $X^3$ which is the counterpart for the condition that Kac-Moody algebra acts in the transversal degrees of freedom only. The condition also states that the components $g_{ri}$ is not changed in the infinitesimal transformation.

   It is possible to choose $f(r)$ freely so that one can perform the choice $f(r) = r^n$ and the notion of radial conformal weight makes sense. The dependence of $c_A$ on transversal coordinates is constrained by the transversality condition only. In particular, a common scale factor having free dependence on the transversal coordinates is possible meaning that $X^3$- local conformal transformations of $H$ are in question.

2. The equation for $g_{ri}$ gives

$$\partial_r \xi^i \quad = \quad \partial_r c_A h_{kl} j^{Ak} h^{ij} \partial_j h^k \ . \tag{4.8.15}$$

   The equation states that $g_{ri}$ are not affected by the symmetry. The radial dependence of $\xi^i$ is fixed by this differential equation. No condition on $\xi^r$ results. These conditions imply that the local gauge transformations are dynamical with the light-like radial coordinate $r$ playing the role of the time variable. One should be able to fix the transformation more or less arbitrarily at the partonic 2-surface $X^2$.

3. The three independent equations for $g_{ij}$ give

$$\xi^\alpha \partial_\alpha g_{ij} + g_{kj} \partial_i \xi^k + g_{ki} \partial_j \xi^k \quad = \quad \partial_i c_A h_{kl} j^{Ak} \partial_j h^l \ . \tag{4.8.16}$$

   These are 3 differential equations for 3 functions $\xi^\alpha$ on 2 independent variables $x^i$ with $r$ appearing as a parameter. Note however that the derivatives of $\xi^r$ do not appear in the equation. At least formally equations are not over-determined so that solutions should exist for arbitrary choices of $c_A$ as functions of $X^3$ coordinates satisfying the orthogonality conditions. If this is the case, the Kac-Moody algebra can be regarded as a local algebra in $X^3$ subject to the orthogonality constraint.

   This algebra contains as a subalgebra the analog of Kac-Moody algebra for which all $c_A$ except the one associated with time translation and fixed by the orthogonality condition depends on the radial coordinate $r$ only. The larger algebra decomposes into a direct sum of representations of this algebra.

### Commutators of infinitesimal symmetries

The commutators of infinitesimal symmetries need not be what one might expect since the vector fields $\xi^\mu$ are functionals $c_A$ and of the induced metric and also $c_A$ depends on induced metric via the orthogonality condition. What this means that $j^{A,k}$ in principle acts also to $\phi_B$ in the commutator $[c_A J^A, c_B J^B]$.

$$\left[c_A J^A, c_B J^B\right] \;=\; c_A c_B J^{[A,B]} + J^A \circ c_B J^B - J^B \circ c_A J^A \ , \tag{4.8.17}$$

where $\circ$ is a short hand notation for the change of $c_B$ induced by the effect of the conformal transformation $J^A$ on the induced metric.

Luckily, the conditions in the case $g_{rr} = g_{ir} = 0$ state that the components $g_{rr}$ and $g_{ir}$ of the induced metric are unchanged in the transformation so that the condition for $c_A$ resulting from $g_{rr}$ component of the metric is not affected. Also the conditions coming from $g_{ir} = 0$ remain unchanged. Therefore the commutation relations of local algebra apart from constraint from transversality result.

The commutator algebra of infinitesimal symmetries should also close in some sense. The orthogonality to the light-like tangent vector creates here a problem since the commutator does not obviously satisfy this condition automatically. The problem can be solved by following the recipes of non-covariant quantization of string model.

1. Make a choice of gauge by choosing time translation $P^0$ in a preferred $M^4$ coordinate frame to be the preferred generator $J^{A_0} \equiv P^0$, whose coefficient $\Phi_{A_0} \equiv \Psi(P^0)$ is solved from the orthogonality condition. This assumption is analogous with the assumption that time coordinate is non-dynamical in the quantization of strings. The natural basis for the algebra is obtained by allowing only a single generator $J^A$ besides $P^0$ and putting $d_A = 1$.

2. This prescription must be consistent with the well-defined radial conformal weight for the $J^A \neq P^0$ in the sense that the proportionality of $d_A$ to $r^n$ for $J^A \neq P^0$ must be consistent with commutators. SU(3) part of the algebra is of course not a problem. From the Lorentz vector property of $P^k$ it is clear that the commutators resulting in a repeated commutation have well-defined radial conformal weights only if one restricts $SO(3,1)$ to $SO(3)$ commuting with $P^0$. Also $D$ could be allowed without losing well-defined radial conformal weights but the argument below excludes it. This picture conforms with the earlier identification of the Kac-Moody algebra.

   Conformal algebra contains besides Poincare algebra and the dilation $D = m^k\partial_{m^k}$ the mutually commuting generators $K^k = (m^r m_r \partial_{m^k} - 2m^k m^l \partial_{m^l})/2$. The commutators involving added generators are

   $$\begin{array}{ll} \left[D, K^k\right] = -K^k \ , & \left[D, P^k\right] = P^k \ , \\ \left[K^k, K^l\right] = 0 \ , & \left[K^k, P^l\right] = m^{kl} D - M^{kl} \ . \end{array} \tag{4.8.18}$$

   From the last commutation relation it is clear that the inclusion of $K^k$ would mean loss of well-defined radial conformal weights.

3. The coefficient $dm^0/dr$ of $\Psi(P^0)$ in the equation

   $$\Psi(P^0)\frac{dm^0}{dr} = -J^{Ak} h_{kl} \partial_r h^l$$

   is always non-vanishing due to the light-likeness of $r$. Since $P^0$ commutes with generators of $SO(3)$ (but not with $D$ so that it is excluded!), one can *define* the commutator of two generators as a commutator of the remaining part and identify $\Psi(P^0)$ from the condition above.

4. Of course, also the more general transformations act as Kac-Moody type symmetries but the interpretation would be that the sub-algebra plays the same role as $SO(3)$ in the case of Lorentz group: that is gives rise to generalized spin degrees of freedom whereas the entire algebra divided by this sub-algebra would define the coset space playing the role of orbital degrees of freedom. In

fact, also the Kac-Moody type symmetries for which $c_A$ depends on the transversal coordinates of $X^3$ would correspond to orbital degrees of freedom. The presence of these orbital degrees of freedom arranging super Kac-Moody representations into infinite multiplets labeled by function basis for $X^2$ means that the number of degrees of freedom is much larger than in string models.

5. It is possible to replace the preferred time coordinate $m^0$ with a preferred light-like coordinate. There are good reasons to believe that orbifold singularity for phases of matter involving non-standard value of Planck constant corresponds to a preferred light-ray going through the tip of $\delta M^4_\pm$. Thus it would be natural to assume that the preferred $M^4$ coordinate varies along this light ray or its dual. The Kac-Moody group $SO(3)\times E^3$ respecting the radial conformal weights would reduce to $SO(2)\times E^2$ as in string models. $E^2$ would act in tangent plane of $S^2_\pm$ along this ray defining also $SO(2)$ rotation axis.

#### Hamiltonians

The action of these transformations on Kähler action is well-defined and one can deduce the conserved quantities having identification as configuration space Hamiltonians. Hamiltonians also correspond to closed 2-forms. The condition that the Hamiltonian reduces to a dual of closed 2-form is satisfied because $X^2$-local conformal transformations of $M^4_\pm\times CP_2$ are in question ($X^2$-locality does not imply any additional conditions).

#### The action of Kac-Moody algebra on spinors and fermionic representations of Kac-Moody algebra

One can imagine two interpretations for the action of generalized Kac-Moody transformations on spinors.

1. The basic goal is to deduce the fermionic Noether charge associated with the bosonic Kac-Moody symmetry and this can be done by a standard recipe. The first contribution to the charge comes from the transformation of modified gamma matrices appearing in the modified Dirac action associated with fermions. Second contribution comes from spinor rotation.

2. Both SO(3) and SU(3) rotations have a standard action as spin rotation and electro-weak rotation allowing to define the action of the Kac-Moody algebra $J^A$ on spinors.

#### How central extension term could emerge?

The central extension term of Kac-Moody algebra could correspond to a symplectic extension which can emerge from the freedom to add a constant term to Hamiltonians as in the case of super-symplectic algebra. The expression of the Hamiltonians as closed forms could allow to understand how the central extension term emerges.

In principle one can construct a representation for the action of Kac-Moody algebra on fermions a representations as a fermionic bilinear and the central extension of Kac-Moody algebra could emerge in this construction just as it appears in Sugawara construction.

#### About the interpretation of super Kac-Moody symmetries

Also the light like 3-surfaces $X^3_l$ of $H$ defining elementary particle horizons at which Minkowskian signature of the metric is changed to Euclidian and boundaries of space-time sheets can act as causal determinants, and thus contribute to the configuration space metric. In this case the symmetries correspond to the isometries of the imbedding space localized with respect to the complex coordinate of the 2-surface $X^2$ determining the light like 3-surface $X^3_l$ so that Kac-Moody type symmetry results. Also the condition $\sqrt{g_3}=0$ for the determinant of the induced metric seems to define a conformal symmetry associated with the light like direction.

If is enough to localize only the $H$-isometries with respect to $X^3_l$, the purely bosonic part of the Kac-Moody algebra corresponds to the isometry group $M^4\times SO(3,1)\times SU(3)$. The physical interpretation of these symmetries is not so obvious as one might think. The point is that one can generalize the formulas characterizing the action of infinitesimal isometries on spinor fields of finite-dimensional Kähler manifold to the level of the configuration space. This gives rise to bosonic generators containing also a sigma-matrix term bilinear in fermionic oscillator operators. This representation need not be equivalent with the purely fermionic representations provided by induced Dirac action. Thus one has two groups of local color charges and the challenge is to find a physical interpretation for them.

The following arguments support one possible identification.

1. The hint comes from the fact that $U(2)$ in the decomposition $CP_2 = SU(3)/U(2)$ corresponds in a well-defined sense electro-weak algebra identified as a holonomy algebra of the spinor connection. Hence one could argue that the $U(2)$ generators of either $SU(3)$ algebra might be identifiable as generators of local $U(2)_{ew}$ gauge transformations whereas non-diagonal generators would correspond to Higgs field. This interpretation would conform with the idea that Higgs field is a genuine scalar field rather than a composite of fermions.

2. Since $X^3_l$-local $SU(3)$ transformations represented by fermionic currents are characterized by central extension they would naturally correspond to the electro-weak gauge algebra and Higgs bosons. This is also consistent with the fact that both leptons and quarks define fermionic Kac Moody currents.

3. The fact that only quarks appear in the gamma matrices of the configuration space supports the view that action of the generators of $X^3_l$-local color transformations on configuration space spinor fields represents local color transformations. If the action of $X^3_l$-local $SU(3)$ transformations on configuration space spinor fields has trivial central extension term the identification as a representation of local color symmetries is possible.

The topological explanation of the family replication phenomenon is based on an assignment of 2-dimensional boundary to a 3-surface characterizing the elementary particle. The precise identification of this surface has remained open and one possibility is that the 2-surface $X^2$ defining the light light-like surface associated with an elementary particle horizon is in question. This assumption would conform with the notion of elementary particle vacuum functionals defined in the zero modes characterizing different conformal equivalences classes for $X^2$.

### The relationship of the Super-Kac Moody symmetry to the standard super-conformal invariance

Super-Kac Moody symmetry can be regarded as $N = 4$ complex super-symmetry with complex $H$-spinor modes of $H$ representing the 4 physical helicities of 8-component leptonic and quark like spinors acting as generators of complex dynamical super-symmetries. The super-symmetries generated by the covariantly constant right handed neutrino appear with *both* $M^4$ helicities: it however seems that covariantly constant neutrino does not generate any global super-symmetry in the sense of particle-sparticle mass degeneracy. Only righthanded neutrino spinor modes (apart from covariantly constant mode) appear in the expressions of configuration space gamma matrices forming a subalgebra of the full super-algebra.

$N = 2$ real super-conformal algebra is generated by the energy momentum tensor $T(z)$, $U(1)$ current $J(z)$, and super generators $G^{\pm}(z)$ carrying $U(1)$ charge. Now $U(1)$ current would correspond to right-handed neutrino number and super generators would involve contraction of covariantly constant neutrino spinor with second quantized induced spinor field. The further facts that $N = 2$ algebra is associated naturally with Kähler geometry, that the partition functions associated with $N = 2$ super-conformal representations are modular invariant, and that $N = 2$ algebra defines so called chiral ring defining a topological quantum field theory [A38], lend a further support for the belief that $N = 2$ super-conformal algebra acts in super-symplectic degrees of freedom.

The values of $c$ and conformal weights for $N = 2$ super-conformal field theories are given by

$$\begin{aligned} c &= \frac{3k}{k+2} \ , \\ \Delta_{l,m}(NS) &= \frac{l(l+2)-m^2}{4(k+2)} \ , \quad l = 0,1,...,k \ , \\ q_m &= \frac{m}{k+2} \ , \quad m = -l, -l+2, ...., l-2, l \ . \end{aligned} \tag{4.8.19}$$

$q_m$ is the fractional value of the $U(1)$ charge, which would now correspond to a fractional fermion number. For $k = 1$ one would have $q = 0, 1/3, -1/3$, which brings in mind anyons. $\Delta_{l=0,m=0} = 0$ state would correspond to a massless state with a vanishing fermion number. Note that $SU(2)_k$ Wess-Zumino model has the same value of $c$ but different conformal weights. More information about conformal algebras can be found from the appendix of [A38].

For Ramond representation $L_0 - c/24$ or equivalently $G_0$ must annihilate the massless states. This occurs for $\Delta = c/24$ giving the condition $k = 2\left[l(l+2) - m^2\right]$ (note that $k$ must be even and that $(k, l, m) = (4, 1, 1)$ is the simplest non-trivial solution to the condition). Note the appearance of a fractional vacuum fermion number $q_{vac} = \pm c/12 = \pm k/4(k+2)$. I have proposed that NS and Ramond algebras could combine to a larger algebra containing also lepto-quark type generators but this not necessary.

The conformal algebra defined as a direct sum of Ramond and NS $N = 4$ complex sub-algebras associated with quarks and leptons might further extend to a larger algebra if lepto-quark generators acting effectively as half odd-integer Virasoro generators can be allowed. The algebra would contain spin and electro-weak spin as fermionic indices. Poincare and color Kac-Moody generators would act as symplectically extended isometry generators on configuration space Hamiltonians expressible in terms of Hamiltonians of $X^3_l \times CP_2$. Electro-weak and color Kac-Moody currents have conformal weight $h = 1$ whereas $T$ and $G$ have conformal weights $h = 2$ and $h = 3/2$.

The experience with $N = 4$ complex super-conformal invariance suggests that the extended algebra requires the inclusion of also second quantized induced spinor fields with $h = 1/2$ and their super-partners with $h = 0$ and realized as fermion-antifermion bilinears. Since $G$ and $\Psi$ are labeled by $2 \times 4$ spinor indices, super-partners would correspond to $2 \times (3 + 1) = 8$ massless electro-weak gauge boson states with polarization included. Their inclusion would make the theory highly predictive since induced spinor and electro-weak fields are the fundamental fields in TGD.

### 4.8.5 Coset space structure for configuration space as a symmetric space

The key ingredient in the theory of symmetric spaces is that the Lie-algebra of $G$ has the following decomposition

$$\begin{aligned} &g = h + t \ , \\ &[h, h] \subset h \ , \quad [h, t] \subset t \ , \quad [t, t] \subset h \ . \end{aligned}$$

In present case this has highly nontrivial consequences. The commutator of *any* two infinitesimal generators generating nontrivial deformation of 3-surface belongs to $h$ and thus vanishing norm in the configuration space metric at the point which is left invariant by $H$. In fact, this same condition follows from Ricci flatness requirement and guarantees also that $G$ acts as isometries of the configuration space. This generalization is supported by the properties of the unitary representations of Lorentz group at the light cone boundary and by number theoretical considerations.

The algebras suggesting themselves as candidates are symplectic algebra of $\delta M^{\pm} \times CP_2$ and Kac-Moody algebra mapping light-like 3-surfaces to light-like 3-surfaces to be discussed in the next section.

The identification of the precise form of the coset space structure is however somewhat delicate.

1. The essential point is that both symplectic and Kac-Moody algebras allow representation in terms of $X^3_l$-local Hamiltonians. The general expression for the Hamilton of Kac-Moody algebra is

$$H = \sum \Phi_A(x) H^A \ . \tag{4.8.20}$$

   Here $H^A$ are Hamiltonians of $SO(3) \times SU(3)$ acting in $\delta X^3_l \times CP_2$. For symplectic algebra any Hamiltonian is allowed. If $x$ corresponds to any point of $X^3_l$, one must assume a slicing of the causal diamond $CD$ by translates of $\delta M^4_{\pm}$.

2. For symplectic generators the dependence of form on $r^{\Delta}$ on light-like coordinate of $\delta X^3_l \times CP_2$ is allowed. $\Delta$ is complex parameter whose modulus squared is interpreted as conformal weight. $\Delta$ is identified as analogous quantum number labeling the modes of induced spinor field.

3. One can wonder whether the choices of the $r_M = constant$ sphere $S^2$ is the only choice. The Hamiltonin-Jacobi coordinate for $X^4_{X^3_l}$ suggest an alternative choice as $E^2$ in the decomposition of $M^4 = M^2(x) \times E^2(x)$ required by number theoretical compactification and present for known extremals of Kähler action with Minkowskian signature of induced metric. In this case $SO(3)$ would be replaced with $SO(2)$. It however seems that the radial light-like coordinate $u$ of $X^4(X^3_l)$ would remain the same since any other curve along light-like boundary would be space-like.

4. The vector fields for representing Kac-Moody algebra must vanish at the partonic 2-surface $X^2 \subset \delta M^4_\pm \times CP_2$. The corresponding vector field must vanish at each point of $X^2$:

$$j^k = \sum \Phi_A(x) J^{kl} H^A_l = 0 \ . \tag{4.8.21}$$

   This means that the vector field corresponds to $SO(2) \times U(2)$ defining the isotropy group of the point of $S^2 \times CP_2$.

   This expression could be deduced from the idea that the surfaces $X^2$ are analogous to origin of $CP_2$ at which $U(2)$ vector fields vanish. Configuration space at $X^2$ could be also regarded as the analog of the origin of local $S^2 \times CP_2$. This interpretation is in accordance with the original idea which however was given up in the lack of proper realization. The same picture can be deduced from braiding in which case the Kac-Moody algebra corresponds to local $SO(2) \times U(2)$ for each point of the braid at $X^2$. The condition that Kac-Moody generators with positive conformal weight annihilate physical states could be interpreted by stating effective 2-dimensionality in the sense that the deformations of $X^3_l$ preserving its light-likeness do not affect the physics. Note however that Kac-Moody type Virasoro generators do not annihilate physical states.

5. Kac-Moody algebra generator must leave induced Kähler form invariant at $X^2$. This is of course trivial since the action leaves each point invariant. The conditions of Cartan decomposition are satisfied. The commutators of the Kac-Moody vector fields with symplectic generators are non-vanishing since the action of symplectic generator on Kac-Moody generator restricted to $X^2$ gives a non-vanishing result belonging to the symplectic algebra. Also the commutators of Kac-Moody generators are Kac-Moody generators.

### 4.8.6 The relationship between super-symplectic and Super Kac-Moody algebras, Equivalence Principle, and justification of p-adic thermodynamics

The relationship between super-symplectic algebra ($SS$) acting at light-cone boundary and Super Kac-Moody algebra ($SKM$) acting on light-like 3-surfaces has remained somewhat enigmatic due to the lack of physical insights. This is not the only problem. The question to precisely what extent Equivalence Principle (EP) remains true in TGD framework and what might be the precise mathematical realization of EP is waiting for an answer. Also the justification of p-adic thermodynamics for the scaling generator $L_0$ of Virasoro algebra -in obvious conflict with the basic wisdom that this generator should annihilate physical states- is lacking. It seems that these three problems could have a common solution.

#### New vision about the relationship between $SSV$ and $SKMV$

Consider now the new vision about the relationship between $SSV$ and $SKMV$.

1. The isometries of $H$ assignable with $SKM$ are also symplectic transformations [K10] (note that I have used the attribute "canonical" instead of "symplectic" previously). Hence might consider the possibility that $SKM$ could be identified as a subalgebra of $SS$. If this makes sense, a generalization of the coset construction obtained by replacing finite-dimensional Lie group with infinite-dimensional symplectic group suggests itself. The differences of $SSV$ and $SKMV$ elements would annihilate physical states and commute/anticommute with $SKMV$. Also the generators $O_n$, $n > 0$, for both algebras would annihilate the physical states so that the differences of the elements would annihilate automatically physical states for $n > 0$.

2. The super-generator $G_0$ contains the Dirac operator $D$ of $H$. If the action of $SSV$ and $SKMV$ Dirac operators on physical states are identical then cm of degrees of freedom disappear from the differences $G_0(SCV) - G_0(SKMV)$ and $L_0(SCV) - L_0(SKMV)$. One could interpret the identical action of the Dirac operators as the long sought-for precise realization of Equivalence Principle (EP) in TGD framework. EP would state that the total inertial four-momentum and color quantum numbers assignable to $SS$ (imbedding space level) are equal to the gravitational four-momentum and color quantum numbers assignable to $SKM$ (space-time level). Note that since super-symplectic transformations correspond to the isometries of the "world of classical worlds" the assignment of the attribute "inertial" to them is natural.

### Consistency with p-adic thermodynamics

The consistency with p-adic thermodynamics provides a strong reality test and has been already used as a constraint in attempts to understand the super-conformal symmetries in partonic level.

1. In physical states the p-adic thermal expectation value of the $SKM$ and $SS$ conformal weights would be non-vanishing and identical and mass squared could be identified equivalently either as the expectation value of $SKM$ or $SS$ scaling generator $L_0$. There would be no need to give up Super Virasoro conditions for $SCV - SKMV$.

2. There is consistency with p-adic mass calculations for hadrons [K35] since the non-perturbative $SS$ contributions and perturbative $SKM$ contributions to the mass correspond to space-time sheets labeled by different p-adic primes. The earlier statement that $SS$ is responsible for the dominating non-perturbative contributions to the hadron mass transforms to a statement reflecting $SS{-}SKM$ duality. The perturbative quark contributions to hadron masses can be calculated most conveniently by using p-adic thermodynamics for $SKM$ whereas non-perturbative contributions to hadron masses can be calculated most conveniently by using p-adic thermodynamics for $SS$. Also the proposal that the exotic analogs of baryons resulting when baryon looses its valence quarks [K32] remains intact in this framework.

3. The results of p-adic mass calculations depend crucially on the number $N$ of tensor factors contributing to the Super-Virasoro algebra. The required number is $N = 5$ and during years I have proposed several explanations for this number. It seems that holonomic contributions that is electro-weak and spin contributions must be regarded as contributions separate from those coming from isometries. $SKM$ algebras in electro-weak degrees and spin degrees of of freedom, would give 2+1=3 tensor factors corresponding to $U(2)_{ew} \times SU(2)$. $SU(3)$ and $SO(3)$ (or $SO(2) \subset SO(3)$ leaving the intersection of light-like ray with $S^2$ invariant) would give 2 additional tensor factors. Altogether one would indeed have 5 tensor factors.

There are some further questions which pop up in mind immediately.

1. Why mass squared corresponds to the thermal expectation value of the net conformal weight? This option is forced among other things by Lorentz invariance but it is not possible to provide a really satisfactory answer to this question yet. In the coset construction there is no reason to require that the mass squared equals to the integer value conformal weight for SKM algebra. This allows the possibility that mass squared has same value for states with different values of SKM conformal weights appearing in the thermal state and equals to the average of the conformal weight.

2. The coefficient of proportionality can be however deduced from the observation that the mass squared values for $CP_2$ Dirac operator correspond to definite values of conformal weight in p-adic mass calculations. It is indeed possible to assign to partonic 2-surface $X^2$ $CP_2$ partial waves correlating strongly with the net electro-weak quantum numbers of the parton so that the assignment of ground state conformal weight to $CP_2$ partial waves makes sense.

3. In the case of $M^4$ degrees of freedom it is strictly speaking not possible to talk about momentum eigen states since translations take parton out of $\delta H_+$. This would suggests that 4-momentum must be assigned with the tip of the light-cone containing the particle but this is not consistent with zero energy ontology. Hence it seems that one must restrict the translations of $X_l^3$ to

time like translations in the direction of geometric future at $\delta M^4_+ \times CP_2$. The decomposition of the partonic 3-surface $X^3_l$ to regions $X^3_{l,i}$ carrying non-vanishing induced Kähler form and the possibility to assign $M^2(x) \subset M^4$ to the tangent space of $X^4(X^3_l)$ at points of $X^3_l$ suggests that the points of number theoretic braid to which oscillator operators can be assigned can carry four-momentum in the plane defined by $M^2(x)$. One could assume that the four-momenta assigned with points in given region $X^3_i$ are collinear but even this restriction is not necessary.

4. The additivity of conformal weight means additivity of mass squared at parton level and this has been indeed used in p-adic mass calculations. This implies the conditions

$$(\sum_i p_i)^2 = \sum_i m_i^2 \tag{4.8.22}$$

The assumption $p_i^2 = m_i^2$ makes sense only for massless partons moving collinearly. In the QCD based model of hadrons only longitudinal momenta and transverse momentum squared are used as labels of parton states, which together with the presence of preferred plane $M^2$ would suggest that one has

$$\begin{aligned} p^2_{i,||} &= m_i^2 \ , \\ -\sum_i p^2_{i,\perp} + 2\sum_{i,j} p_i \cdot p_j &= 0 \ . \end{aligned} \tag{4.8.23}$$

The masses would be reduced in bound states: $m_i^2 \to m_i^2 - (p_T^2)_i$. This could explain why massive quarks can behave as nearly massless quarks inside hadrons.

**How it is possible to have negative conformal weights for ground states?**

p-Adic mass calculations require negative conformal weights for ground states [K29] . The only elegant solution of the problems caused by this requirement seems to be p-adic: the conformal weights are positive in the real sense but as p-adic numbers their dominating part is negative integer (in the real sense), which can be compensated by the conformal weights of Super Virasoro generators.

1. If $\pm\lambda_i^2$ as such corresponds to a ground state conformal weight and if $\lambda_i$ is real the ground state conformal weight positive in the real sense. In complex case (instanton term) the most natural formula is $h = \pm|\lambda|^2$.

2. The first option is based on the understanding of conformal excitations in terms of CP breaking instanton term added to the modified Dirac operator. In this case the conformal weights are identified as $h = n - |\lambda_k|^2$ and the minus sign comes from the Euclidian signature of the effective metric for the modified Dirac operator. Ground state conformal weight would be non-vanishing for non-zero modes of $D(X^3_l)$. Massless bosons produce difficulties unless one has $h = |\lambda_i(1) - \lambda_i(2)|^2$, where $i = 1,2$ refers to the two wormhole throats. In this case the difference can vanish and its non-vanishing would be due to the symmetric breaking. This scenario is assumed in p-adic mass calculations. Fermions are predicted to be always massive since zero modes of $D(X^2)$ represent super gauge degrees of freedom.

3. In the context of p-adic thermodynamics a loop hole opens allowing $\lambda_i$ to be real. In spirit of rational physics suppose that one has in natural units $h = \lambda_i^2 = xp^2 - n$, where $x$ is integer. This number is positive and large in the real sense. In p-adic sense the dominating part of this number is $-n$ and can be compensated by the net conformal weight $n$ of Super Virasoro generators acting on the ground state. $xp^2$ represents the small Higgs contribution to the mass squared proportional to $(xp^2)_R \simeq x/p^2$ ($_R$ refers to canonical identification ). By the basic features of the canonical identification $p > x \simeq p$ should hold true for gauge bosons for which Higgs contribution dominates. For fermions $x$ should be small since p-adic mass calculations are

consistent with the vanishing of Higgs contribution to the fermion mass. This would lead to the earlier conclusion that $xp^2$ and hence $B_K$ is large for bosons and small for fermions and that the size of fermionic (bosonic) wormhole throat is large (small). This kind of picture is consistent with the p-adic modular arithmetics and suggests by the cutoff for conformal weights implied by the fact that both the number of fermionic oscillator operators and the number of points of number theoretic braid are finite. This solution is however tricky and does not conform with number theoretical universality.

### 4.8.7 Comparison of TGD and stringy views about super-conformal symmetries

The best manner to represent TGD based view about conformal symmetries is by comparison with the conformal symmetries of super string models.

#### Basic differences between the realization of super conformal symmetries in TGD and in super-string models

The realization super-symmetries in TGD framework differs from that in string models in several fundamental aspects.

1. In TGD framework super-symmetry generators acting as configuration space gamma matrices carry either lepton or quark number. Majorana condition required by the hermiticity of super generators which is crucial for super string models would be in conflict with the conservation of baryon and lepton numbers and is avoided. This is made possible by the realization of bosonic generators represented as Hamiltonians of symplectic transformations rather than vector fields generating them. This kind of representation applies also in Kac-Moody sector since the local transversal isometries localized in $X_l^3$ and respecting light-likeness condition can be regarded as $X^2$ local symplectic transformations, whose Hamiltonians generate also isometries. The fermionic representations of super-symplectic and super Kac-Moody generators can be identified as Noether charges in standard manner.

2. Super-symmetry generators can be identified as configuration space gamma matrices carrying quark and lepton numbers and the notion of super-space is not needed at all. Therefore no super-variant of geometry is needed. The distinction between Ramond and N-S representations important for $N = 1$ super-conformal symmetry and allowing only ground state weight 0 an 1/2 disappears. Indeed, for $N = 2$ super-conformal symmetry it is already possible to generate spectral flow transforming these Ramond and N-S representations to each other ($G_n$ is not Hermitian anymore). This means that the interpretation of $\lambda_i^2$ ($\lambda_i$ is generalized eigenvalue of $D_K(X^2)$) as ground state conformal weight does not lead to difficulties.

3. Kac-Moody and symplectic algebras generate larger algebra obtained by making symplectic algebra $X^2$ local. This realization of super symmetries is what distinguishes between TGD and super string models and leads to a totally different physical interpretation of super-conformal symmetries. What makes spinor field mode a generator of gauge super-symmetry is that is c-number and not an eigenmode of $D_K(X^2)$ and thus represents non-dynamical degrees of freedom. If the number of eigen modes of $D_K(X^2)$ is indeed finite means that most of spinor field modes represent super gauge degrees of freedom. One must be here somewhat cautious since bound state in the Coulomb potential associated with electric part of induced electro-weak gauge field might give rise to an infinite number of bound states which eigenvalues converging to a fixed eigenvalue (as in the case of hydrogen atom).

4. The finite number of spinor modes means that the representations of super-conformal algebras reduces to finite-dimensional ones in TGD framework and the notion of number theoretic braid indeed implies this. The physical interpretation is in terms of finite measurement resolution.

#### Basic super-conformal symmetries

The identification of explicit representations of super conformal algebras was for a long time plagued by the lack of appropriate formalism. The modified Dirac operator $D_K$ associated with Kähler action resolves this problem if one accepts the implications of number theoretic compactification supported by what is known about preferred extremals of Kähler action and one can identify the charges associated with symplectic and Kac-Moody algebra as Noether charges. Fermionic generators can in turn be identified from the condition that they anticommute to $X^2$ local Hamiltonians of corresponding bosonic transformations. In case of Super Virasoro algebra Sugaware construction allows to construct super generators $G$.

1. Covariantly constant right handed neutrino is the fundamental generator of dynamical super conformal symmetries and appears in both leptonic and quark-like realizations of gamma matrices. $\Gamma$ matrices have also Super Kac-Moody counterparts and reduce in special case to symplectic ones. Also super currents whose anti-commutators give products of corresponding Hamiltonians can be defined so that both ordinary product and Poisson bracket give rise to quark and lepton like realizations of super-symmetries. Besides this there are also electric and magnetic representations of the gamma matrices.

2. The zero modes of $D_K(X^2)$ which do not depend on the light-like radial coordinate of $X_l^3$ define super conformal symmetries for which any c-number spinor field generates super conformal symmetry. These symmetries are pure gauge symmetries but also them can be parameterized by Hamiltonians and by functions depending only on the coordinates of the transverse section $X^2$ so that one obtains also now both function algebra and symplectic algebra localized with respect to $X^2$. Similar picture applies in both super-symplectic and super Kac-Moody sector. In particular, one can deduce canonical expressions for the super currents associated with these super symmetries. Since all charge states are possible for the generators of these super symmetries, these super symmetries naturally correspond to those assignable to electro-weak degrees of freedom.

3. The notion of $X^2$ local super-symmetry makes sense if the choice of coordinates $x$ for $X^2$ is specified by the inherent properties of $X^2$ so that same coordinates $x$ apply for all surfaces obtained as deformations of $X^2$. The regions, where induced Kähler form is non-vanishing define good candidates for coordinate patches. The Hamilton-Jacobi coordinates associated with the decomposition of $M^4$ are a natural choice. Also geodesic coordinates can be considered. The redundancy related to rotations of coordinate axis around origin can be reduced by choosing second axis so that it connects the origin to nearest point of the number theoretic braid.

4. The diffeomorphisms of light-like coordinate of $\delta M^4_\pm$ and $X_l^3$ playing the role of conformal transformations. One can construct fermionic representations of as Noether charges associated with modified Dirac action. The problem is however that that super-generators cannot be derived in this manner so that these transformations cannot be regarded as symplectic transformations. The manner to circumvent the difficulty is to construct fermionic super charges $\Gamma_A$ as gamma matrices for both super symplectic and super Kac-Moody algebras in terms of generators $j^{Ak}\Gamma_k$ and corresponding Kac-Moody algebra elements $T^A$ as fermionic super charges. From these operators super generators $G$ can be constructed by the standard Sugawara construction allowing to interpret operators $G = T^A\Gamma_A$ as Dirac operators at the level of configuration space. By coset construction the actions of super-symplectic and super Kac-Moody Dirac operators are identical. Internal consistency requires that the Virasoro generators obtained as anticommutator $L = \{G, G^\dagger\}$ are equal to the Virasoro generators derived as fermionic Noether charges.

#### Finite measurement resolution and cutoff in the spectrum of conformal weights

The basic properties of Kähler action imply that the number generalized eigenvalues $\lambda_i$ of $D_K(X^2)$ is finite. The interpretation is that the notion of finite measurement resolution is coded by Kähler action to space-time dynamics. This has also implications for the representations of super-conformal algebras.

1. The fermionic representations of various super-algebras involve only finite number of oscillator operators. Hence some kind of cutoff in the number of states reflecting the finiteness of the measurement resolution is unavoidable. A cutoff reduce integers as labels of the generators of super-conformal algebras to a finite number of integers. Finite field $G(p, 1)$ for some prime $p$

would be a natural candidate. Since p-adic integers modulo $p$ are in question the cutoff could relate closely to effective p-adicity and p-adic length scale-hypothesis.

2. The interpretation of the eigenvalues of the modified Dirac operator as ground state conformal weights raises the question how to represent states with conformal weights $n + \lambda_i^2$, $n > 0$. The notion of number theoretic braid allows to circumvent the difficulty. Since canonical anti-commutation relations fail, one must replace the integral representations of super-conformal generators with discrete sums over the points of number theoretic braid, the resulting representations of super-conformal algebras must reduce to representation of finite-dimensional algebras. The cutoff on conformal weight must result from the fact that the higher Virasoro generators are expressible in terms of lower ones. The cutoff is not a problem since $n < 3$ cutoff for conformal weights gives an excellent accuracy in p-adic mass calculations. A not-very-educated guess but the only one that one can imagine is that for $p \simeq 2^k$, $n_{max} = k$ defines the cutoff on allowed conformal weights.

### What are the counter parts of stringy conformal fields in TGD framework?

The experience with string models would suggest the conformal symmetries associated with the complex coordinates of $X^2$ as a candidate for conformal super-symmetries. One can imagine two counterparts of the stringy coordinate $z$ in TGD framework.

1. Super-symplectic and super Kac-Moody symmetries are local with respect to $X^2$ in the sense that the coefficients of generators depend on the invariant $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ rather than being completely free [K10] . Thus the real variable $J$ replaces complex coordinate and effective 1-dimensionality holds true also now but in different sense than for conformal field theories.

2. The slicing of $X^2$ by string world sheets $Y^2$ and partonic 2-surfaces $X^2$ implied by number theoretical compactification implies string-parton duality and involves the super conformal fermionic gauge symmetries associated with the coordinates $u$ and $w$ in the dual dimensional reductions to stringy and partonic dynamics. These coordinates define the natural analogs of stringy coordinate.

3. An further identification for TGD parts of conformal fields is inspired by $M^8 - H$ duality. Conformal fields would be fields in configuration space. The counterpart of $z$ coordinate could be the hyper-octonionic $M^8$ coordinate $m$ appearing as argument in the Laurent series of configuration space Clifford algebra elements. $m$ would characterize the position of the tip of $CD$ and the fractal hierarchy of $CD$s within $CD$s would give a hierarchy of Clifford algebras and thus inclusions of hyper-finite factors of type $II_1$. Reduction to hyper-quaternionic field -that is field in $M^4$ center of mass degrees of freedom- would be needed to obtained associativity. The arguments $m$ at various level might correspond to arguments of N-point function in quantum field theory.

### Generalized coset representation

$X^2$ local super-symplectic algebra as super Kac-Moody algebra as sub-algebra. Since $X^2$ locality corresponds to a full 2-D gauge invariance, one can conclude that SKM is in well defined sense sub-algebra of super-symplectic algebra so that generalized coset construction makes sense and generalizes Equivalence Principle in the sense that not only four-momenta but all analogous quantum numbers associated with SKM and SS algebras are identical.

1. In this framework the ground state conformal weights associated with both super-symplectic and super Kac-Moody algebras can be identified as squares of the eigenvalues $\lambda_i$ of $D_K(X^2)$. This identification together with p-adic mass thermodynamics predicts that $\lambda_i^2$ gives to mass squared a contribution analogous to the square of Higgs vacuum expectation. This identification would resolve the long-standing problem of identifying the values of these ground state conformal weights for super-conformal algebras and give a direct connection with Higgs mechanism.

2. The identification of SKM as a sub-algebra of super-symplectic algebra becomes more convincing if the light-like coordinate $r$ allows lifting to a light-like coordinate of $H$. This is achieved if $r$

is identified as coordinate associated with a light-like curve whose tangent at point $x \in X_l^3$ is light-like vector in $M^2(x) \subset T(X^4(X^3)$. With this interpretation of SKM algebra as sub-algebra of super-symplectic algebra becomes natural.

3. The existence of a lifting of $SS$ and $SKM$ algebras to entire $H$ would solve the problems. The lifting problem is obviously non-trivial only in $M^4$ degrees of freedom. Suppose that the existence of an integrable distribution of planes $M^2(x)$ and their orthogonal complements $E^2(x)$ belonging to the tangent space of $M^4$ projection $P_{M^4}(X^4(X^3))$ characterizes the preferred extremals with Minkowskian signature of induced metric. In this case the lifting of the super-symplectic and super Kac-Moody algebras to entire $H$ is possible. The local degrees of freedom contributing to the configuration space metric would belong to the integrable distribution of orthogonal complements $E^2(x)$ of $M^2(x)$ having physical interpretation as planes of physical polarizations.

# Chapter 5

# Does the Modified Dirac Equation Define the Fundamental Action Principle?

## 5.1 Introduction

Although quantum criticality in principle predicts the possible values of Kähler coupling strength, one might hope that there exists even more fundamental approach involving no coupling constants and predicting even quantum criticality and realizing quantum gravitational holography. The Dirac determinant associated with the modified Dirac action is an excellent candidate in this respect.

The original working hypothesis was that Dirac determinant defines the vacuum functional of the theory having interpretation as the exponent of Kähler function of world of classical worlds (WCW) and that Kähler function reduces to Kähler action for a preferred extremal of Kähler action.

Two alternative choices represented themselves as candidates for the modified Dirac action: either the 3-D Chern-Simons Dirac action or 4-D Kähler action with imaginary measurement interaction term added. Quite recently it became clear that the addition of a measurement interaction term to either Chern-Simons action or Kähler action resolves a bundle of conceptual problems. The question which option is correct is not completely settled yet although it seems that the measurement interaction term assigned to Chern-Simons-Dirac action creates more problems that it solves.

### 5.1.1 What are the basic equations of quantum TGD?

A good place to start is to as what might the basic equations of quantum TGD. There are two kinds of equations at the level of space-time surfaces.

1. Purely classical equations define the dynamics of the space-time sheets as preferred extremals of Kähler action. Preferred extremals are quantum critical in the sense that second variation vanishes for critical deformations representing zero modes. This condition guarantees that corresponding fermionic currents are conserved. There is infinite hierarchy of these currents and they define fermionic counterparts for zero modes. Space-time sheets can be also regarded as hyper-quaternionic surfaces. What these statements precisely mean has become clear only during this year. A rigorous proof for the equivalence of these two identifications is still lacking.

2. The purely quantal equations are associated with the representations of various super-conformal algebras and with the modified Dirac equation. The requirement that there are deformations of the space-time surface -actually infinite number of them- giving rise to conserved fermionic charges implies quantum criticality at the level of Kähler action in the sense of critical deformations. The precise form of the modified Dirac equation is not however completely fixed without further input. Quantal equations involve also generalized Feynman rules for $M$-matrix generalizing $S$-matrix to a "complex square root" of density matrix and defined by time-like entanglement coefficients between positive and negative energy parts of zero energy states is certainly the basic goal of quantum TGD.

3. The notion of weak electric-magnetic duality leads to a detailed understanding of how TGD reduces to almost topological quantum field theory. If Kähler current defines Beltrami flow it is possible to find a gauge in which Coulomb contribution to Kähler action vanishes so that it reduces to Chern-Simons term. If light-like 3-surfaces and ends of space-time surface are extremals of Chern-Simons action also effective 2-dimensionality is realized. The condition that the theory reduces to almost topological QFT and the hydrodynamical character of field equations leads to a detailed ansatz for the general solution of field equations and also for the solutions of the modified Dirac equation relying on the notion of Beltrami flow for which the flow parameter associated with the flow lines defined by a conserved current extends to a global coordinate. This makes the theory is in well-defined sense completely integrable. Direct connection with massless theories emerges: every conserved Beltrami currents corresponds to a pair of scalar functions with the first one satisfying massless d'Alembert equation in the induced metric. The orthogonality of the gradients of these functions allows interpretation in terms of polarization and momentum directions. The Beltrami flow property can be also seen as one aspect of quantum criticality since the conserved currents associated with critical deformations define this kind of pairs.

4. The hierarchy of Planck constants provides also a fresh view to the quantum criticality. The original justification for the hierarchy of Planck constants came from the indications that Planck constant could have large values in both astrophysical systems involving dark matter and also in biology. The realization of the hierarchy in terms of the singular coverings and possibly also factor spaces of $CD$ and $CP_2$ emerged from consistency conditions. It however seems that TGD actually predicts this hierarchy of covering spaces. The extreme non-linearity of the field equations defined by Kähler action means that the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is 1-to-many. This leads naturally to the introduction of the covering space of $CD \times CP_2$, where $CD$ denotes causal diamond defined as intersection of future and past directed light-cones.

At the level of WCW there is the generalization of the Dirac equation which can be regarded as a purely classical Dirac equation. The modified Dirac operators associated with quarks and leptons carry fermion number but the Dirac equations are well-defined. An orthogonal basis of solutions of these Dirac operators define in zero energy ontology a basis of zero energy states. The $M$-matrices defining entanglement between positive and negative energy parts of the zero energy state define what can be regarded as analogs of thermal S-matrices. The M-matrices associated with the solution basis of the WCW Dirac equation define by their orthogonality unitary U-matrix between zero energy states. This matrix finds the proper interpretation in TGD inspired theory of consciousness. WCW Dirac equation as the analog of super-Virasoro conditions for the "gamma fields" of superstring models defining super counterparts of Virasoro generators was the main focus during earlier period of quantum TGD but has not received so much attention lately and will not be discussed in this chapter.

Quantum classical correspondence requires a coupling between quantum and classical and this coupling should also give rise to a generalization of quantum measurement theory. The big question mark is how to realize this coupling. The addition of a measurement interaction term to the modified Dirac action turned out to do the job [K9, K13] and solves a handful of problems of quantum TGD and unifies various visions about the physics predicted by quantum TGD.

### 5.1.2 Modified Dirac equation for induced classical spinor fields

The basic vision is that WCW geometry reduces to the second quantization of induced spinor fields. This means that WCW gamma matrices are linear combinations of fermionic oscillator operators and the vacuum functional of the theory is identifiable as Dirac determinant. An unproven conjecture is that this determinant equals to the exponent of Kähler action for its preferred extremal.

The motivation for the modified Dirac action came from the observation that the counterpart of the ordinary Dirac equation is internally consistent only if the space-time surfaces are minimal surfaces. One can however assign to any general coordinate invariant action principle for space-time surfaces a unique modified Dirac action, which is internally consistent and super-symmetric. Space-time geometry must carry information about conserved quantum charges assignable to partonic 2-surfaces and it took considerable to to realize that this is achieved via a measurement interaction terms linear in conserved charges. It took still some time to conclude that Kähler action with a measurement interaction term is is required in order the code information about quantum numbers to the space-time geometry.

### Preferred extremals as critical extremals

The study of the modified Dirac equation leads to a detailed view about criticality. Quantum criticality [D5] fixes the values of Kähler coupling strength as the analog of critical temperature. Quantum criticality implies that second variation of Kähler action vanishes for critical deformations and the existence of conserved current except in the case of Cartan algebra of isometries. Quantum criticality allows to fix the values of couplings appearing in the measurement interaction by using the condition $K \rightarrow K + f + \overline{f}$. p-Adic coupling constant evolution can be understood also and corresponds to scale hierarchy for the sizes of causal diamonds ($CD$s). The discovery that the hierarchy of Planck constants realized in terms of singular covering spaces of $CD \times CP_2$ can be understood in terms of the extremely non-linear dynamics of Kähler action implying 1-to-many correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates led to a further very concrete understanding of the criticality at space-time level and its relationship to zero energy ontology [K24] .

### Inclusion of the measurement interaction term

One can pose several conditions on the measurement interaction term of Dirac action. The term should be linear in the measured charges which must commute and act on their eigenstates. The effective 2-dimensionality requires that the measurement interaction term is 3-dimensional and this allows only the Dirac action associated with the generalized Chern-Simons action [B1] . Measurement interaction term must define fermionic 3-D propagators along wormhole throats. This is necessary because 4-D Dirac equation is satisfied always and cannot define the fermionic propagator. For Chern-Simons term off mass shell propagation is possible since 3-D Chern-Simons Dirac equation need not to be satisfied.

1. The basic vision is that the addition of the measurement interaction term induces a $U(1)$ gauge transformation $K \rightarrow K + f + \overline{f}$ of the Kähler function of WCW. Here $f$ is holomorphic function of WCW ("world of classical worlds") complex coordinates and arbitrary function of zero mode coordinates. Altough WCW Kähler metric is not affected, Kähler function changes and this means that preferred extremal changes also and therefore codes information about the values of the measured observables.

2. The measurement interaction is assumed to be linear in the measured charges which must commute and therefore belong to the Cartan algebra. Cartan algebra plays a key role not only at quantum level but also at the level of space-time geometry since quantum critical conserved currents vanish for the Cartan algebra of isometries and the measurement interaction terms giving rise to conserved currents are possible only for Cartan algebras. Furthermore, modified Dirac equation makes sense only for the eigen states of Cartan algebra generators. The hierarchy of Planck constants realized in terms of the book like structure of the generalized imbedding space assigns to each $CD$ (causal diamond) preferred Cartan algebra: in case of Poincare algebra there are two of them corresponding to linear and cylindrical $M^4$ coordinates. The origin of the hierarchy of Planck constants can be now understood from the basic quantum TGD and it relates directly with criticality [K24] .

3. The values of Cartan charges are feeded to 3-D Chern-Simons Dirac action via the measurement interaction term. Measurement interaction term corresponds to a term resulting from the $U(1)$ transformation $\phi$ of the $CP_2$ Kähler potential. Since this term is assigned only with the Chern-Simons Dirac action, it does not reduce to a mere gauge transformation with a trivial effect. This picture is consistent with the reduction of TGD to almost topological QFT [B11] implied by electric-magnetic duality and the vanishing of the Coulomb interaction term in Kähler action [K24] .

4. One can require that the propagating states are generalized eigenstates of the modified Dirac equation. The generalized eigenvalues are of form $D_{C-S}\Psi = \lambda^k \gamma_k \Psi$, where only the covariantly constant $M^4$ gamma matrices can appear. $\lambda^k$ is completely analogous to four-momentum and the propagator is formally massless propagator so that ordinary twistor formalism should apply.

The identification with actual four-momentum does not however make sense. This suggests that also massless gauge theories could make sense if the four-momenta do not correspond to the actual four-momenta.

### CP breaking and matter-antimatter asymmetry

Chern-Simons Dirac action used to defined measurement interaction term breaks CP and T symmetries and therefore provides a first principle description for the breaking of these symmetries. CP breaking could also reflect to the discretization of the relative coordinate between the tips of the $CD$. One could label the positions of the lower tip of $CD$ by $M^4$ and the relative positions of the upper tip by a discrete space consisting of discrete variants of hyperboloids with proper time coordinate coming as powers of 2. This $CP$ and $T$ breaking would be apparent and due to the fixing the rest system to the observer assigned with the "lower" boundary of $CD$ serving as a role of medium forcing the $CP$ breaking at the level sub-$CD$s. One can of course argue that the CP breaking induced by Chern-Simons action gives the special role for the "lower" boundary of $CD$. In fact, the breaking of Lorentz invariance at the level of $CD$ (but not at the level of WCW) could even make possible a spontaneous breaking of CPT symmetry.

### What one should still continue to be worried about?

The construction of WCW spinor structure in terms of induced spinor fields has been continual shifting between various options. 3-D or 4-D modified Dirac action at the fundamental level? Does the idea about TGD as almost topological QFT make sense or not? Is the identification of Kähler function as Dirac determinant really needed? Does it even make sense?

The reduction to almost topological QFT based on weak electric-magnetic duality gives the explicit form of the WCW Kähler function and one understand how the measurement interaction term affects it. This is of utmost importance for the construction of quantum TGD since WCW Kähler metric becomes directly calculable. The progress in some aspects however forces always to challenge the basic assumptions so that there is no hope about the end of endless confusiomn.

1. The basic idea has been that a correlation between 4-D geometry of the space-time sheet and quantum numbers would be achieved by the identification of the exponent of Kähler function as a Dirac determinant. The effect of the measurement interaction to the Kähler function is however induced by the same gauge transformation of the induced Kähler gauge potential appearing in Chern-Simons action as appears in Chern-Simons Dirac action. Therefore Dirac determinant is not needed to calculate the Kähler function and one can ask whether the identification of Kähler function as a Dirac determinant has any practical value.

2. One can still worry whether the measurement interaction is really needed. The propagator reduces formally to massless Dirac propator in which the analog of four-momentum is expressible in terms of quantum numbers propagating in the line. This would be a fantastic news for a believer in the twistor program since also massive case and virtual momenta could be treated. One could however argue that the road involving minimum amount of calculations is the safest one: why not to identify the four-momentum with the physical four-momentum and try to resolve the resulting problems?

3. The teasing hen-egg question still remains.
Does the 4-D Kähler-Dirac action with Chern-Simons term define preferred extremals giving Kähler function as a Kähler action reducing to Chern-Simons term? In this case the induced spinor fields in the interior of space-time surface would be present and one would have a symmetry in the sense that one could use the restriction of the induced spinor field and Chern-Simons action for any light-like 3-surface to constructe the quantum theory.
Or should 3-D Chern-Simons-Dirac action be interpreted as the Kähler function of WCW to which one directly assigns the modified Dirac action making possible to construct the spinor structure of WCW and does electric-magnetic duality make possible the assignment of preferred extremal of Kähler action to a given 3-surface. In this case the induced spinor fields in the interior of space-time surface would not be needed at all and wormhole throats and ends of the space-time surface would play a special role as carriers of spinorial shock waves.

### 5.1.3 Identification of configuration space gamma matrices as super Hamiltonians

The basic super-algebra corresponds to the fermionic oscillator operators and can be regarded as a generalization $\mathcal{N}$ super algebras by replacing $\mathcal{N}$ with the number of solutions of the modified Dirac equation which can be infinite. This leads to QFT SUSY limit of TGD different in many respects crucially from standard SUSYs.

Configuration space gamma matrices identified as super generators of super-symplectic and are expressible in terms of these oscillator operators. Super-symplectic and super charges are assumed to be expressible as integrals over 2-dimensional partonic surfaces $X^2$ and interior degrees of freedom of $X^4$ can be regarded as zero modes representing classical variables in one-one correspondence with quantal degrees of freedom at $X_l^3$ as indeed required by quantum measurement theory.

## 5.2 Modified Dirac equation

In the following the problems of the ordinary Dirac action are discussed and the notion of modified Dirac action is introduced.

### 5.2.1 Problems associated with the ordinary Dirac action

Minimal 2-surface represents a situation in which the representation of surface reduces to a complex-analytic map. This implies that induced metric is hermitian so that it has no diagonal components in complex coordinates $(z, \overline{z})$ and the second fundamental form has only diagonal components of type $H^k_{zz}$. This implies that minimal surface is in question since the trace of the second fundamental form vanishes. At first it seems that the same must happen also in the more general case with the consequence that the space-time surface is a minimal surface. Although many basic extremals of Kähler action are minimal surfaces, it seems difficult to believe that minimal surface property plus extremization of Kähler action could really boil down to the absolute minimization of Kähler action or some other general principle selecting preferred extremals as Bohr orbits [K10, K54] .

This brings in mind a similar long-standing problem associated with the Dirac equation for the induced spinors. The problem is that right-handed neutrino generates super-symmetry only provided that space-time surface and its boundary are minimal surfaces. Although one could interpret this as a geometric symmetry breaking, there is a strong feeling that something goes wrong. Induced Dirac equation and super-symmetry fix the variational principle but this variational principle is not consistent with Kähler action.

One can also question the implicit assumption that Dirac equation for the induced spinors is consistent with the super-symmetry of the configuration space geometry. Super-symmetry would obviously require that for vacuum extremals of Kähler action also induced spinor fields represent vacua. This is however not the case. This super-symmetry is however assumed in the construction of the configuration space geometry so that there is internal inconsistency.

### 5.2.2 Super-symmetry forces modified Dirac equation

The above described three problems have a common solution. Nothing prevents from starting directly from the hypothesis of a super-symmetry generated by covariantly constant right-handed neutrino and finding a Dirac action which is consistent with this super-symmetry. Field equations can be written as

$$\begin{aligned} D_\alpha T^\alpha_k &= 0 \ , \\ T^\alpha_k &= \frac{\partial}{\partial {h^k}_\alpha} L_K \ . \end{aligned} \tag{5.2.1}$$

If super-symmetry is present one can assign to this current its super-symmetric counterpart

$$\begin{aligned} J^{\alpha k} &= \overline{\nu_R} \Gamma^k T^\alpha_l \Gamma^l \Psi \ , \\ D_\alpha J^{\alpha k} &= 0 \ . \end{aligned} \tag{5.2.2}$$

having a vanishing divergence. The isometry currents currents and super-currents are obtained by contracting $T^{\alpha_k}$ and $J^{\alpha_k}$ with the Killing vector fields of super-symmetries. Note also that the super current

$$J^\alpha = \overline{\nu_R} T^\alpha_l \Gamma^l \Psi \quad (5.2.3)$$

has a vanishing divergence.

By using the covariant constancy of the right-handed neutrino spinor, one finds that the divergence of the super current reduces to

$$D_\alpha J^{\alpha k} = \overline{\nu_R} \Gamma^k T^\alpha_l \Gamma^l D_\alpha \Psi \ . \quad (5.2.4)$$

The requirement that this current vanishes is guaranteed if one assumes that modified Dirac equation

$$\begin{aligned} \hat{\Gamma}^\alpha D_\alpha \Psi &= 0 \ , \\ \hat{\Gamma}^\alpha &= T^\alpha_l \Gamma^l \ . \end{aligned} \quad (5.2.5)$$

This equation must be derivable from a modified Dirac action. It indeed is. The action is given by

$$L = \overline{\Psi} \hat{\Gamma}^\alpha D_\alpha \Psi \ . \quad (5.2.6)$$

Thus the variational principle exists. For this variational principle induced gamma matrices are replaced with effective induced gamma matrices and the requirement

$$D_\mu \hat{\Gamma}^\mu = 0 \quad (5.2.7)$$

guaranteing that super-symmetry is identically satisfied if the bosonic field equations are satisfied. For the ordinary Dirac action this condition would lead to the minimal surface property. What sounds strange that the essentially hydrodynamical equations defined by Kähler action have fermionic counterpart: this is very far from intuitive expectations raised by ordinary Dirac equation and something which one might not guess without taking super-symmetry very seriously.

### 5.2.3 How can one avoid minimal surface property?

These observations suggest how to avoid the emergence of the minimal surface property as a consequence of field equations. It is not induced metric which appears in field equations. Rather, the effective metric appearing in the field equations is defined by the anti-commutators of $\hat{\gamma}_\mu$

$$\hat{g}_{\mu\nu} = \{\hat{\Gamma}_\mu, \hat{\Gamma}_\nu\} = 2T^k_\mu T_{\nu k} \ . \quad (5.2.8)$$

Here the index raising and lowering is however performed by using the induced metric so that the problems resulting from the non-invertibility of the effective metric are avoided. It is this dynamically generated effective metric which must appear in the number theoretic formulation of the theory.

Field equations state that space-time surface is minimal surface with respect to the effective metric. Note that a priori the choice of the bosonic action principle is arbitrary. The requirement that effective metric defined by energy momentum tensor has only non-diagonal components except in the case of non-light-like coordinates, is satisfied for the known solutions of field equations.

### 5.2.4 Does the modified Dirac action define the fundamental action principle?

There is quite fundamental and elegant interpretation of the modified Dirac action as a fundamental action principle discussed also in [K54] . In this approach vacuum functional can be defined as the Grassmannian functional integral associated with the exponent of the modified Dirac action. This definition is invariant with respect to the scalings of the Dirac action so that theory contains no free parameters.

An alternative definition is as a Dirac determinant which might be calculated in TGD framework without applying the poorly defined functional integral. There are good reasons to expect that the Dirac determinant exponent of Kähler function for a preferred Bohr orbit like extremal of the Kähler action with the value of Kähler coupling strength coming out as a prediction. Hence the dynamics of the modified Dirac action at light-like partonic 3-surfaces $X_l^3$, even when restricted to almost-topological dynamics induced by Chern-Simons action, would dictate the dynamics at the interior of the space-time sheet.

The knowledge of the symplectic currents and super-currents, together with the anti-commutation relations stating that the fermionic super-currents $S_A$ and $S_B$ associated with Hamiltonians $H_A$ and $H_B$ anti-commute to a bosonic current $H_{[A,B]}$, allows in principle to deduce the anti-commutation relations satisfied by the induced spinor field. In fact, these conditions replace the usual anti-commutation relations used to quantize free spinor field. Since the normal ordering of the Dirac action would give Kähler action,

Kähler coupling strength would be determined completely by the anti-commutation relations of the super-symplectic algebra. Kähler coupling strength would be dynamical and the selection of preferred extremals of Kähler action would be more or less equivalent with quantum criticality because criticality corresponds to conformal invariance and the hyper-quaternionic version of the super-conformal invariance results only for the extrema of Kähler action. p-Adic (or possibly more general) coupling constant evolution and quantum criticality would come out as a prediction whereas in the case that Kähler action is introduced as primary object, the value of Kähler coupling strength must be fixed by quantum criticality hypothesis.

The mixing of the $M^4$ chiralities of the imbedding space spinors serves as a signal for particle massivation and breaking of super-conformal symmetry. The induced gamma matrices for the space-time surfaces which are deformations of $M^4$ indeed contain a small contribution from $CP_2$ gamma matrices: this implies a mixing of $M^4$ chiralities even for the modified Dirac action so that there is no need to introduce this mixing by hand.

### 5.2.5 Which Dirac action?

Which modified Dirac action should one choose? The four-dimensional modified Dirac action associated with Kähler action or 3-D Dirac action associated with Chern-Simons ( Chern-Simons ) action? Or something else? To express the number of proposed answers to this question requires the fingers of both hands.

1. The first guess inspired by TGD as almost-TQFT (for topological QFTs see [A66] ) was that Chern-Simons action is enough. The idea was that one starts from Chern-Simons action and end up with Kähler function defined by a preferred extremal of Kähler action defining Kähler function with exponent of Kähler function identified as a Dirac determinant. The difficulties related to the definition of the Dirac determinant however forced to give up this approach. It must be emphasized that Dirac determinant is a potentially problematic notion quite generally.

2. TGD reduces to almost topological QFT if the Kähler action for the preferred extremals reduces to Chern-Simons action associated with the ends of the space-time surface and with the light-like wormhole throats at which the signature of the induced metric changes. The assumption that Kähler current defines so called Beltrami flow indeed allows to find a gauge for Kähler gauge potential in which the Coulomb term in Kähler action vanishes and the boundary term reduces to generalized Chern-Simons action if the weak form of electric-magnetic duality holds true. Both Dirac-Kähler action and Dirac-Chern-Simons action are needed in the fermionic sector. It must be emphasized that the identification of the exponent of Kähler function as a Dirac determinant is not necessary anymore although it could make sense.

One ends up with a very detailed ansatz for preferred extremals and one should rigorously show that these surfaces are critical in the sense of having infinite number of deformations giving a vanishing second variation of Kähler action. One should also demonstrate that preferred extremals are hyper-quaternionic- a property forced by number theoretical vision [K54] .

3. Quantum classical correspondence requires the coding of the quantum numbers of positive and negative energy parts of the zero energy state to the space-time geometry. Chern-Simons term with a measurement interaction term allows to obtain a coupling between the space-time geometry and the quantum numbers of super-conformal representations and stringy propagators. It took some time to realize that measurement interaction term can be regarded as a gauge part of the Kähler gauge potential. Since it is however present only for the Chern-Simons part of the action it affects the physics. For instance, the value of Kähler function identified both as Dirac determinant and directly as Chern-Simons term is affected and therefore also the preferred extremal is affected.

4. Kähler Dirac operator $D_K$ annihilates the induced spinor fields in the interior of space-time surface. At the ends of the space-time surface induces spinors are generalized eigenstates of $D_{C-S}+Q$, where $Q$ represents the measurement interaction which effectively reduces to a gauge part in Kähler gauge potential. The generalized eigenvalue is the analog of the action of ordinary Dirac operator $p_k\gamma^k$ but the pseudo-momentum $\lambda_k$ replacing $p_k$ is not the real momentum. For instance, by number theoretic considerations this pseudo-momentum is in preferred plane $M^2$ of $M^4$ assigned to a given $CD$ in zero energy ontology. $\lambda_k$ is quantized and Dirac propagator defined by Chern-Simons Dirac action reduces effectively to a massless propagator suggesting that twistor formalism can be applied almost as such.

5. An interesting conjecture is that any light-like 3-surfaces in the slicing of space-time surface by light-like 3-surfaces "parallel" to wormhole throat is physically equivalent and the Kähler functions obtained as Dirac determinants differ only by a gauge transformation for Kähler function: $K \rightarrow K+f+\overline{f}$, where $f$ is holomorphic function of complex coordinates of WCW and a priori arbitrary function of zero modes. Also measurement interaction term induces only this kind of gauge change but replaces the space-time surface with a new one. A physically attractive realization of the slicings of space-time surface by 3-surfaces and string world sheets is discussed in [K25] by starting from the observation that TGD could define a natural realization of braids, braid cobordisms, and 2-knots.

## 5.3 Quantum criticality and modified Dirac action

The precise mathematical formulation of quantum criticality has remained one of the basic challenges of quantum TGD. The question leading to a considerable progress in the problem was simple: Under what conditions the modified Dirac action allows to assign conserved fermionic currents with the deformations of the space-time surface? The answer was equally simple: These currents exists only if these deformations correspond to vanishing second variations of Kähler action - which is what criticality is. The vacuum degeneracy of Kähler action strongly suggests that the number of critical deformations is always infinite and that these deformations define an infinite inclusion hierarchy of super-conformal algebras. This inclusion hierarchy would correspond to a fractal hierarchy of breakings of super-conformal symmetry generalizing the symmetry breaking hierarchies of gauge theories. These super-conformal inclusion hierarchies would realize the inclusion hierarchies for hyper-finite factors of type $II_1$.

### 5.3.1 Quantum criticality and fermionic representation of conserved charges associated with second variations of Kähler action

It is rather obvious that TGD allows a huge generalizations of conformal symmetries. The development of the understanding of conservation laws has been slow. In the approach based on $D_{C-S}$ the non-conservation of gauge charges posed the basic problem and led to the introduction of the gauge part $A_a$ of Kähler gauge potential. Modified Dirac action provides excellent candidates for quantum counterparts of Noether charges. Unfortunately, the isometry charges vanish for Cartan algebras. The only manner to obtain non-trivial isometry charges is to add a direct coupling to the charges in Cartan algebra as will be found later.

**Conservation of the fermionic current requires the vanishing of the second variation of Kähler action**

1. The modified Dirac action assigns to a deformation of the space-time surface a conserved charge expressible as bilinears of fermionic oscillator operators only if the first variation of the modified Dirac action under this deformation vanishes. The vanishing of the first variation for the modified Dirac action is equivalent with the vanishing of the second variation for the Kähler action. This can be seen by the explicit calculation of the second variation of the modified Dirac action and by performing partial integration for the terms containing derivatives of $\Psi$ and $\overline{\Psi}$ to give a total divergence representing the difference of the charge at upper and lower boundaries of the causal diamond plus a four-dimensional integral of the divergence term defined as the integral of the quantity

$$\begin{aligned} \Delta S_D &= \overline{\Psi}\Gamma^k D_\alpha J_k^\alpha \Psi \ , \\ J_k^\alpha &= \frac{\partial^2 L_K}{\partial h_\alpha^k \partial h_\beta^l}\delta h_\beta^k + \frac{\partial^2 L_K}{\partial h_\alpha^k \partial h^l}\delta h^l \ . \end{aligned} \tag{5.3.1}$$

Here $h_\beta^k$ denote partial derivative of the imbedding space coordinate with respect to space-time coordinates. This term must vanish:

$$D_\alpha J_k^\alpha = 0 \ .$$

The condition states the vanishing of the second variation of Kähler action. This can of course occur only for preferred deformations of $X^4$. One could consider the possibility that these deformations vanish at light-like 3-surfaces or at the boundaries of CD. Note that covariant divergence is in question so that $J_k^\alpha$ does not define conserved classical charge in the general case.

2. It is essential that the modified Dirac equation holds true so that the modified Dirac action vanishes: this is needed to cancel the contribution to the second variation coming from the determinant of the induced metric. The condition that the modified Dirac equation is satisfied for the deformed space-time surface requires that also $\Psi$ suffers a transformation determined by the deformation. This gives

$$\delta\Psi = -\frac{1}{D}\times\Gamma^k J_k^\alpha \Psi \ . \tag{5.3.2}$$

Here $1/D$ is the inverse of the modified Dirac operator defining the counterpart of the fermionic propagator.

3. The fermionic conserved currents associated with the deformations are obtained from the standard conserved fermion current

$$J^\alpha = \overline{\Psi}\Gamma^\alpha\Psi \ . \tag{5.3.3}$$

Note that this current is conserved only if the space-time surface is extremal of Kähler action: this is also needed to guarantee Hermiticity and same form for the modified Dirac equation for

$\Psi$ and its conjugate as well as absence of mass term essential for super-conformal invariance. Note also that ordinary divergence rather only covariant divergence of the current vanishes.

The conserved currents are expressible as sums of three terms. The first term is obtained by replacing modified gamma matrices with their increments in the deformation keeping $\Psi$ and its conjugate constant. Second term is obtained by replacing $\Psi$ with its increment $\delta\Psi$. The third term is obtained by performing same operation for $\delta\overline{\Psi}$.

$$J^{\alpha} = \overline{\Psi}\Gamma^k J_k^{\alpha}\Psi + \overline{\Psi}\hat{\Gamma}^{\alpha}\delta\Psi + \delta\overline{\Psi}\hat{\Gamma}^{\alpha}\Psi \ . \tag{5.3.4}$$

These currents provide a representation for the algebra defined by the conserved charges analogous to a fermionic representation of Kac-Moody algebra.

4. Also conserved super charges corresponding to super-conformal invariance are obtained. The first class of super currents are obtained by replacing $\Psi$ or $\overline{\Psi}$ right handed neutrino spinor or its conjugate in the expression for the conserved fermion current and performing the above procedure giving two terms since nothing happens to the covariantly constant right handed-neutrino spinor. Second class of conserved currents is defined by the solutions of the modified Dirac equation interpreted as c-number fields replacing $\Psi$ or $\overline{\Psi}$ and the same procedure gives three terms appearing in the super current.

5. The existence of vanishing of second variations is analogous to criticality in systems defined by a potential function for which the rank of the matrix defined by second derivatives of the potential function vanishes at criticality. Quantum criticality becomes the prerequisite for the existence of quantum theory since fermionic anti-commutation relations in principle can be fixed from the condition that the algebra in question is equivalent with the algebra formed by the vector fields defining the deformations of the space-time surface defining second variations. Quantum criticality in this sense would also select preferred extremals of Kähler action as analogs of Bohr orbits and the the spectrum of preferred extremals would be more or less equivalent with the expected existence of infinite-dimensional symmetry algebras.

### About the general structure of the algebra of conserved charges

Some general comments about the structure of the algebra of conserved charges are in order.

1. Any Cartan algebra of the isometry group $P \times SU(3)$ (there are two types of them for $P$ corresponding to linear and cylindrical Minkowski coordinates) defines critical deformations (one could require that the isometries respect the geometry of $CD$). The corresponding charges are conserved but vanish since the corresponding conjugate coordinates are cyclic for the Kähler metric and Kähler form so that the conserved current is proportional to the gradient of a Killing vector field which is constant in these coordinates. Therefore one cannot represent isometry charges as fermionic bilinears. Four-momentum and color quantum numbers are defined for Kähler action as classical conserved quantities but this is probably not enough. This can be seen as a problem.

   (a) Four-momentum and color Cartan algebra emerge naturally in the representations of super-conformal algebras. In the case of color algebra the charges in the complement of the Cartan algebra can be constructed in standard manner as extension of those for the Cartan algebra using free field representation of Kac-Moody algebras. In string theories four-momentum appears linearly in bosonic Kac-Moody generators and in the Sugawara representation [A68] of super Virasoro generators as bilinears of bosonic Kac-Moody generators and fermionic super Kac-Moody generators. Also now quantized transversal parts for $M^4$ coordinates could define a second quantized field having interpretation as an operator acting on spinor fields of WCW. The angle coordinates conjugate to color isospin and hypercharge take the role of $M^4$ coordinates in case of $CP_2$.

(b) Somehow one should be able to feed the information about the super-conformal representation of the isometry charges to the modified Dirac action by adding to it a term coupling fermionic current to the Cartan charges in general coordinate invariant and isometry invariant manner. As will be shown later, this is possible. The interpretation is as measurement interaction guaranteing also the stringy character of the fermionic propagators. The values of the couplings involved are fixed by the condition of quantum criticality assumed in the sense that Kähler function of WCW suffers only a $U(1)$ gauge transformation $K \rightarrow K + f + \overline{f}$, where $f$ is a holomorphic function of WCW coordinates depending also on zero modes.

(c) The simplest addition involves the modified gamma matrices defined by a Chern-Simon term at the light-like wormhole throats and is sum of Chern-Simons Dirac action and corresponding coupling term linear in Cartan charges assignable to the partonic 2-surfaces at the ends of the throats. Hence the modified Dirac equation in the interior of the space-time sheet is not affected and nothing changes as far as quantum criticality in interior is considered.

2. The action defined by four-volume gives a first glimpse about what one can expect. In this case modified gamma matrices reduce to the induced gamma matrices. Second variations satisfy d'Alembert type equation in the induced metric so that the analogs of massless fields are in question. Mass term is present only if some dimensions are compact. The vanishing of excitations at light-like boundaries is a natural boundary condition and might well imply that the solution spectrum could be empty. Hence it is quite possible that four-volume action leads to a trivial theory.

3. For the vacuum extremals of Kähler action the situation is different. There exists an infinite number of second variations and the classical non-determinism suggests that deformations vanishing at the light-like boundaries exist. For the canonical imbedding of $M^4$ the equation for second variations is trivially satisfied. If the $CP_2$ projection of the vacuum extremal is one-dimensional, the second variation contains a on-vanishing term and an equation analogous to massless d'Alembert equation for the increments of $CP_2$ coordinates is obtained. Also for the vacuum extremals of Kähler action with 2-D $CP_2$ projection all terms involving induced Kähler form vanish and the field equations reduce to d'Alembert type equations for $CP_2$ coordinates. A possible interpretation is as the classical analog of Higgs field. For the deformations of non-vacuum extremals this would suggest the presence of terms analogous to mass terms: these kind of terms indeed appear and are proportional to $\delta s_k$. $M^4$ degrees of freedom decouple completely and one obtains QFT type situation.

4. The physical expectation is that at least for the vacuum extremals the critical manifold is infinite-dimensional. The notion of finite measurement resolution suggests infinite hierarchies of inclusions of hyper-finite factors of type $II_1$ possibly having interpretation in terms of inclusions of the super conformal algebras defined by the critical deformations.

5. The properties of Kähler action give support for this expectation. The critical manifold is infinite-dimensional in the case of vacuum extremals. Canonical imbedding of $M^4$ would correspond to maximal criticality analogous to that encountered at the tip of the cusp catastrophe. The natural guess would be that as one deforms the vacuum extremal the previously critical degrees of freedom are transformed to non-critical ones. The dimension of the critical manifold could remain infinite for all preferred extremals of the Kähler action. For instance, for cosmic string like objects any complex manifold of $CP_2$ defines cosmic string like objects so that there is a huge degeneracy is expected also now. For $CP_2$ type vacuum extremals $M^4$ projection is arbitrary light-like curve so that also now infinite degeneracy is expected for the deformations.

### Critical super algebra and zero modes

The relationship of the critical super-algebra to configuration space geometry is interesting.

1. The vanishing of the second variation plus the identification of Kähler function as a Kähler action for preferred extremals means that the critical variations are orthogonal to all deformations of

the space-time surface with respect to the configuration space metric and thus correspond to zero modes. This conforms with the fact that configuration space metric vanishes identically for canonically imbedded $M^4$. Zero modes do not seem to correspond to gauge degrees of freedom so that the super-conformal algebra associated with the zero modes has genuine physical content.

2. Since the action of $X^4$ local Hamiltonians of $\delta M^4_\times CP_2$ corresponds to the action in quantum fluctuating degrees of freedom, critical deformations cannot correspond to this kind of Hamiltonians.

3. The notion of finite measurement resolution suggests that the degrees of freedom which are below measurement resolution correspond to vanishing gauge charges. The sub-algebras of critical super-conformal algebra for which charges annihilate physical states could correspond to this kind of gauge algebras.

4. The conserved super charges associated with the vanishing second variations cannot give configuration space metric as their anti-commutator. This would also lead to a conflict with the effective 2-dimensionality stating that the configuration space line-element is expressible as sum of contribution coming from partonic 2-surfaces as also with fermionic anti-commutation relations.

**Connection with quantum criticality**

The vanishing of the second variation for some deformations means that the system is critical, in the recent case quantum critical. Basic example of criticality is bifurcation diagram for cusp catastrophe. For some mysterious reason I failed to realize that quantum criticality realized as the vanishing of the second variation makes possible a more or less unique identification of preferred extremals and considered alternative identifications such as absolute minimization of Kähler action which is just the opposite of criticality. Both the super-symmetry of $D_K$ and conservation Dirac Noether currents for modified Dirac action have thus a connection with quantum criticality.

1. Finite-dimensional critical systems defined by a potential function $V(x^1, x^2, ..)$ are characterized by the matrix defined by the second derivatives of the potential function and the rank of system classifies the levels in the hierarchy of criticalities. Maximal criticality corresponds to the complete vanishing of this matrix. Thom's catastrophe theory classifies these hierarchies, when the numbers of behavior and control variables are small (smaller than 5). In the recent case the situation is infinite-dimensional and the criticality conditions give additional field equations as existence of vanishing second variations of Kähler action.

2. The vacuum degeneracy of Kähler action allows to expect that this kind infinite hierarchy of criticalities is realized. For a general vacuum extremal with at most 2-D $CP_2$ projection the matrix defined by the second variation vanishes because $J_{\alpha\beta} = 0$ vanishes and also the matrix $(J^\alpha_{\ k} + J_k^{\ \alpha})(J^\beta_{\ l} + J_l^{\ \beta})$ vanishes by the antisymmetry $J^\alpha_{\ k} = -J_k^{\ \alpha}$. Recall that the formulation of Equivalence Principle in string picture demonstrated that the reduction of stringy dynamics to that for free strings requires that second variation with respect to $M^4$ coordinates vanish. This condition would guarantee the conservation of fermionic Noether currents defining gravitational four-momentum and other Poincare quantum numbers but not those for gravitational color quantum numbers. Encouragingly, the action of $CP_2$ type vacuum extremals having random light-like curve as $M^4$ projection have vanishing second variation with respect to $M^4$ coordinates (this follows from the vanishing of Kähler energy momentum tensor, second fundamental form, and Kähler gauge current). In this case however the momentum is vanishing.

3. Conserved bosonic and fermionic Noether charges would characterize quantum criticality. In particular, the isometries of the imbedding space define conserved currents represented in terms of the fermionic oscillator operators if the second variations defined by the infinitesimal isometries vanish for the modified Dirac action. For vacuum extremals the dimension of the critical manifold is infinite: maybe there is hierarchy of quantum criticalities for which this dimension decreases step by step but remains always infinite. This hierarchy could closely relate to the hierarchy of inclusions of hyper-finite factors of type $II_1$. Also the conserved charges associated with super-symplectic and Super Kac-Moody algebras would require infinite-dimensional critical manifold defined by the spectrum of second variations.

4. Phase transitions are characterized by the symmetries of the phases involved with the transitions, and it is natural to expect that dynamical symmetries characterize the hierarchy of quantum criticalities. The notion of finite quantum measurement resolution based on the hierarchy of Jones inclusions indeed suggests the existence of a hierarchy of dynamical gauge symmetries characterized by gauge groups in ADE hierarchy [K18] with degrees of freedom below the measurement resolution identified as gauge degrees of freedom.

5. Does this criticality have anything to do with the criticality against the phase transitions changing the value of Planck constant? If the geodesic sphere $S^2_I$ for which induced Kähler form vanishes corresponds to the back of the $CP_2$ book (as one expects), this could be the case. The homologically non-trivial geodesic sphere $S^12_{II}$ is as far as possible from vacuum extremals. If it corresponds to the back of $CP_2$ book, cosmic strings would be quantum critical with respect to phase transition changing Planck constant. They cannot however correspond to preferred extremals.

### 5.3.2 Preferred extremal property as classical correlate for quantum criticality, holography, and quantum classical correspondence

The Noether currents assignable to the modified Dirac equation are conserved only if the first variation of the modified Dirac operator $D_K$ defined by Kähler action vanishes. This is equivalent with the vanishing of the second variation of Kähler action -at least for the variations corresponding to dynamical symmetries having interpretation as dynamical degrees of freedom which are below measurement resolution and therefore effectively gauge symmetries.

The vanishing of the second variation in interior of $X^4(X^3_l)$ is what corresponds exactly to quantum criticality so that the basic vision about quantum dynamics of quantum TGD would lead directly to a precise identification of the preferred extremals. Something which I should have noticed for more than decade ago! The question whether these extremals correspond to absolute minima remains however open.

The vanishing of second variations of preferred extremals -at least for deformations representing dynamical symmetries, suggests a generalization of catastrophe theory of Thom, where the rank of the matrix defined by the second derivatives of potential function defines a hierarchy of criticalities with the tip of bifurcation set of the catastrophe representing the complete vanishing of this matrix. In the recent case this theory would be generalized to infinite-dimensional context. There are three kind of variables now but quantum classical correspondence (holography) allows to reduce the types of variables to two.

1. The variations of $X^4(X^3_l)$ vanishing at the intersections of $X^4(X^3_l)$ wth the light-like boundaries of causal diamonds $CD$ would represent behavior variables. At least the vacuum extremals of Kähler action would represent extremals for which the second variation vanishes identically (the "tip" of the multi-furcation set).

2. The zero modes of Kähler function would define the control variables interpreted as classical degrees of freedom necessary in quantum measurement theory. By effective 2-dimensionality (or holography or quantum classical correspondence) meaning that the configuration space metric is determined by the data coming from partonic 2-surfaces $X^2$ at intersections of $X^3_l$ with boundaries of $CD$, the interiors of 3-surfaces $X^3$ at the boundaries of $CD$s in rough sense correspond to zero modes so that there is indeed huge number of them. Also the variables characterizing 2-surface, which cannot be complexified and thus cannot contribute to the Kähler metric of configuration space represent zero modes. Fixing the interior of the 3-surface would mean fixing of control variables. Extremum property would fix the 4-surface and behavior variables if boundary conditions are fixed to sufficient degree.

3. The complex variables characterizing $X^2$ would represent third kind of variables identified as quantum fluctuating degrees of freedom contributing to the configuration space metric. Quantum classical correspondence requires 1-1 correspondence between zero modes and these variables. This would be essentially holography stating that the 2-D "causal boundary" $X^2$ of $X^3(X^2)$ codes for the interior. Preferred extremal property identified as criticality condition would realize the holography by fixing the values of zero modes once $X^2$ is known and give rise to

the holographic correspondence $X^2 \to X^3(X^2)$. The values of behavior variables determined by extremization would fix then the space-time surface $X^4(X^3_l)$ as a preferred extremal.

4. Clearly, the presence of zero modes would be absolutely essential element of the picture. Quantum criticality, quantum classical correspondence, holography, and preferred extremal property would all represent more or less the same thing. One must of course be very cautious since the boundary conditions at $X^3_l$ involve normal derivative and might bring in delicacies forcing to modify the simplest heuristic picture.

5. There is a possible connection with the notion of self-organized criticality [B10] introduced to explain the behavior of systems like sand piles. Self-organization in these systems tends to lead "to the edge". The challenge is to understand how system ends up to a critical state, which by definition is unstable. Mechanisms for this have been discovered and based on phase transitions occurring in a wide range of parameters so that critical point extends to a critical manifold. In TGD Universe quantum criticality suggests a universal mechanism of this kind. The criticality for the preferred extremals of Kähler action would mean that classically all systems are critical in well-defined sense and the question is only about the degree of criticality. Evolution could be seen as a process leading gradually to increasingly critical systems. One must however distinguish between the criticality associated with the preferred extremals of Kähler action and the criticality caused by the spin glass like energy landscape like structure for the space of the maxima of Kähler function.

## 5.4 Handful of problems with a common resolution

Theory building could be compared to pattern recognition or to a solving a crossword puzzle. It is essential to make trials, even if one is aware that they are probably wrong. When stares long enough to the letters which do not quite fit, one suddenly realizes what one particular crossword must actually be and it is soon clear what those other crosswords are. In the following I describe an example in which this analogy is rather concrete. Let us begin by listing the problems.

1. The condition that modified Dirac action allows conserved charges leads to the condition that the symmetries in question give rise to vanishing second variations of Kähler action. The interpretation is as quantum criticality and there are good arguments suggesting that the critical symmetries define an infinite-dimensional super-conformal algebra forming an inclusion hierarchy related to a sequence of symmetry breakings closely related to a hierarchy of inclusions of hyper-finite factors of types $II_1$ and $III_1$. This means an enormous generalization of the symmetry breaking patterns of gauge theories.

   There is however a problem. For the translations of $M^4$ and color hyper charge and isospin (more generally, any Cartan algebra of $P \times SU(3)$) the resulting fermionic charges vanish. The trial for the crossword in absence of nothing better would be the following argument. By the abelianity of these charges the vanishing of quantal representation of four-momentum and color Cartan charges is not a problem and that classical representation of these charges or their super-conformal representation is enough.

2. Modified Dirac equation is satisfied in the interior of space-time surface always. This means that one does not obtain off-mass shell propagation at all in 4-D sense. Effective 2-dimensionality suggests that off mass shell propagation takes place along wormhole throats. The reduction to almost topological QFT with Kähler function reducing to Chern-Simonst type action implied by the weak form of electric-magnetic duality and a proper gauge choice for the induced Kähler gauge potential implies effective 3-dimensionality at classical level. This inspires the question whether Chern-Simons type action resulting from an instanton term could define the modified gamma matrices appearing in the 3-D modified Dirac action associated with wormhole throats and ends of the space-time sheet at the boundaries of $CD$. The assumption that modified Dirac equation is satisfied also at the ends and wormhole throats would realize effective 2-dimensionality as conditions on the boundary values of the 4-D Dirac equation but would would not allow off mass shell propagation. Therefore one could argue that effective 2-dimensionality holds true only for incoming and outgoing particles. The reduction of Kähler action to generalized Chern-Simons term means that the maxima of Kähler function should correspond to

extrema of this action. The presence of also the Chern-Simons term corresponding to $J+J_1$ would give these extrema.

3. Quantum classical correspondence requires that the geometry of the space-time sheet should correlate with the quantum numbers characterizing positive (negative) energy part of the quantum state. One could argue that by multiplying WCW spinor field by a suitable phase factor depending on the charges of the state, the correspondence follows from stationary phase approximation. This crossword looks unconvincing. A more precise connection between quantum and classical is required.

4. In quantum measurement theory classical macroscopic variables identified as degrees of freedom assignable to the interior of the space-time sheet correlate with quantum numbers. Stern Gerlach experiment is an excellent example of the situation. The generalization of the imbedding space concept by replacing it with a book like structure implies that imbedding space geometry at given page and for given causal diamond ($CD$) carries information about the choice of the quantization axes (preferred plane $M^2$ of $M^4$ *resp.* geodesic sphere of $CP_2$ associated with singular covering/factor space of $CD$ *resp.* $CP_2$ ). This is a big step but not enough. Modified Dirac action as such does not seem to provide any hint about how to achieve this correspondence. One could even wonder whether dissipative processes or at least the breaking of $T$ and $CP$ characterizing the outcome of quantum jump sequence should have space-time correlate. How to achieve this?

Each of these problems makes one suspect that something is lacking from the modified Dirac action: there should exist an elegant manner to feed information about quantum numbers of the state to the modified Dirac action in turn determining vacuum functional as an exponent Kähler function identified as Kähler action for the preferred extremal assumed to be dictated by by quantum criticality and equivalently by hyper-quaternionicity.

This observation leads to what might be the correct question. Could a general coordinate invariant and Poincare invariant modification of the modified Dirac action consistent with the vacuum degeneracy of Kähler action allow to achieve this information flow somehow? In the following one manner to achieve this modification is discussed. It must be however emphasized that I have considered many alternatives and the one discussed below finds its justification only from the fact that it is the simplest one found hitherto.

### 5.4.1 The identification of the measurement interaction term

The idea is simple: add to the modified Dirac action a term which is analogous to the Dirac action in $M^4\times CP_2$. One can consider two options according to whether the term is assigned with interior or with a 3-D light-like 3-surface and last years have been continual argumentation about which option is the correct one.

1. The additional term would be essentially the analog of the ordinary Dirac action at the imbedding space level.

$$\begin{aligned} S_{int} &= \sum_A Q_A \int \overline{\Psi} g^{AB} j_{B\alpha}\hat{\Gamma}^{\alpha}\Psi\sqrt{g}d^4x \ , \\ g_{AB} &= j^k_A h_{kl} j^l_B \ , \ g^{AB}g_{BC}=\delta^A_C \ , \\ j_{B\alpha} &= j^k_B h_{kl}\partial_\alpha h^l \ . \end{aligned} \tag{5.4.1}$$

The sum is over isometry charges $Q_A$ interpreted as quantal charges and $j^{Ak}$ denotes the Killing vector field of the isometry. $g^{AB}$ is the inverse of the tensor $g_{AB}$ defined by the local inner products of Killing vectors fields in $M^4$ and $CP_2$. The space-time projections of the Killing vector fields $j_{B\alpha}$ have interpretation as classical color gauge potentials in the case of $SU(3)$. In $M^4$ degrees of freedom and for Cartan algebra of $SU(3)$ $j_{B\alpha}$ reduce to the gradients of linear $M^4$ coordinates in case of translations. Modified gamma matrices could be assigned to Kähler action or its instanton term or with Chern-Simons action.

2. The added term containing quantal charges must make sense in the modified Dirac equation. This requires that the physical state is an eigenstate of momentum and color charges. This allows only color hyper-charge and color isospin so that there is no hope of obtaining exactly the stringy formula for the propagator. The modified Dirac operator is given by

$$\begin{aligned} D_{tot} &= D + D_{int} = \hat{\Gamma}^\alpha D_\alpha + \hat{\Gamma}^\alpha \sum_A Q_A g^{AB} j_{B\alpha} \\ &= \hat{\Gamma}^\alpha (D_\alpha + \partial_\alpha \phi) \ , \ \partial_\alpha \phi = \sum_A Q_A g^{AB} j_{B\alpha} \ . \end{aligned} \tag{5.4.2}$$

The conserved fermionic isometry currents are

$$J^{A\alpha} = \sum_B Q_B \overline{\Psi} g^{BC} j^k_C h_{kl} j^l_A \hat{\Gamma}^\alpha \Psi = Q_A \overline{\Psi} \hat{\Gamma}^\alpha \Psi \ . \tag{5.4.3}$$

Here the sum is restricted to a Cartan sub-algebra of Poincare group and color group.

3. An important restriction is that by four-dimensionality of $M^4$ and $CP_2$ the rank of $g_{AB}$ is 4 so that $g^{AB}$ exists only when one considers only four conserved charges. In the case of $M^4$ this is achieved by a restriction to translation generators $Q_A = p_A$. $g_{AB}$ reduces to Minkowski metric and Killing vector fields are constants. The Cartan sub-algebra could be however replaced by any four commuting charges in the case of Poincare algebra (second one corresponds to time translation plus translation, boost and rotation in given direction). In the case of $SU(3)$ one must restrict the consideration either to $U(2)$ sub-algebra or its complement. $CP_2 = SU(3)/SU(2)$ decomposition would suggest the complement as the correct choice. One can indeed build the generators of $U(2)$ as commutators of the charges in the complement. On the other hand, Cartan algebra is enough in free field construction of Kac-Moody algebras.

4. What is remarkable that for the Cartan algebra of $M^4 \times SU(3)$ the measurement interaction term is equivalent with the addition of gauge part $\partial_\alpha \phi$ of the induced Kähler gauge potential $A_\alpha$. This property might hold true for any measurement interaction term. This also suggests that the change in Kähler function is only the transformation $A_\alpha \to A_\alpha + \partial_\alpha \phi$, $\partial_\alpha \phi = \sum_A Q_A g^{AB} j_{B\alpha}$.

5. Recall that the $\phi$ for $U(1)$ gauge transformations respecting the vanishing of the Coulomb interaction term of Kähler action [K24] the current $j^\alpha_K \phi$ is conserved, which implies that the change of the Kähler action is trivial. These properties characterize the gauge transformations respecting the gauge in which Coulombic interaction term of the Kähler action vanishes so that Kähler action reduces to 3-dimensional generalized Chern-Simons term if the weak form of electric-magnetic duality holds true guaranteing among other things that the induced Kähler field is not too singular at the wormhole throats [K24] . The scalar function assignable to the measurement interaction terms does not have this property and this is what is expected since it must change the value of the Kähler function and therefore affect the preferred extremal.

   The reduction to 3-D form however gives a non-trivial WCW metric in $M^4$ degrees of freedom only if one replaces $CP_2$ Kähler form $J$ with the sum $J + J_1$, where $J_1$ is the Kähler form of the $r_M = constant$ sphere so that the time-like line connecting the tips of $CD$ carries monopole charge [K24] . This enriches dramatically the vacuum sector of the theory giving better hopes about a realistic description of gravitation in long length scales. The basic non-vacuum extremals of Kähler action are not lost.

Concerning the precise form of the modified Dirac action the basic clue comes from the observation that the measurement interaction term corresponds to the addition of a gauge part to the induced $CP_2$ Kähler gauge potential $A_\alpha$. The basic question is what part of the action one assigns the measurement interaction term.

1. One could define the measurement interaction term using either the four-dimensional instanton term or its reduction to Chern-Simons terms. The part of Dirac action defined by the instanton term in the interior does not reduce to a 3-D form unless the Dirac equation defined by the instanton term is satisfied : this cannot be true. Hence Chern-Simons term is the only possibility.

   The classical field equations associated with the Chern-Simons term cannot be assumed since they would imply that the $CP_2$ projection of the wormhole throat and space-like 3-surface are 2-dimensional. This might hold true for space-like 3-surfaces at the ends of $CD$ and incoming and outgoing particles but not for off mass shell particles. This is however not a problem since $D_\alpha\hat{\Gamma}^\alpha_{C-S}$ for the modified gamma matrices for Chern-Simons action does not contain second derivatives. This is due to the topological character of this term. For Kähler action second derivatives are present and this forces extremal property of Kähler action in the modified Dirac Kähler action so that classical physics resuts as a consistency condition.

2. If one assigns measurement interaction term to both $D_K$ and $D_{C-S}$ the measurement interaction corresponds to a mere gauge transformation for $AS_\alpha$ and is trivial. Therefore it seems that one must choose between $D_K$ or $D_{C-S}$. At least formally the measurement interaction term associated with $D_K$ is gauge equivalent with its negative $D_{C-S}$. The addition of the measurement interaction to $D_K$ changes the basis for the 4-D induced spinors by the phase $exp(-iQK\phi)$ and therefore also the basis for the generalized eigenstates of $D_{C-S}$ and this brings in effectively the measurement interaction term affecting the Dirac determinant.

3. The definition of Dirac determinant should be in terms of Chern-Simons action induced by the instanton term and identified as a product of the generalized eigenvalues of this operator. The modified Dirac equation for $\Psi$ is consistent with that for its conjugate if the coefficient of the instanton term is real and one uses the Dirac action $\overline{\Psi}(D^{\rightarrow} - D^{\leftarrow})\Psi$ giving modified Dirac equation as

$$D_{C-S}\Psi + \frac{1}{2}(D_\alpha\hat{\Gamma}^\alpha_{C-S})\Psi = 0 \ . \tag{5.4.4}$$

   As noticed, the divergence of gamma matrices does not contain second derivatives in the case of Chern-Simons action. In the case of Kähler action they occur unless field equations equivalent with the vanishing of the divergence term are satisfied.

   Also the fermionic current is conserved in this case, which conforms with the idea that fermions flow along the light-like 3-surfaces. If one uses the action $\overline{\Psi}D^{\rightarrow}\Psi$, $\overline{\Psi}$ does not satisfy the Dirac equation following from the variational principle and fermion current is not conserved. Also if the Chern-Simons term is imaginary - as a naive idea about dissipation would suggest- the Dirac equation fails to be consistent with the conjugation.

4. Off mass shell states appear in the lines of the generalized Feynman diagrams and for these $D_{C-S}$ cannot annihilate the spinor field. The generalized eigenmodes lf $D_{C-S}$ should be such that one obtains the counterpart of Dirac propagator which is purely algebraic and does not therefore depend on the coordinates of the throat. This is satisfied if the generalized eigenvalues are expressible in terms of covariantly constant combinations of gamma matrices and here only $M^4$ gamma matrices are possible. Therefore the eigenvalue equation reqads as

$$D\Psi = \lambda^k\gamma_k\Psi \ , \ D = D_{C-S} + D_\alpha\hat{\Gamma}^\alpha_{C-S} \ , \ D_{C-S} = \hat{\Gamma}^\alpha_{C-S}D_\alpha \ . \tag{5.4.5}$$

   Here the covariant derivatives $D_\alpha$ contain the measurement interaction term as an apparent gauge term. Covariant constancy allows to take the square of this equation and one has

$$(D^2 + \left[D, \lambda^k\gamma_k\right])\Psi = \lambda^k\lambda_k\Psi \ . \tag{5.4.6}$$

The commutator term is analogous to magnetic moment interaction. The generalized eigenvalues correspond to $\lambda = \sqrt{\lambda^k \lambda_k}$ and Dirac determinant is defined as a product of the eigenvalues. $\lambda$ is completely analogous to mass. For incoming lines this mass would vanish so that all incoming particles irrespective their actual quantum numbers would be massless in this sense and the propagator is indeed that for a massless particle. Note that the eigen-modes define the boundary values for the solutions of $D_K\Psi = 0$ so that the values of $\lambda$ indeed define the counterpart of the momentum space.

This transmutation of massive particles to effectively massless ones might make possible the application of the twistor formalism as such in TGD framework [K61] . $N = 4$ SUSY is one of the very few gauge theory which might be UV finite but it is definitely unphysical due to the masslessness of the basic quanta. Could the resolution of the interpretational problems be that the four-momenta appearing in this theory do not directly correspond to the observed four-momenta?

### 5.4.2 Objections

The alert reader has probably raised several critical questions. Doesn't the need to solve $\lambda_k$ as functions of incoming quantum numbers plus the need to construct the measurement interactions makes the practical application of the theory hopelessly difficult? Could the resulting pseudo-momentum $\lambda_k$ correspond to the actual four-momentum? Could one drop the measurement interaction term altogether and assume that the quantum classical correspondence is through the identification of the eigenvalues as the four-momenta of the on mass shell particles propagating at the wormhole throats? Could one indeed assume that the momenta have a continuous spectrum and thus do not depend on the boundary conditions at all? Usually the thinking is just the opposite and in the general case would lead to to singular eigen modes.

1. Only the information about four-momentum would be feeded into the space-time geometry. TGD however allows much more general measurement interaction terms and it would be very strange if the space-time geometry would not correlate also with the other quantum numbers. Mass formulas would of course contain information also about other quantum numbers so that this claim is not quite justified.

2. Number theoretic considerations and also the construction of octonionic variant of Dirac equation [K52] force the conclusion that the spectrum of pseudo four-momentum is restricted to a preferred plane $M^2$ of $M^4$ and this excludes the interpretation of $\lambda^k$ as a genuine four-momentum. It also improves the hopes that the sum over pseudo-momenta does not imply divergences.

3. Dirac determinant would depend on the mass spectrum only and could not be identified as exponent of Kähler function. Note that the original guideline was the dream about stringy propagators. This is achieved for $\lambda_A \lambda^A = n$ in suitable units. This spectrum would of coure also imply that Dirac determinant defined in terms of $\zeta$ function regularization is independent of the space-time surface and could not be identified with the exponent of Kähler function. One must of course take the identification of exponent of Kähler function as Dirac determinant as an additional conjecture which is not necessary for the calculation of Kähler function if the weak form of electric-magnetic duality is accepted.

4. All particles would behave as massless particles and this would not be consistent with the proposed Feynman diagrammatics inspired by zero energy ontology. Since wormhole throats carry on mass shell particles with positive or negative energy so that the net momentum can be also space-like propagators diverge for massless particles. One might overcome this problem by assuming small thermal mass (from p-adic thermodynamics [K34] ) and this is indeed assumed to reduce the number of generalized Feynman diagrams contributing to a given reaction to finite number.

Second objection of the skeptic reader relates to the delicacies of $U(1)$ gauge invariance. The modified Dirac action seems to break gauge symmetries and this breaking of gauge symmetry is absolutely essential for the dependence of the Dirac determinant on the quantum numbers. It however seems that this breaking of gauge invariance is only apparent.

1. One must distinguish between genuine $U(1)$ gauge transformations carried out for the induced Kähler gauge potential $A_\alpha$ and apparent gauge transformations of the Kähler gauge potential $A_k$ of $S^2 \times CP_2$ induced by symplectic transformations deforming the space-time surface and affect also induced metric. This delicacy of $U(1)$ gauge symmetry explains also the apparent breaking of $U(1)$ gauge symmetry of Chern-Simons Dirac action due to the presence of explicit terms $A_k$ and $A_\alpha$.

2. $CP_2$ Kähler gauge potential is obtained in complex coordinates from Kähler function as $(K_{\xi^i}, K_{\overline{\xi^i}}) = (\partial_{\xi^i}K, -\partial_{\overline{\xi^i}}K)$. Gauge transformations correspond to the additions $K \rightarrow K + f + \overline{f}$, where $f$ is a holomorphic function. Kähler gauge potential has a unique gauge in which the Kähler function of $CP_2$ is $U(2)$ invariant and contains no holomorphic part. Hence $A_k$ is defined in a preferred gauge and is a gauge invariant quantity in this sense. Same applies to $S^2$ part of the Kähler potential.

3. $A_\alpha$ should be also gauge invariant under gauge transformation respecting the vanishing of Coulombic interaction energy. The allowed gauge transformations $A_\alpha \rightarrow A_\alpha + \partial_\alpha\phi$ must satisfy $D_\alpha(j^\alpha_K\phi) = 0$. If the scalar function $\phi$ reduces to constant at the wormhole throats and at the ends of the space-time surface $D_{C-S}$ is gauge invariant. The gauge transformations for which $\phi$ does not satisfy this condition are identified as representations of critical deformations of space-time surface so that the change of $A_\alpha$ would code for this kind of deformation and indeed affect the modified Dirac operator and Kähler function (the change would be due to the change of zero modes).

### 5.4.3 Some details about the modified Dirac equation defined by Chern-Simons action

First some general comments about $D_{C-S}$ are in order.

1. Quite generally, there is vacuum avoidance in the sense that $\Psi$ must vanish in the regions where the modified gamma matrices vanish. A physical analogy for the system consider is a charged particle in an external magnetic field. The effective metric defined by the anticommutators of the modified gamma matrices so that standard intuitions might not help much. What one would naively expect would be analogs of bound states in magnetic field localized into regions inside which the magnetic field is nonvanishing.

2. If only $CP_2$ Kähler form appears in the Kähler action, the modified Dirac action defined by the Chern-Simons term is non-vanishing only when the dimension of the $CP_2$ projection of the 3-surface is $D(CP_2) \geq 2$ and the induced Kähler field is non-vanishing. This conforms with the properties of Kähler action. The solutions of the modified Dirac equation with a vanishing eigenvalue $\lambda$ would naturally correspond to incoming and outgoing particles. $D(CP_2) \leq 2$ is inconsistent with the weak form of electric-magnetic duality unless one allows the presence of also $S^2$ symplectic form $J_1$ in the conditions (the value of Planck constant would be infinite [K24] ). The extrema of Chern-Simons action have $D(CP_2) \leq 2$ and vanishing Chern-Simons density so that they would naturally represent on mass shell particles appearing as incoming and outgoing particles. This conforms with the interpretation of the basic extremals as free particles (massless extremals and cosmic strings with 2-D $CP_2$ projection). One could say that CP breaking is not present for free particles but unavoidably accompanies the propagator lines.

3. If a reduction to almost topological QFT is assumed [K24] , a realistic WCW metric requires the replacement of $J$ with $J + J_1$, where $J_1$ is $S^2$ Kähler form. An analogous replacement must be carried out also for the Chern-Simons term. In this case on can have a non-vanishing $\Psi$ also for 1-dimensional $CP_2$ projection. On the other hand, one can have also 3-D $CP_2$ projection for vacuum regions and $\Psi$ must vanish in these regions.

The explicit expression of $D_{C-S}$ is given by

$$\begin{aligned}
D &= \hat{\Gamma}^{\mu} D_{\mu} + \frac{1}{2} D_{\mu}\hat{\Gamma}^{\mu} \ , \\
\hat{\Gamma}^{\mu} &= \frac{\partial L_{C-S}}{\partial_{\mu} h^k}\Gamma_k = \epsilon^{\mu\alpha\beta}\left[2J_{kl}\partial_{\alpha}h^l A_{\beta} + J_{\alpha\beta}A_k\right]\Gamma^k D_{\mu} \ , \\
D_{\mu}\hat{\Gamma}^{\mu} &= B_K^{\alpha}(J_{k\alpha} + \partial_{\alpha}A_k) \ , \\
B_K^{\alpha} &= \epsilon^{\alpha\beta\gamma}J_{\beta\gamma} \ , \ J_{k\alpha} = J_{kl}\partial_{\alpha}s^l \ , \ \hat{\epsilon}^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma}\sqrt{g_3} \ .
\end{aligned} \tag{5.4.7}$$

Note $\hat{\epsilon}^{\alpha\beta\gamma} =$ does not depend on the induced metric.

The extremals of Chern-Simons action satisfy

$$B_K^{\alpha}(J_{kl} + \partial_l A_k)\partial_{\alpha}h^l = 0 \ , \ B_K^{\alpha} = \epsilon^{\alpha\beta\gamma}J_{\beta\gamma} \ . \tag{5.4.8}$$

For non-vanishing Kähler magnetic field $B^{\alpha}$ these equations hold true when $CP_2$ projection is 2-dimensional and $S^2$ projection is 1-dimensional or vice versa. This implies a vanishing of Chern-Simons action for both options. Consider for the simplicity the case when $S^2$ projection is 1-dimensional.

1. Suppose that one can assign a global coordinate to the flow lines of the Kähler magnetic field. In this case one might hope that ordinary intuitions about motion in constant magnetic field might be helpful. The repetition of the discussion of [K24] leads to the condition $B \wedge dB = 0$ implying that a Beltrami flow for which current flows along the field lines and Lorentz forces vanishes is in question. This need not be the generic case.

2. With this assumption the modified Dirac operator reduces to a one-dimensional Dirac operator

$$D = \hat{\epsilon}^{r\alpha\beta}\left[2J_{kl}\partial_{\alpha}h^l A_{\beta} + J_{\alpha\beta}A_k\right]\Gamma^k D_r \ . \tag{5.4.9}$$

3. The general solutions of the modified Dirac equation is covariantly constant with respect to the coordinate $r$:

$$D_r\Psi = 0 \ . \tag{5.4.10}$$

   The solution to this condition can be written immediately in terms of a non-integrable phase factor $Pexp(i\int A_r dr)$, where integration is along curve with constant transversal coordinates. If $\hat{\Gamma^v}$ is light-like vector field also $\hat{\Gamma^v}\Psi_0$ defines a solution of $D_{C-S}$. This solution corresponds to a zero mode for $D_{C-S}$ and does not contribute to the Dirac determinant. Note that the dependence of these solutions on transversal coordinates of $X_l^3$ is arbitrary.

4. The formal solution associated with a general eigenvalue can be constructed by integrating the eigenvalue equation separately along all coordinate curves. This makes sense if $r$ indeed assigned to light-like curves indeed defines a global coordinate. What is strange that there is no correlation between the behaviors with respect longitudinal coordinate and transversal coordinates. System would be like a collection of totally uncorrelated point like particles reflecting the flow of the current along flux lines. It is difficult to say anything about the spectrum of the generalized eigenvalues in this case: it might be that the boundary conditions at the ends of the flow lines fix the allowed values of $\lambda$. Clearly, the Beltrami flow property is what makes this case very special.

### 5.4.4 A connection with quantum measurement theory

It is encouraging that isometry charges and also other charges could make themselves visible in the geometry of space-time surface as they should by quantum classical correspondence. This suggests an interpretation in terms of quantum measurement theory.

1. The interpretation resolves the problem caused by the fact that the choice of the commuting isometry charges is not unique. Cartan algebra corresponds naturally to the measured observables. For instance, one could choose the Cartan algebra of Poincare group to consist of energy and momentum, angular momentum and boost (velocity) in particular direction as generators of the Cartan algebra of Poincare group. In fact, the choices of a preferred plane $M^2 \subset M^4$ and geodesic sphere $S^2 \subset CP_2$ allowing to fix the measurement sub-algebra to a high degree are implied by the replacement of the imbedding space with a book like structure forced by the hierarchy of Planck constants. Therefore the hierarchy of Planck constants seems to be required by quantum measurement theory. One cannot overemphasize the importance of this connection.

2. One can add similar couplings of the net values of the measured observables to the currents whose existence and conservation is guaranteed by quantum criticality. It is essential that one maps the observables to Cartan algebra coupled to critical current characterizing the observable in question. The coupling should have interpretation as a replacement of the induced Kähler gauge potential with its gauge transform. Quantum classical correspondence encourages the identification of the classical charges associated with Kähler action with quantal Cartan charges. This would support the interpretation in terms of a measurement interaction feeding information to classical space-time physics about the eigenvalues of the observables of the measured system. The resulting field equations remain second order partial differential equations since the second order partial derivatives appear only linearly in the added terms.

3. What about the space-time correlates of electro-weak charges? The earlier proposal explains this correlation in terms of the properties of quantum states: the coupling of electro-weak charges to Chern-Simons term could give the correlation in stationary phase approximation. It would be however very strange if the coupling of electro-weak charges with the geometry of the space-time sheet would not have the same universal description based on quantum measurement theory as isometry charges have.

   (a) The hint as how this description could be achieved comes from a long standing un-answered question motivated by the fact that electro-weak gauge group identifiable as the holonomy group of $CP_2$ can be identified as $U(2)$ subgroup of color group. Could the electro-weak charges be identified as classical color charges? This might make sense since the color charges have also identification as fermionic charges implied by quantum criticality. Or could electro-weak charges be only represented as classical color charges by mapping them to classical color currents in the measurement interaction term in the modified Dirac action? At least this question might make sense.

   (b) It does not make sense to couple both electro-weak and color charges to the same fermion current. There are also other fundamental fermion currents which are conserved. All the following currents are conserved.

$$\begin{aligned} J^{\alpha} &= \overline{\Psi} O \hat{\Gamma}^{\alpha} \Psi \\ O &\in \{1 \ , \ J \equiv J_{kl}\Sigma^{kl} \ , \ \Sigma_{AB} \ , \ \Sigma_{AB} J\} \ . \end{aligned} \tag{5.4.11}$$

   Here $J_{kl}$ is the covariantly constant $CP_2$ Kähler form and $\Sigma_{AB}$ is the (also covariantly) constant sigma matrix of $M^4$ (flatness is absolutely essential).

   (c) Electromagnetic charge can be expressed as a linear combination of currents corresponding to $O = 1$ and $O = J$ and vectorial isospin current corresponds to $J$. It is natural to couple of electromagnetic charge to the the projection of Killing vector field of color hyper charge and coupling it to the current defined by $O_{em} = a+bJ$. This allows to interpret the puzzling finding that electromagnetic charge can be identified as anomalous color hyper-charge for induced spinor fields made already during the first years of TGD. There exist no conserved axial isospin currents in accordance with CVC and PCAC hypothesis which belong to the basic stuff of the hadron physics of old days.

(d) Color charges would couple naturally to lepton and quark number current and the $U(1)$ part of electro-weak charges to the $n = 1$ multiple of quark current and $n = 3$ multiple of the lepton current (note that leptons *resp.* quarks correspond to $t = 0$ *resp.* $t = \pm 1$ color partial waves). If electro-weak *resp.* couplings to $H$-chirality are proportional to 1 *resp.* $\Gamma_9$, the fermionic currents assigned to color and electro-weak charges can be regarded as independent. This explains why the possibility of both vectorial and axial couplings in 8-D sense does not imply the doubling of gauge bosons.

(e) There is also an infinite variety of conserved currents obtained as the quantum critical deformations of the basic fermion currents identified above. This would allow in principle to couple an arbitrary number of observables to the geometry of the space-time sheet by mapping them to Cartan algebras of Poincare and color group for a particular conserved quantum critical current. Quantum criticality would therefore make possible classical space-time correlates of observables necessary for quantum measurement theory.

(f) The coupling constants associated with the deformations would appear in the couplings. Quantum criticality ($K \rightarrow K + f + \overline{f}$ condition) should predict the spectrum of these couplings. In the case of momentum the coupling would be proportional to $\sqrt{G/\hbar_0} = kR/\hbar_0$ and $k \sim 2^{11}$ should follow from quantum criticality. p-Adic coupling constant evolution should follow from the dependence on the scale of $CD$ coming as powers of 2.

4. Quantum criticality implies fluctuations in long length and time scales and it is not surprising that quantum criticality is needed to produce a correlation between quantal degrees of freedom and macroscopic degrees of freedom. Note that quantum classical correspondence can be regarded as an abstract form of entanglement induced by the entanglement between quantum charges $Q_A$ and fermion number type charges assignable to zero modes.

5. Space-time sheets can have an arbitrary number of wormhole contacts so that the interpretation in terms of measurement theory coupling short and long length scales suggests that the measurement interaction terms are localizable at the wormhole throats. This would favor Chern-Simons term or possibly instanton term if reducible to Chern-Simons terms. The breaking of CP and T might relate to the fact that state function reductions performed in quantum measurements indeed induce dissipation and breaking of time reversal invariance.

6. The experimental arrangement quite concretely splits the quantum state to a quantum superposition of space-time sheets such that each eigenstate of the measured observables in the superposition corresponds to different space-time sheet already before the realization of state function reduction. This relates interestingly to the question whether state function reduction really occurs or whether only a branching of wave function defined by WCW spinor field takes place as in multiverse interpretation in which different branches correspond to different observers. TGD inspired theory consciousness requires that state function reduction takes place. Maybe multiversalist might be able to find from this picture support for his own beliefs.

7. One can argue that "free will" appears not only at the level of quantum jumps but also as the possibility to select the observables appearing in the modified Dirac action dictating in turn the Kähler function defining the Kähler metric of WCW representing the "laws of physics". This need not to be the case. The choice of $CD$ fixes $M^2$ and the geodesic sphere $S^2$: this does not fix completely the choice of the quantization axis but by isometry invariance rotations and color rotations do not affect Kähler function for given $CD$ and for a given type of Cartan algebra. In $M^4$ degrees of freedom the possibility to select the observables in two manners corresponding to linear and cylindrical Minkowski coordinates could imply that the resulting Kähler functions are different. The corresponding Kähler metrics do not differ if the real parts of the Kähler functions associated with the two choices differ by a term $f(Z) + \overline{f(Z)}$, where $Z$ denotes complex coordinates of WCW, the Kähler metric remains the same. The function $f$ can depend also on zero modes. If this is the case then one can allow in given CD superpositions of WCW spinor fields for which the measurement interactions are different. This condition is expected to pose non-trivial constraints on the measurement action and quantize coupling parameters appearing in it.

### 5.4.5 New view about gravitational mass and matter antimatter asymmetry

The physical interpretation of the additional term in the modified Dirac action might force quite a radical revision of the ideas about matter and antimatter.

1. The term $p_A\partial_\alpha m^A$ contracted with the fermion current is analogous to a gauge potential coupling to fermion number. Since the additional terms in the modified Dirac operator induce stringy propagation, a natural interpretation of the coupling to the induced spinor fields is in terms of gravitation. One might perhaps say that the measurement of four momentum induces gravitational interaction. Besides momentum components also color charges take the role of gravitational charges. As a matter fact, any observable takes this role via coupling to the projections of Killing vector fields of Cartan algebra. The analogy of color interactions with gravitational interactions is indeed one of the oldest ideas in TGD.

2. The coupling to four-momentum is through fermion number (both quark number and lepton number). For states with a vanishing fermion number isometry charges therefore vanish. In this framework matter antimatter asymmetry would be due to the fact that matter (antimatter) corresponds to positive (negative) energy parts of zero energy states for massive systems so that the contributions to the net gravitational four-momentum are of same sign. Could antimatter be unobservable to us because it resides at negative energy space-time sheets? As a matter fact, I proposed already years ago that gravitational mass is essentially the magnitude of the inertial mass but gave up this idea.

3. Bosons do not couple at all to gravitation if they are purely local bound states of fermion and anti-fermion at the same space-time sheet (say represented by generators of super Kac-Moody algebra). Therefore the only possible identification of gauge bosons is as wormhole contacts. If the fermion and anti-fermion at the opposite throats of the contact correspond to positive and negative energy states the net gravitational energy receives a positive contribution from both sheets. If both correspond to positive (negative) energy the contributions to the net four-momentum have opposite signs. It is not yet clear which identification is the correct one.

### 5.4.6 Generalized eigenvalues of $D_{C-S}$ and General Coordinate Invariance

The fixing of light-like 3-surface to be the wormhole throat at which the signature of induced metric changes from Minkowskian to Euclidian corresponds to a convenient fixing of gauge. General Coordinate Invariance however requires that any light-like surface $Y_l^3$ parallel to $X_l^3$ in the slicing is equally good choice. In particular, it should give rise to same Kähler metric but not necessarily the same exponent of Kähler function identified as the product of the generalized eigenvalues of $D_{C,S}$ at $Y_l^3$.

General Coordinate Invariance requires that the components of Kähler metric of configuration space defined in terms of Kähler function as

$$G_{k\bar{l}} = \partial_k\partial_{\bar{l}}K = \sum_i \partial_k\partial_{\bar{l}}\lambda_i$$

remain invariant under this flow. Here complex coordinate are of course associated with the configuration space. This is the case if the flow corresponds to the addition of sum of holomorphic function $f(z)$ and its conjugate $\overline{f}(z))$ which is anti-holomorphic function to $K$. This boils down to the scaling of eigenvalues $\lambda_i$ by

$$\lambda_i \rightarrow exp(f_i(z) + \overline{f_i(z)})\lambda_i \quad . \tag{5.4.12}$$

If the eigenvalues are interpreted as vacuum conformal weights, general coordinate transformations correspond to a spectral flow scaling the eigenvalues in this manner. This in turn would induce spectral flow of ground state conformal weights if the squares of $\lambda_i$ correspond to ground state conformal weights.

## 5.5 Quaternions, octonions, and modified Dirac equation

Classical number fields define one vision about quantum TGD. This vision about quantum TGD has evolved gradually and involves several speculative ideas.

1. The hard core of the vision is that space-time surfaces as preferred extremals of Kähler action can be identified as what I have called hyper-quaternionic surfaces of $M^8$ or $M^4 \times CP_2$. This requires only the mapping of the modified gamma matrices to octonions or to a basis of subspace of complexified octonions. This means also the mapping of spinors to octonionic spinors. There is no need to assume that imbedding space-coordinates are octonionic.

2. I have considered also the idea that quantum TGD might emerge from the mere associativity.

   (a) Consider Clifford algebra of WCW. Treat "vibrational" degrees of freedom in terms second quantized spinor fields and add center of mass degrees of freedom by replacing 8-D gamma matrices with their octonionic counterparts - which can be constructed as tensor products of octonions providing alternative representation for the basis of 7-D Euclidian gamma matrix algebra - and of 2-D sigma matrices. Spinor components correspond to tensor products of octonions with 2-spinors: different spin states for these spinors correspond to leptons and baryons.

   (b) Construct a local Clifford algebra by considering Clifford algebra elements depending on point of $M^8$ or $H$. The octonionic 8-D Clifford algebra and its local variant are non-accociative. Associative sub-algebra of 8-D Clifford algebra is obtained by restricting the elements so any quaternionic 4-plane. Doing the same for the local algebra means restriction of the Clifford algebra valued functions to any 4-D hyper-quaternionic sub-manifold of $M^8$ or $H$ which means that the gamma matrices span complexified quaternionic algebra at each point of space-time surface. Also spinors must be quaternionic.

   (c) The assignment of the 4-D gamma matrix sub-algebra at each point of space-time surface can be done in many manners. If the gamma matrices correspond to the tangent space of space-time surface, one obtains just induced gamma matrices and the standard definition of quaternionic sub-manifold. In this case induced 4-volume is taken as the action principle. If Kähler action defines the space-time dynamics, the modified gamma matrices do not span the tangent space in general.

   (d) An important additional element is involved. If the $M^4$ projection of the space-time surface contains a preferred subspace $M^2$ at each point, the quaternionic planes are labeled by points of $CP_2$ and one can equivalently regard the surfaces of $M^8$ as surfaces of $M^4 \times CP_2$ (number-theoretical "compactification"). This generalizes: $M^2$ can be replaced with a distribution of planes of $M^4$ which integrates to a 2-D surface of $M^4$ (for instance, for string like objects this is necessarily true). The presence of the preferred local plane $M^2$ corresponds to the fact that octonionic spin matrices $\Sigma_{AB}$ span 14-D Lie-algebra of $G_2 \subset SO(7)$ rather than that 28-D Lie-algebra of $SO(7,1)$ whereas octonionic imaginary units provide 7-D fundamental representation of $G_2$. Also spinors must be quaternionic and this is achieved if they are created by the Clifford algebra defined by induced gamma matrices from two preferred spinors defined by real and preferred imaginary octonionic unit. Therefore the preferred plane $M^3 \subset M^4$ and its local variant has direct counterpart at the level of induced gamma matrices and spinors.

   (e) This framework implies the basic structures of TGD and therefore leads to the notion of world of classical worlds (WCW) and from this one ends up with the notion WCW spinor field and WCW Clifford algebra and also hyper-finite factors of type $\mathrm{II}_1$ and $\mathrm{III}_1$. Note that $M^8$ is exceptional: in other dimensions there is no reason for the restriction of the local Clifford algebra to lower-dimensional sub-manifold to obtain associative algebra.

The above line of ideas leads naturally to (hyper-)quaternionic sub-manifolds and to basic quantum TGD (note that the "hyper" is un-necessary if one accepts just the notion of quaternionic sub-manifold formulated in terms of modified gamma matrices). One can pose some further questions.

1. Quantum TGD reduces basically to the second quantization of the induced spinor fields. Could it be that the theory is integrable only for 4-D hyper-quaternionic space-time surfaces in $M^8$

(equivalently in $M^4\times CP_2$) in the sense than one can solve the modified Dirac equation exactly only in these cases?

2. The construction of quantum TGD -including the construction of vacuum functional as exponent of Kähler function reducing to Kähler action for a preferred extremal - should reduce to the modified Dirac equation defined by Kähler action. Could it be that the modified Dirac equation can be solved exactly only for Kähler action.

3. Is it possible to solve the modified Dirac equation for the octonionic gamma matrices and octonionic spinors and map the solution as such to the real context by replacing gamma matrices and sigma matrices with their standard counterparts? Could the associativity conditions for octospinors and modified Dirac equation allow to pin down the form of solutions to such a high degree that the solution can be constructed explicitly?

4. Octonionic gamma matrices provide also a non-associative representation for 8-D version of Pauli sigma matrices and encourage the identification of 8-D twistors as pairs of octonionic spinors conjectured to be highly relevant also for quantum TGD. Does the quaternionicity condition imply that octo-twistors reduce to something closely related to ordinary twistors as the fact that 2-D sigma matrices provide a matrix representation of quaternions suggests?

In the following I will try to answer these questions by developing a detailed view about the octonionic counterpart of the modified Dirac equation and proposing explicit solution ansätze for the modes of the modified Dirac equation.

### 5.5.1 The replacement of $SO(7,1)$ with $G_2$

The basic implication of octonionization is the replacement of $SO(7,1)$ as the structure group of spinor connection with $G_2$. This has some rather unexpected consequences.

#### Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1\times\sigma_1 \ \ , \ \ \gamma^i = \gamma^i\otimes\sigma_2 \ \ , i=1,..,7 \ \ . \tag{5.5.1}$$

7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing $\gamma^7$ as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i , i=1,...,6 \ \ , \ \ \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6}\gamma^{6)}_i \ \ . \tag{5.5.2}$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1\times\sigma_1 \ \ , \ \ \gamma_i = e_i\otimes\sigma_2 \ \ . \tag{5.5.3}$$

where $e_i$ are the octonionic units. $e_i^2=-1$ guarantees that the $M^4$ signature of the metric comes out correctly. Note that $\gamma_7=\prod\gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane $M^2$.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = e_i \times \sigma_3 \ , \ \Sigma_{ij} = f_{ij}{}^k e_k \otimes 1 \ . \tag{5.5.4}$$

These matrices span $G_2$ algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be $\Sigma_{01}$ and $\Sigma_{23}$ and belong to a quaternionic sub-algebra.

4. The lower dimension of the $G_2$ algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [A19] one finds $e_4e_5 = e_1$ and $e_6e_7 = -e_1$ and analogous expressions for the cyclic permutations of $e_4, e_5, e_6, e_7$. From the expression of the left handed sigma matrix $I_L^3 = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of $CP_2$ [L1] , [L1] ) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of $\Sigma_{ij}$ in the quaternionic sub-algebra.

**Some physical implications of $SO(7,1) \rightarrow G_2$ reduction**

This has interesting physical implications if one believes that the octonionic description is equivalent with the standard one.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonization. The right handed part is neutral containing only photon and $Z^0$ so that the gauge field becomes Abelian. $Z^0$ and photon fields become proportional to each other ($Z^0 \rightarrow sin^2(\theta_W)\gamma$) so that classical $Z^0$ field disappears from the dynamics, and one would obtain just electrodynamics. This might provide a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that $CP_2$ coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical $W$ boson fields.

   Also the realization of $M^8 - H$ duality led to the conclusion $M^8$ spinor connection should have only neutral components. The isospin matrix associated with the electromagnetic charge is $e_1 \times 1$ and represents the preferred imaginary octonionic unit so that that the image of the electro-weak gauge algebra respects associativity condition. An open question is whether octonionization is part of $M^8$-H duality or defines a completely independent duality. The objection is that information is lost in the mapping so that it becomes questionable whether the same solutions to the modified Dirac equation can work as a solution for ordinary Clifford algebra.

2. If $SU(2)_R$ were mapped to zero only left handed parts of the gauge fields would remain. All classical gauge fields would remain in the spectrum so that information would not be lost. The identification of the electro-weak gauge fields as three covariantly constant quaternionic units would be possible in the case of $M^8$ allowing Hyper-Kähler structure [A10] , which has been speculated to be a hidden symmetry of quantum TGD at the level of WCW. This option would lead to difficulties with associativity since the action of the charged gauge potentials would lead out from the local quaternionic subspace defined by the octonionic spinor.

3. The gauge potentials and gauge fields defined by $CP_2$ spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to $M^4$ degrees of freedom! Since the resulting interactions are of gravitational character, one might say that electro-weak interactions are mapped to manifestly gravitational interactions. Since $SU(2)$ corresponds to rotational group one cannot say that spinor connection would give rise only to left or right handed couplings, which would be obviously a disaster.

### Octo-spinors and their relation to ordinary imbedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\begin{aligned} \Psi_{L,i} &= e_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \\ \Psi_{q,i} &= e_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \end{aligned} \tag{5.5.5}$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit $e_1$ corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1+1+3+\overline{3}$ as representations of $SU(3)\subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1\pm ie_1\} \ , & e_R \text{ and } \nu_R \text{ with spin } 1/2 \ , \\ \{e_2\pm ie_3\} \ , & e_R \text{ and } \nu_L \text{ with spin -}1/2 \ , \\ \{e_4\pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ , \\ \{e_6\pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ . \end{array} \tag{5.5.6}$$

Instead of spin one could consider helicity. All these spinors are eigenstates of $e_1$ (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon=\pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon=1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon=-1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions. The only difference between quarks and leptons is that the induced Kähler gauge potentials couple to them differently.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic modified Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit $e_1$ so that the preferred subspace $M^2$ can corresponds to a sub-manifold $M^2\subset M^4$.

## 5.5.2 Octonionic counterpart of the modified Dirac equation

The solution ansatz for the octonionic counterpart of the modified Dirac equation discussed below makes sense also for ordinary modified Dirac equation which raises the hope that the same ansatz, and even same solution could provide a solution in both cases.

### The general structure of the modified Dirac equation

In accordance with quantum holography and the notion of generalized Feynman diagram, the modified Dirac equation involves two equations which must be consistent with each other.

1. There is 3-dimensional generalized eigenvalue equation for which the modified gamma matrices are defined by Chern-Simons action defined by the sum $J_{tot}=J+J_1$ of Kähler forms of $S^2$ and $CP_2$ [K9, K19] .

$$\begin{aligned} D_3\Psi &= \left[D_{C-S}+Q_{C-S}\right]\Psi = \lambda^k\gamma_k\Psi \ , \\ Q_{C-S} &= Q_\alpha\hat{\Gamma}^\alpha_{C-S} \ , \ Q_\alpha = Q_A g^{AB} j_{B\alpha} \ . \end{aligned} \tag{5.5.7}$$

The gamma matrices $\gamma_k$ are $M^4$ gamma matrices in standard Minkowski coordinates and thus constant. Given eigenvalue $\lambda_k$ defines pseudo momentum which is some function of the genuine momenta $p_k$ and other quantum numbers via the boundary conditions associated with the generalized eigenvalue equation.

The charges $Q_A$ correspond to real four-momentum and charges in color Cartan algebra. The term $Q$ can be rather general since it provides a representation for the measurement interaction by mapping observables to Cartan algebra of isometry group and to the infinite hierarchy of conserved currents implied by quantum criticality. The operator $O$ characterizes the quantum critical conserved current. The surface $Y_l^3$ can be chosen to be any light-like 3-surface "parallel" to the wormhole throat in the slicing of $X^4$: this means an additional symmetry. Formally the measurement interaction term can be regarded as an addition of a gauge term to the Kähler gauge potential associated with the Kähler form $J_{tot}$ of $S^2 \times CP_2$.

The square of the equation gives the spinor analog of d'Alembert equation and generalized eigenvalue as the analog of mass squared. The propagator associated with the wormhole throats is formally massless Dirac propagator so that standard twistor formalism applies also without the octonionic representation of the gamma matrices although the physical particles propagating along the opposite wormhole throats are massive on mass shell particles with both signs of energy [K19] .

2. Second equation is the 4-D modified Dirac equation defined by Kähler action.

$$D_K\Psi = 0 . \tag{5.5.8}$$

The dimensional reduction of this operator to a sum corresponding to $D_{K,3}$ acting on light-like 3-surfaces and 1-D operator $D_{K,1}$ acting on the coordinate labeling the 3-D light-like 3-surfaces in the slicing would allow to assign eigenvalues to $D_{K,3}$ as analogs of energy eigenvalues for ordinary Schrödinger equation. One proposal has been that Dirac determinant could be identified as the product of these eigen values. Another and more plausible identification is as the product of pseudo masses assignable to $D_3$ defined by Chern-Simons action [B1] . It must be however made clear that the identification of the exponent of the Kähler function to Chern-Simons term makes the identification as Dirac determinant un-necessary.

3. There are two options depending on whether one requires that the eigenvalue equation applies only on the wormhole throats and at the ends of the space-time surface or for all 3-surfaces in the slicing of the space-time surface by light-like 3-surfaces. In the latter case the condition that the pseudo four-momentum is same for all the light-like 3-surfaces in the slicing gives a consistency condition stating that the commutator of the two Dirac operators vanishes for the solutions in the case of preferred extremals, which depend on the momentum and color quantum numbers also:

$$[D_K, D_3]\,\Psi = 0 . \tag{5.5.9}$$

This condition is quite strong and there is no deep reason for it since $\lambda_k$ does not correspond to the physical conserved momentum so that its spectrum could depend on the light-like 3-surface in the slicing. On the other hand, if the eigenvalues of $D_3$ belong to the preferred hyper-complex plane $M^2$, $D_3$ effectively reduces to a 2-dimensional algebraic Dirac operator $\lambda^k\gamma_k$ commuting with $D_K$: the values of $\lambda^k$ cannot depend on slice since this would mean that $D_K$ does not commute with $D_3$.

#### About the hyper-octonionic variant of the modified Dirac equation

What gives excellent hopes that the octonionic variant of modified Dirac equation could lead to a provide precise information about the solution spectrum of modified Dirac equation is the condition that everything in the equation should be associative. Hence the terms which are by there nature non-associative should vanish automatically.

1. The first implication is that the besides octonionic gamma matrices also octonionic spinors should belong to the local quaternionic plane at each point of the space-time surface. Spinors are also generated by quaternionic Clifford algebra from two preferred spinors defining a preferred plane in the space of spinors. Hence spinorial dynamics seems to mimic very closely the space-time dynamics and one might even hope that the solutions of the modified Dirac action could be seen as maps of the space-time surface to surfaces of the spinor space. The reduction to quaternionic sub-algebra suggest that some variant of ordinary twistors emerges in this manner in matrix representation.

2. The octonionic sigma matrices span $G_2$ where as ordinary sigma matrices define $SO(7,1)$. On the other hand, the holonomies are identical in the two cases if right-handed charge matrices are mapped to zero so that there are indeed hopes that the solutions of the octonionic Dirac equation cannot be mapped to those of ordinary Dirac equation. If left-handed charge matrices are mapped to zero, the resulting theory is essentially the analog of electrodynamics coupled to gravitation at classical level but it is not clear whether this physically acceptable. It is not clear whether associativity condition leaves only this option under consideration.

3. The solution ansatz to the modified Dirac equation is expected to be of the form $\Psi = D_K(\Psi_0 u_0 + \Psi_1 u_1)$, where $u_0$ and $u_1$ are constant spinors representing real unit and the preferred unit $e_1$. Hence constant spinors associated with right handed electron and neutrino and right-handed $d$ and $u$ quark would appear in $\Psi$ and $\Psi_i$ could correspond to scalar coefficients of spinors with different charge. This ansatz would reduce the modified Dirac equation to $D_K^2\Psi_i = 0$ since there are no charged couplings present. The reduction of a d'Alembert type equation for single scalar function coupling to $U(1)$ gauge potential and $U(1)$ "gravitation" would obviously mean a dramatic simplification raising hopes about integrable theory.

4. The condition $D_K^2\Psi = 0$ involves products of three octonions and involves derivatives of the modified gamma matrices which might belong to the complement of the quaternionic sub-space. The restriction of $\Psi$ to the preferred hyper-complex plane $M^2$ simplifies the situation dramatically but $(D_K^2)D_K\Psi = D_K(D_K^2)\Psi = 0$ could still fail. The problem is that the action of $D_K$ is not algebraic so that one cannot treat reduce the associativity condition to $(AA)A = A(AA)$.

### 5.5.3 Could the notion of octo-twistor make sense?

Twistors have led to dramatic successes in the understanding of Feynman diagrammatics of gauge theories, $N = 4$ SUSYs, and $N = 8$ supergravity [B37, B43, B35] . This motivated the question whether they might be applied in TGD framework too [K61] - at least in the description of the QFT limit. The basic problem of the twistor program is how to overcome the difficulties caused by particle massivation and TGD framework suggests possible clues in this respect.

1. In TGD it is natural to regard particles as massless particles in 8-D sense and to introduce 8-D counterpart of twistors by relying on the geometric picture in which twistors correspond to a pair of spinors characterizing light-like momentum ray and a point of $M^8$ through which the ray traverses. Twistors would consist of a pair of spinors and quark and lepton spinors define the natural candidate for the spinors in question. This approach would allow to handle massive on-mass-shell states but cannot cope with virtual momenta massive in 8-D sense.

2. The emergence of pseudo momentum $\lambda_k$ from the generalized eigenvalue equation for $D_{C-S}$ suggest a dramatically simpler solution to the problem. Since propagators are effectively massless propagators for pseudo momenta, which are functions of physical on shell momenta (with both signs of energy in zero energy ontology) and of other quantum numbers, twistor formalism can be applied in its standard form. An attractive assumption is that also $\lambda^k$ are conserved in the vertices but a good argument justifying this is lacking. One can ask whether also $N = 4$ SUSY, $N = 8$ super-gravity, and even QCD could have similar interpretation.

This picture should apply also in the case of octotwistors with minor modifications and one might hope that octotwistors could provide new insights about what happens in the real case.

1. In the case of ordinary Clifford algebra unit matrix and six-dimensional gamma matrices $\gamma_i$, $i = 1, ..., 6$ and $\gamma_7 = \prod_i \gamma_i$ would define the variant of Pauli sigma matrices as $\sigma_0 = 1$, $\sigma_k = \gamma_k$, $k = 1, .., 7$ The problem is that masslessness condition does not correspond to the vanishing of the determinant for the matrix $p_k\sigma^k$.

2. In the case of octo-twistors Pauli sigma matrices $\sigma^k$ would correspond to hyper-octonion units $\{\sigma_0, \sigma_k\} = \{1, ie^k\}$ and one could assign to $p_k\sigma^k$ a matrix by the linear map defined by the multiplication with $P = p_k\sigma^k$. The matrix is of form $P_{mn} = p^k f_{kmn}$, where $f_{kmn}$ are the structure constants characterizing multiplication by hyper-octonion. The norm squared for octonion is the fourth root for the determinant of this matrix. Since $p_k\sigma^k$ maps its octonionic conjugate to zero so that the determinant must vanish (as is easy to see directly by reducing the situation to that for hyper-complex numbers by considering the hyper-complex plane defined by $P$).

3. Associativity condition for the octotwistors requires that the gamma matrix basis appearing in the generalized eigenvalue equation for Chern-Simons Dirac operator must differs by a local $G_2$ rotation from the standard hyper-quaternionic gamma matrix for $M^4$ so that it is always in the local hyper-quaternionic plane. This suggests that octo-twistor can be mapped to an ordinary twistor by mapping the basis of hyper-quaternions to Pauli sigma matrices. A stronger condition guaranteing the commutativity of $D_3$ with $\lambda^k\gamma_k$ is that $\lambda_k$ belongs to a preferred hyper-complex plane $M^2$ assignable to a given $CD$. Also the two spinors should belong to this plane for the proposed solution ansatz for the modified Dirac equation. Quaternionization would also allow to assign momentum to the spinors in standard manner.

   The spectrum of pseudo-momenta would be 2-dimensional (continuum at worst) and this should certainly improve dramatically the convergence properties for the sum over the non-conserved pseudo-momenta in propagators which in the worst possible of worlds might destroy the manifest finiteness of the theory based on the generalized Feynman diagrams with the throats of wormholes carrying always on mass shell momenta. This effective 2-dimensionality should apply also in the real case and would have no catastrophic consequences since pseudo momenta are in question.

   As a matter fact, the assumption the decomposition of quark momenta to longitudinal and transversal parts in perturbative QCD might have interpretation in terms of pseudo-momenta if they are conserved.

4. $M^8 - H$ duality suggests a possible interpretation of the pseudo-momenta as $M^8$ momenta which by purely number theoretical reasons must be commutative and thus belong to $M^2$ hyper-complex plane. One ends up with the similar outcome as one constructs a representation for the quantum states defined by WCW spinor fields as superpositions of real units constructed as ratios of infinite hyper-octonionic integers with precisely defined number theoretic anatomy and transformation properties under standard model symmetries having number theoretic interpretation [K52] .

## 5.6 Weak form electric-magnetic duality and its implications

The notion of electric-magnetic duality [B8] was proposed first by Olive and Montonen and is central in $\mathcal{N} = 4$ supersymmetric gauge theories. It states that magnetic monopoles and ordinary particles are two different phases of theory and that the description in terms of monopoles can be applied at the limit when the running gauge coupling constant becomes very large and perturbation theory fails to converge. The notion of electric-magnetic self-duality is more natural since for $CP_2$ geometry Kähler form is self-dual and Kähler magnetic monopoles are also Kähler electric monopoles and Kähler coupling strength is by quantum criticality renormalization group invariant rather than running coupling constant. The notion of electric-magnetic (self-)duality emerged already two decades ago in the attempts to formulate the Kähler geometric of world of classical worlds. Quite recently a considerable step of progress took place in the understanding of this notion [K10] . What seems to be essential is that one adopts a weaker form of the self-duality applying at partonic 2-surfaces. What this means will be discussed in the sequel.

Every new idea must be of course taken with a grain of salt but the good sign is that this concept leads to precise predictions. The point is that elementary particles do not generate monopole fields in macroscopic length scales: at least when one considers visible matter. The first question is whether elementary particles could have vanishing magnetic charges: this turns out to be impossible. The next question is how the screening of the magnetic charges could take place and leads to an identification of the physical particles as string like objects identified as pairs magnetic charged wormhole throats connected by magnetic flux tubes.

1. The first implication is a new view about electro-weak massivation reducing it to weak confinement in TGD framework. The second end of the string contains particle having electroweak isospin neutralizing that of elementary fermion and the size scale of the string is electro-weak scale would be in question. Hence the screening of electro-weak force takes place via weak confinement realized in terms of magnetic confinement.

2. This picture generalizes to the case of color confinement. Also quarks correspond to pairs of magnetic monopoles but the charges need not vanish now. Rather, valence quarks would be connected by flux tubes of length of order hadron size such that magnetic charges sum up to zero. For instance, for baryonic valence quarks these charges could be $(2,-1,-1)$ and could be proportional to color hyper charge.

3. The highly non-trivial prediction making more precise the earlier stringy vision is that elementary particles are string like objects in electro-weak scale: this should become manifest at LHC energies.

4. The weak form electric-magnetic duality together with Beltrami flow property of Kähler leads to the reduction of Kähler action to Chern-Simons action so that TGD reduces to almost topological QFT and that Kähler function is explicitly calculable. This has enormous impact concerning practical calculability of the theory.

5. One ends up also to a general solution ansatz for field equations from the condition that the theory reduces to almost topological QFT. The solution ansatz is inspired by the idea that all isometry currents are proportional to Kähler current which is integrable in the sense that the flow parameter associated with its flow lines defines a global coordinate. The proposed solution ansatz would describe a hydrodynamical flow with the property that isometry charges are conserved along the flow lines (Beltrami flow). A general ansatz satisfying the integrability conditions is found. The solution ansatz applies also to the extremals of Chern-Simons action and and to the conserved currents associated with the modified Dirac equation defined as contractions of the modified gamma matrices between the solutions of the modified Dirac equation. The strongest form of the solution ansatz states that various classical and quantum currents flow along flow lines of the Beltrami flow defined by Kähler current (Kähler magnetic field associated with Chern-Simons action). Intuitively this picture is attractive. A more general ansatz would allow several Beltrami flows meaning multi-hydrodynamics. The integrability conditions boil down to two scalar functions: the first one satisfies massless d'Alembert equation in the induced metric and the the gradients of the scalar functions are orthogonal. The interpretation in terms of momentum and polarization directions is natural.

6. The general solution ansatz works for induced Kähler Dirac equation and Chern-Simons Dirac equation and reduces them to ordinary differential equations along flow lines. The induced spinor fields are simply constant along flow lines of indued spinor field for Dirac equation in suitable gauge. Also the generalized eigen modes of the modified Chern-Simons Dirac operator can be deduced explicitly if the throats and the ends of space-time surface at the boundaries of $CD$ are extremals of Chern-Simons action. Chern-Simons Dirac equation reduces to ordinary differential equations along flow lines and one can deduce the general form of the spectrum and the explicit representation of the Dirac determinant in terms of geometric quantities characterizing the 3-surface (eigenvalues are inversely proportional to the lengths of strands of the flow lines in the effective metric defined by the modified gamma matrices).

### 5.6.1 Could a weak form of electric-magnetic duality hold true?

Holography means that the initial data at the partonic 2-surfaces should fix the configuration space metric. A weak form of this condition allows only the partonic 2-surfaces defined by the wormhole throats at which the signature of the induced metric changes. A stronger condition allows all partonic 2-surfaces in the slicing of space-time sheet to partonic 2-surfaces and string world sheets. Number theoretical vision suggests that hyper-quaternionicity *resp.* co-hyperquaternionicity constraint could be enough to fix the initial values of time derivatives of the imbedding space coordinates in the space-time regions with Minkowskian *resp.* Euclidian signature of the induced metric. This is a condition on modified gamma matrices and hyper-quaternionicity states that they span a hyper-quaternionic sub-space.

#### Definition of the weak form of electric-magnetic duality

One can also consider alternative conditions possibly equivalent with this condition. The argument goes as follows.

1. The expression of the matrix elements of the metric and Kähler form of $WCW$ in terms of the Kähler fluxes weighted by Hamiltonians of $\delta M^4_\pm$ at the partonic 2-surface $X^2$ looks very attractive. These expressions however carry no information about the 4-D tangent space of the partonic 2-surfaces so that the theory would reduce to a genuinely 2-dimensional theory, which cannot hold true. One would like to code to the WCW metric also information about the electric part of the induced Kähler form assignable to the complement of the tangent space of $X^2 \subset X^4$.

2. Electric-magnetic duality of the theory looks a highly attractive symmetry. The trivial manner to get electric magnetic duality at the level of the full theory would be via the identification of the flux Hamiltonians as sums of of the magnetic and electric fluxes. The presence of the induced metric is however troublesome since the presence of the induced metric means that the simple transformation properties of flux Hamiltonians under symplectic transformations -in particular color rotations- are lost.

3. A less trivial formulation of electric-magnetic duality would be as an initial condition which eliminates the induced metric from the electric flux. In the Euclidian version of 4-D YM theory this duality allows to solve field equations exactly in terms of instantons. This approach involves also quaternions. These arguments suggest that the duality in some form might work. The full electric magnetic duality is certainly too strong and implies that space-time surface at the partonic 2-surface corresponds to piece of $CP_2$ type vacuum extremal and can hold only in the deep interior of the region with Euclidian signature. In the region surrounding wormhole throat at both sides the condition must be replaced with a weaker condition.

4. To formulate a weaker form of the condition let us introduce coordinates $(x^0, x^3, x^1, x^2)$ such $(x^1, x^2)$ define coordinates for the partonic 2-surface and $(x^0, x^3)$ define coordinates labeling partonic 2-surfaces in the slicing of the space-time surface by partonic 2-surfaces and string world sheets making sense in the regions of space-time sheet with Minkowskian signature. The assumption about the slicing allows to preserve general coordinate invariance. The weakest condition is that the generalized Kähler electric fluxes are apart from constant proportional to Kähler magnetic fluxes. This requires the condition

$$J^{03}\sqrt{g_4} = KJ_{12} \ . \quad (5.6.1)$$

A more general form of this duality is suggested by the considerations of [K24] reducing the hierarchy of Planck constants to basic quantum TGD and also reducing Kähler function for preferred extremals to Chern-Simons terms [B1] at the boundaries of $CD$ and at light-like wormhole throats. This form is following

$$J^{n\beta}\sqrt{g_4} = K\epsilon \times \epsilon^{n\beta\gamma\delta} J_{\gamma\delta}\sqrt{g_4} \ . \quad (5.6.2)$$

Here the index $n$ refers to a normal coordinate for the space-like 3-surface at either boundary of $CD$ or for light-like wormhole throat. $\epsilon$ is a sign factor which is opposite for the two ends of $CD$. It could be also opposite of opposite at the opposite sides of the wormhole throat. Note that the dependence on induced metric disappears at the right hand side and this condition eliminates the potentials singularity due to the reduction of the rank of the induced metric at wormhole throat.

5. Information about the tangent space of the space-time surface can be coded to the configuration space metric with loosing the nice transformation properties of the magnetic flux Hamiltonians if Kähler electric fluxes or sum of magnetic flux and electric flux satisfying this condition are used and $K$ is symplectic invariant. Using the sum

$$J_e + J_m = (1+K)J_{12} \ , \tag{5.6.3}$$

where $J$ denotes the Kähler magnetic flux, , makes it possible to have a non-trivial configuration space metric even for $K=0$, which could correspond to the ends of a cosmic string like solution carrying only Kähler magnetic fields. This condition suggests that it can depend only on Kähler magnetic flux and other symplectic invariants. Whether local symplectic coordinate invariants are possible at all is far from obvious, If the slicing itself is symplectic invariant then $K$ could be a non-constant function of $X^2$ depending on string world sheet coordinates. The light-like radial coordinate of the light-cone boundary indeed defines a symplectically invariant slicing and this slicing could be shifted along the time axis defined by the tips of $CD$.

### Electric-magnetic duality physically

What could the weak duality condition mean physically? For instance, what constraints are obtained if one assumes that the quantization of electro-weak charges reduces to this condition at classical level?

1. The first thing to notice is that the flux of $J$ over the partonic 2-surface is analogous to magnetic flux

$$Q_m = \frac{e}{\hbar}\oint BdS = n \ .$$

$n$ is non-vanishing only if the surface is homologically non-trivial and gives the homology charge of the partonic 2-surface.

2. The expressions of classical electromagnetic and $Z^0$ fields in terms of Kähler form [L1] , [L1] read as

$$\begin{aligned} \gamma &= \frac{eF_{em}}{\hbar} = 3J - sin^2(\theta_W)R_{03} \ , \\ Z^0 &= \frac{g_Z F_Z}{\hbar} = 2R_{03} \ . \end{aligned} \tag{5.6.4}$$

Here $R_{03}$ is one of the components of the curvature tensor in vielbein representation and $F_{em}$ and $F_Z$ correspond to the standard field tensors. From this expression one can deduce

$$J = \frac{e}{3\hbar}F_{em} + sin^2(\theta_W)\frac{g_Z}{6\hbar}F_Z \ . \tag{5.6.5}$$

3. The weak duality condition when integrated over $X^2$ implies

$$\begin{aligned} \frac{e^2}{3\hbar}Q_{em}+\frac{g_Z^2p}{6}Q_{Z,V} &= K\oint J=Kn \ , \\ Q_{Z,V} &= \frac{I_V^3}{2}-Q_{em} \ , \ p=sin^2(\theta_W) \ . \end{aligned} \tag{5.6.6}$$

Here the vectorial part of the $Z^0$ charge rather than as full $Z^0$ charge $Q_Z=I_L^3+sin^2(\theta_W)Q_{em}$ appears. The reason is that only the vectorial isospin is same for left and right handed components of fermion which are in general mixed for the massive states.

The coefficients are dimensionless and expressible in terms of the gauge coupling strengths and using $\hbar=r\hbar_0$ one can write

$$\begin{aligned} \alpha_{em}Q_{em} &+ p\frac{\alpha_Z}{2}Q_{Z,V}=\frac{3}{4\pi}\times rnK \ , \\ \alpha_{em} &= \frac{e^2}{4\pi\hbar_0} \ , \ \alpha_Z=\frac{g_Z^2}{4\pi\hbar_0}=\frac{\alpha_{em}}{p(1-p)} \ . \end{aligned} \tag{5.6.7}$$

4. There is a great temptation to assume that the values of $Q_{em}$ and $Q_Z$ correspond to their quantized values and therefore depend on the quantum state assigned to the partonic 2-surface. The linear coupling of the modified Dirac operator to conserved charges implies correlation between the geometry of space-time sheet and quantum numbers assigned to the partonic 2-surface. The assumption of standard quantized values for $Q_{em}$ and $Q_Z$ would be also seen as the identification of the fine structure constants $\alpha_{em}$ and $\alpha_Z$. This however requires weak isospin invariance.

### The value of $K$ from classical quantization of Kähler electric charge

The value of $K$ can be deduced by requiring classical quantization of Kähler electric charge.

1. The condition that the flux of $F^{03}=(\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge $g_K$ would give the condition $K=g_K^2/\hbar$, where $g_K$ is Kähler coupling constant which should invariant under coupling constant evolution by quantum criticality. Within experimental uncertainties one has $\alpha_K=g_K^2/4\pi\hbar_0=\alpha_{em}\simeq 1/137$, where $\alpha_{em}$ is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

2. The quantization of Planck constants makes the condition highly non-trivial. The most general quantization of $r$ is as rationals but there are good arguments favoring the quantization as integers corresponding to the allowance of only singular coverings of $CD$ andn $CP_2$. The point is that in this case a given value of Planck constant corresponds to a finite number pages of the "Big Book". The quantization of the Planck constant implies a further quantization of $K$ and would suggest that $K$ scales as $1/r$ unless the spectrum of values of $Q_{em}$ and $Q_Z$ allowed by the quantization condition scales as $r$. This is quite possible and the interpretation would be that each of the $r$ sheets of the covering carries (possibly same) elementary charge. Kind of discrete variant of a full Fermi sphere would be in question. The interpretation in terms of anyonic phases [K40] supports this interpretation.

3. The identification of $J$ as a counterpart of $eB/\hbar$ means that Kähler action and thus also Kähler function is proportional to $1/\alpha_K$ and therefore to $\hbar$. This implies that for large values of $\hbar$ Kähler coupling strength $g_K^2/4\pi$ becomes very small and large fluctuations are suppressed in the functional integral. The basic motivation for introducing the hierarchy of Planck constants was indeed that the scaling $\alpha\rightarrow\alpha/r$ allows to achieve the convergence of perturbation theory: Nature itself would solve the problems of the theoretician. This of course does not mean that the physical states would remain as such and the replacement of single particles with anyonic states in order to satisfy the condition for $K$ would realize this concretely.

4. The condition $K = g_K^2/\hbar$ implies that the Kähler magnetic charge is always accompanied by Kähler electric charge. A more general condition would read as

$$K = n \times \frac{g_K^2}{\hbar}, n \in Z \ . \tag{5.6.8}$$

This would apply in the case of cosmic strings and would allow vanishing Kähler charge possible when the partonic 2-surface has opposite fermion and antifermion numbers (for both leptons and quarks) so that Kähler electric charge should vanish. For instance, for neutrinos the vanishing of electric charge strongly suggests $n = 0$ besides the condition that abelian $Z^0$ flux contributing to em charge vanishes.

It took a year to realize that this value of $K$ is natural at the Minkowskian side of the wormhole throat. At the Euclidian side much more natural condition is

$$K = \frac{1}{hbar} \ . \tag{5.6.9}$$

In fact, the self-duality of $CP_2$ Kähler form favours this boundary condition at the Euclidian side of the wormhole throat. Also the fact that one cannot distinguish between electric and magnetic charges in Euclidian region since all charges are magnetic can be used to argue in favor of this form. The same constraint arises from the condition that the action for $CP_2$ type vacuum extremal has the value required by the argument leading to a prediction for gravitational constant in terms of the square of $CP_2$ radius and $\alpha_K$ the effective replacement $g_K^2 \rightarrow 1$ would spoil the argument.

The boundary condition $J_E = J_B$ for the electric and magnetic parts of Kählwer form at the Euclidian side of the wormhole throat inspires the question whether all Euclidian regions could be self-dual so that the density of Kähler action would be just the instanton density. Self-duality follows if the deformation of the metric induced by the deformation of the canonically imbedded $CP_2$ is such that in $CP_2$ coordinates for the Euclidian region the tensor $(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})/\sqrt{g}$ remains invariant. This is certainly the case for $CP_2$ type vacuum extremals since by the light-likeness of $M^4$ projection the metric remains invariant. Also conformal scalings of the induced metric would satisfy this condition. Conformal scaling is not consistent with the degeneracy of the 4-metric at the wormhole

**Reduction of the quantization of Kähler electric charge to that of electromagnetic charge**

The best manner to learn more is to challenge the form of the weak electric-magnetic duality based on the induced Kähler form.

1. Physically it would seem more sensible to pose the duality on electromagnetic charge rather than Kähler charge. This would replace induced Kähler form with electromagnetic field, which is a linear combination of induced K"ahler field and classical $Z^0$ field

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{03} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \tag{5.6.10}$$

Here $Z_0 = 2R_{03}$ is the appropriate component of $CP_2$ curvature form [L1]. For a vanishing Weinberg angle the condition reduces to that for Kähler form.

2. For the Euclidian space-time regions having interpretation as lines of generalized Feynman diagrams Weinberg angle should be non-vanishing. In Minkowskian regions Weinberg angle could however vanish. If so, the condition guaranteing that electromagnetic charge of the partonic 2-surfaces equals to the above condition stating that the em charge assignable to the fermion content of the partonic 2-surfaces reduces to the classical Kähler electric flux at the Minkowskian side of the wormhole throat. One can argue that Weinberg angle must increase smoothly from a vanishing value at both sides of wormhole throat to its value in the deep interior of the Euclidian region.

3. The vanishing of the Weinberg angle in Minkowskian regions conforms with the physical intuition. Above elementary particle length scales one sees only the classical electric field reducing to the induced Kähler form and classical $Z^0$ fields and color gauge fields are effectively absent. Only in phases with a large value of Planck constant classical $Z^0$ field and other classical weak fields and color gauge field could make themselves visible. Cell membrane could be one such system [K42]. This conforms with the general picture about color confinement and weak massivation.

The GRT limit of TGD suggests a further reason for why Weinberg angle should vanish in Minkowskian regions.

1. The value of the Kähler coupling strength mut be very near to the value of the fine structure constant in electron length scale and these constants can be assumed to be equal.

2. GRT limit of TGD with space-time surfaces replaced with abstract 4-geometries would naturally correspond to Einstein-Maxwell theory with cosmological constant which is non-vanishing only in Euclidian regions of space-time so that both Reissner-Nordström metric and $CP_2$ are allowed as simplest possible solutions of field equations [K58]. The extremely small value of the observed cosmological constant needed in GRT type cosmology could be equal to the large cosmological constant associated with $CP_2$ metric multiplied with the 3-volume fraction of Euclidian regions.

3. Also at GRT limit quantum theory would reduce to almost topological QFT since Einstein-Maxwell action reduces to 3-D term by field equations implying the vanishing of the Maxwell current and of the curvature scalar in Minkowskian regions and curvature scalar + cosmological constant term in Euclidian regions. The weak form of electric-magnetic duality would guarantee also now the preferred extremal property and prevent the reduction to a mere topological QFT.

4. GRT limit would make sense only for a vanishing Weinberg angle in Minkowskian regions. A non-vanishing Weinberg angle would make sense in the deep interior of the Euclidian regions where the approximation as a small deformation of $CP_2$ makes sense.

The weak form of electric-magnetic duality has surprisingly strong implications for the basic view about quantum TGD as following considerations show.

### 5.6.2 Magnetic confinement, the short range of weak forces, and color confinement

The weak form of electric-magnetic duality has surprisingly strong implications if one combines it with some very general empirical facts such as the non-existence of magnetic monopole fields in macroscopic length scales.

#### How can one avoid macroscopic magnetic monopole fields?

Monopole fields are experimentally absent in length scales above order weak boson length scale and one should have a mechanism neutralizing the monopole charge. How electroweak interactions become short ranged in TGD framework is still a poorly understood problem. What suggests itself is the neutralization of the weak isospin above the intermediate gauge boson Compton length by neutral Higgs bosons. Could the two neutralization mechanisms be combined to single one?

1. In the case of fermions and their super partners the opposite magnetic monopole would be a wormhole throat. If the magnetically charged wormhole contact is electromagnetically neutral but has vectorial weak isospin neutralizing the weak vectorial isospin of the fermion only the electromagnetic charge of the fermion is visible on longer length scales. The distance of this wormhole throat from the fermionic one should be of the order weak boson Compton length. An interpretation as a bound state of fermion and a wormhole throat state with the quantum numbers of a neutral Higgs boson would therefore make sense. The neutralizing throat would have quantum numbers of $X_{-1/2} = \nu_L\overline{\nu}_R$ or $X_{1/2} = \overline{\nu}_L\nu_R$. $\nu_L\overline{\nu}_R$ would not be neutral Higgs boson (which should correspond to a wormhole contact) but a super-partner of left-handed neutrino obtained by adding a right handed neutrino. This mechanism would apply separately

to the fermionic and anti-fermionic throats of the gauge bosons and corresponding space-time sheets and leave only electromagnetic interaction as a long ranged interaction.

2. One can of course wonder what is the situation situation for the bosonic wormhole throats feeding gauge fluxes between space-time sheets. It would seem that these wormhole throats must always appear as pairs such that for the second member of the pair monopole charges and $I_V^3$ cancel each other at both space-time sheets involved so that one obtains at both space-time sheets magnetic dipoles of size of weak boson Compton length. The proposed magnetic character of fundamental particles should become visible at TeV energies so that LHC might have surprises in store!

### Magnetic confinement and color confinement

Magnetic confinement generalizes also to the case of color interactions. One can consider also the situation in which the magnetic charges of quarks (more generally, of color excited leptons and quarks) do not vanish and they form color and magnetic singles in the hadronic length scale. This would mean that magnetic charges of the state $q_{\pm 1/2} - X_{\mp 1/2}$ representing the physical quark would not vanish and magnetic confinement would accompany also color confinement. This would explain why free quarks are not observed. To how degree then quark confinement corresponds to magnetic confinement is an interesting question.

For quark and antiquark of meson the magnetic charges of quark and antiquark would be opposite and meson would correspond to a Kähler magnetic flux so that a stringy view about meson emerges. For valence quarks of baryon the vanishing of the net magnetic charge takes place provided that the magnetic net charges are $(\pm 2, \mp 1, \mp 1)$. This brings in mind the spectrum of color hyper charges coming as $(\pm 2, \mp 1, \mp 1)/3$ and one can indeed ask whether color hyper-charge correlates with the Kähler magnetic charge. The geometric picture would be three strings connected to single vertex. Amusingly, the idea that color hypercharge could be proportional to color hyper charge popped up during the first year of TGD when I had not yet discovered $CP_2$ and believed on $M^4 \times S^2$.

p-Adic length scale hypothesis and hierarchy of Planck constants defining a hierarchy of dark variants of particles suggest the existence of scaled up copies of QCD type physics and weak physics. For p-adically scaled up variants the mass scales would be scaled by a power of $\sqrt{2}$ in the most general case. The dark variants of the particle would have the same mass as the original one. In particular, Mersenne primes $M_k = 2^k - 1$ and Gaussian Mersennes $M_{G,k} = (1+i)^k - 1$ has been proposed to define zoomed copies of these physics. At the level of magnetic confinement this would mean hierarchy of length scales for the magnetic confinement.

One particular proposal is that the Mersenne prime $M_{89}$ should define a scaled up variant of the ordinary hadron physics with mass scaled up roughly by a factor $2^{(107-89)/2} = 512$. The size scale of color confinement for this physics would be same as the weal length scale. It would look more natural that the weak confinement for the quarks of $M_{89}$ physics takes place in some shorter scale and $M_{61}$ is the first Mersenne prime to be considered. The mass scale of $M_{61}$ weak bosons would be by a factor $2^{(89-61)/2} = 2^{14}$ higher and about $1.6 \times 10^4$ TeV. $M_{89}$ quarks would have virtually no weak interactions but would possess color interactions with weak confinement length scale reflecting themselves as new kind of jets at collisions above TeV energies.

In the biologically especially important length scale range 10 nm -2500 nm there are as many as four Gaussian Mersennes corresponding to $M_{G,k}$, $k = 151, 157, 163, 167$. This would suggest that the existence of scaled up scales of magnetic-, weak- and color confinement. An especially interesting possibly testable prediction is the existence of magnetic monopole pairs with the size scale in this range. There are recent claims about experimental evidence for magnetic monopole pairs [D3] .

### Magnetic confinement and stringy picture in TGD sense

The connection between magnetic confinement and weak confinement is rather natural if one recalls that electric-magnetic duality in super-symmetric quantum field theories means that the descriptions in terms of particles and monopoles are in some sense dual descriptions. Fermions would be replaced by string like objects defined by the magnetic flux tubes and bosons as pairs of wormhole contacts would correspond to pairs of the flux tubes. Therefore the sharp distinction between gravitons and physical particles would disappear.

The reason why gravitons are necessarily stringy objects formed by a pair of wormhole contacts is that one cannot construct spin two objects using only single fermion states at wormhole throats. Of course, also super partners of these states with higher spin obtained by adding fermions and anti-fermions at the wormhole throat but these do not give rise to graviton like states [K20] . The upper and lower wormhole throat pairs would be quantum superpositions of fermion anti-fermion pairs with sum over all fermions. The reason is that otherwise one cannot realize graviton emission in terms of joining of the ends of light-like 3-surfaces together. Also now magnetic monopole charges are necessary but now there is no need to assign the entities $X_{\pm}$ with gravitons.

Graviton string is characterized by some p-adic length scale and one can argue that below this length scale the charges of the fermions become visible. Mersenne hypothesis suggests that some Mersenne prime is in question. One proposal is that gravitonic size scale is given by electronic Mersenne prime $M_{127}$. It is however difficult to test whether graviton has a structure visible below this length scale.

What happens to the generalized Feynman diagrams is an interesting question. It is not at all clear how closely they relate to ordinary Feynman diagrams. All depends on what one is ready to assume about what happens in the vertices. One could of course hope that zero energy ontology could allow some very simple description allowing perhaps to get rid of the problematic aspects of Feynman diagrams.

1. Consider first the recent view about generalized Feynman diagrams which relies zero energy ontology. A highly attractive assumption is that the particles appearing at wormhole throats are on mass shell particles. For incoming and outgoing elementary bosons and their super partners they would be positive it resp. negative energy states with parallel on mass shell momenta. For virtual bosons they the wormhole throats would have opposite sign of energy and the sum of on mass shell states would give virtual net momenta. This would make possible twistor description of virtual particles allowing only massless particles (in 4-D sense usually and in 8-D sense in TGD framework). The notion of virtual fermion makes sense only if one assumes in the interaction region a topological condensation creating another wormhole throat having no fermionic quantum numbers.

2. The addition of the particles $X^{\pm}$ replaces generalized Feynman diagrams with the analogs of stringy diagrams with lines replaced by pairs of lines corresponding to fermion and $X_{\pm 1/2}$. The members of these pairs would correspond to 3-D light-like surfaces glued together at the vertices of generalized Feynman diagrams. The analog of 3-vertex would not be splitting of the string to form shorter strings but the replication of the entire string to form two strings with same length or fusion of two strings to single string along all their points rather than along ends to form a longer string. It is not clear whether the duality symmetry of stringy diagrams can hold true for the TGD variants of stringy diagrams.

3. How should one describe the bound state formed by the fermion and $X^{\pm}$? Should one describe the state as superposition of non-parallel on mass shell states so that the composite state would be automatically massive? The description as superposition of on mass shell states does not conform with the idea that bound state formation requires binding energy. In TGD framework the notion of negentropic entanglement has been suggested to make possible the analogs of bound states consisting of on mass shell states so that the binding energy is zero [K31] . If this kind of states are in question the description of virtual states in terms of on mass shell states is not lost. Of course, one cannot exclude the possibility that there is infinite number of this kind of states serving as analogs for the excitations of string like object.

4. What happens to the states formed by fermions and $X_{\pm 1/2}$ in the internal lines of the Feynman diagram? Twistor philosophy suggests that only the higher on mass shell excitations are possible. If this picture is correct, the situation would not change in an essential manner from the earlier one.

The highly non-trivial prediction of the magnetic confinement is that elementary particles should have stringy character in electro-weak length scales and could behaving to become manifest at LHC energies. This adds one further item to the list of non-trivial predictions of TGD about physics at LHC energies [K32] .

### 5.6.3 Could Quantum TGD reduce to almost topological QFT?

There seems to be a profound connection with the earlier unrealistic proposal that TGD reduces to almost topological quantum theory in the sense that the counterpart of Chern-Simons action assigned with the wormhole throats somehow dictates the dynamics. This proposal can be formulated also for the modified Dirac action action. I gave up this proposal but the following argument shows that Kähler action with weak form of electric-magnetic duality effectively reduces to Chern-Simons action plus Coulomb term.

1. Kähler action density can be written as a 4-dimensional integral of the Coulomb term $j_K^\alpha A_\alpha$ plus and integral of the boundary term $J^{n\beta}A_\beta\sqrt{g_4}$ over the wormhole throats and of the quantity $J^{0\beta}A_\beta\sqrt{g_4}$ over the ends of the 3-surface.

2. If the self-duality conditions generalize to $J^{n\beta} = 4\pi\alpha_K\epsilon^{n\beta\gamma\delta}J_{\gamma\delta}$ at throats and to $J^{0\beta} = 4\pi\alpha_K\epsilon^{0\beta\gamma\delta}J_{\gamma\delta}$ at the ends, the Kähler function reduces to the counterpart of Chern-Simons action evaluated at the ends and throats. It would have same value for each branch and the replacement $\hbar_0 \rightarrow r\hbar_0$ would effectively describe this. Boundary conditions would however give $1/r$ factor so that $\hbar$ would disappear from the Kähler function! The original attempt to realize quantum TGD as an almost topological QFT was in terms of Chern-Simons action but was given up. It is somewhat surprising that Kähler action gives Chern-Simons action in the vacuum sector defined as sector for which Kähler current is light-like or vanishes.

Holography encourages to ask whether also the Coulomb interaction terms could vanish. This kind of dimensional reduction would mean an enormous simplification since TGD would reduce to an almost topological QFT. The attribute "almost" would come from the fact that one has non-vanishing classical Noether charges defined by Kähler action and non-trivial quantum dynamics in $M^4$ degrees of freedom. One could also assign to space-time surfaces conserved four-momenta which is not possible in topological QFTs. For this reason the conditions guaranteeing the vanishing of Coulomb interaction term deserve a detailed analysis.

1. For the known extremals $j_K^\alpha$ either vanishes or is light-like ("massless extremals" for which weak self-duality condition does not make sense [K5] ) so that the Coulombic term vanishes identically in the gauge used. The addition of a gradient to $A$ induces terms located at the ends and wormhole throats of the space-time surface but this term must be cancelled by the other boundary terms by gauge invariance of Kähler action. This implies that the $M^4$ part of WCW metric vanishes in this case. Therefore massless extremals as such are not physically realistic: wormhole throats representing particles are needed.

2. The original naive conclusion was that since Chern-Simons action depends on $CP_2$ coordinates only, its variation with respect to Minkowski coordinates must vanish so that the WCW metric would be trivial in $M^4$ degrees of freedom. This conclusion is in conflict with quantum classical correspondence and was indeed too hasty. The point is that the allowed variations of Kähler function must respect the weak electro-magnetic duality which relates Kähler electric field depending on the induced 4-metric at 3-surface to the Kähler magnetic field. Therefore the dependence on $M^4$ coordinates creeps via a Lagrange multiplier term

$$\int \Lambda_\alpha(J^{n\alpha} - K\epsilon^{n\alpha\beta\gamma}J_{\beta\gamma})\sqrt{g_4}d^3x \quad . \tag{5.6.11}$$

The (1,1) part of second variation contributing to $M^4$ metric comes from this term.

3. This erratic conclusion about the vanishing of $M^4$ part WCW metric raised the question about how to achieve a non-trivial metric in $M^4$ degrees of freedom. The proposal was a modification of the weak form of electric-magnetic duality. Besides $CP_2$ Kähler form there would be the Kähler form assignable to the light-cone boundary reducing to that for $r_M = constant$ sphere - call it $J^1$. The generalization of the weak form of self-duality would be $J^{n\beta} = \epsilon^{n\beta\gamma\delta}K(J_{\gamma\delta} + \epsilon J^1_{\gamma\delta})$. This form implies that the boundary term gives a non-trivial contribution to the $M^4$ part of

the WCW metric even without the constraint from electric-magnetic duality. Kähler charge is not affected unless the partonic 2-surface contains the tip of $CD$ in its interior. In this case the value of Kähler charge is shifted by a topological contribution. Whether this term can survive depends on whether the resulting vacuum extremals are consistent with the basic facts about classical gravitation.

4. The Coulombic interaction term is not invariant under gauge transformations. The good news is that this might allow to find a gauge in which the Coulomb term vanishes. The vanishing condition fixing the gauge transformation $\phi$ is

$$j^{\alpha}_K\partial_{\alpha}\phi = -j^{\alpha}A_{\alpha} \ . \tag{5.6.12}$$

This differential equation can be reduced to an ordinary differential equation along the flow lines $j_K$ by using $dx^{\alpha}/dt = j^{\alpha}_K$. Global solution is obtained only if one can combine the flow parameter $t$ with three other coordinates- say those at the either end of $CD$ to form space-time coordinates. The condition is that the parameter defining the coordinate differential is proportional to the covariant form of Kähler current: $dt = \phi j_K$. This condition in turn implies $d^2t = d(\phi j_K) = d(\phi j_K) = d\phi \wedge j_K + \phi dj_K = 0$ implying $j_K \wedge dj_K = 0$ or more concretely,

$$\epsilon^{\alpha\beta\gamma\delta}j^K_{\beta}\partial_{\gamma}j^K_{\delta} \quad = \quad 0 \ . \tag{5.6.13}$$

$j_K$ is a four-dimensional counterpart of Beltrami field [B29] and could be called generalized Beltrami field.

The integrability conditions follow also from the construction of the extremals of Kähler action [K5] . The conjecture was that for the extremals the 4-dimensional Lorentz force vanishes (no dissipation): this requires $j_K \wedge J = 0$. One manner to guarantee this is the topologization of the Kähler current meaning that it is proportional to the instanton current: $j_K = \phi j_I$, where $j_I = {}^*(J \wedge A)$ is the instanton current, which is not conserved for 4-D $CP_2$ projection. The conservation of $j_K$ implies the condition $j^{\alpha}_I\partial_{\alpha}\phi = \partial_{\alpha}j^{\alpha}\phi$ and from this $\phi$ can be integrated if the integrability condition $j_I \wedge dj_I = 0$ holds true implying the same condition for $j_K$. By introducing at least 3 or $CP_2$ coordinates as space-time coordinates, one finds that the contravariant form of $j_I$ is purely topological so that the integrability condition fixes the dependence on $M^4$ coordinates and this selection is coded into the scalar function $\phi$. These functions define families of conserved currents $j^{\alpha}_K\phi$ and $j^{\alpha}_I\phi$ and could be also interpreted as conserved currents associated with the critical deformations of the space-time surface.

5. There are gauge transformations respecting the vanishing of the Coulomb term. The vanishing condition for the Coulomb term is gauge invariant only under the gauge transformations $A \rightarrow A + \nabla\phi$ for which the scalar function the integral $\int j^{\alpha}_K\partial_{\alpha}\phi$ reduces to a total divergence a giving an integral over various 3-surfaces at the ends of $CD$ and at throats vanishes. This is satisfied if the allowed gauge transformations define conserved currents

$$D_{\alpha}(j^{\alpha}\phi) \quad = \quad 0 \ . \tag{5.6.14}$$

As a consequence Coulomb term reduces to a difference of the conserved charges $Q^e_{\phi} = \int j^0\phi\sqrt{g_4}d^3x$ at the ends of the CD vanishing identically. The change of the imons type term is trivial if the total weighted Kähler magnetic flux $Q^m_{\phi} = \sum\int J\phi dA$ over wormhole throats is conserved. The existence of an infinite number of conserved weighted magnetic fluxes is in accordance with the electric-magnetic duality. How these fluxes relate to the flux Hamiltonians central for WCW geometry is not quite clear.

6. The gauge transformations respecting the reduction to almost topological QFT should have some special physical meaning. The measurement interaction term in the modified Dirac interaction corresponds to a critical deformation of the space-time sheet and is realized as an addition of a gauge part to the Kähler gauge potential of $CP_2$. It would be natural to identify this gauge transformation giving rise to a conserved charge so that the conserved charges would provide a representation for the charges associated with the infinitesimal critical deformations not affecting Kähler action. The gauge transformed Kähler potential couples to the modified Dirac equation and its effect could be visible in the value of Kähler function and therefore also in the properties of the preferred extremal. The effect on WCW metric would however vanish since $K$ would transform only by an addition of a real part of a holomorphic function. Kähler function is identified as a Dirac determinant for Chern-Simons Dirac action and the spectrum of this operator should not be invariant under these gauge transformations if this picture is correct. This is is achieved if the gauge transformation is carried only in the Dirac action corresponding to the Chern-Simons term: this assumption is motivated by the breaking of time reversal invariance induced by quantum measurements. The modification of Kähler action can be guessed to correspond just to the Chern-Simons contribution from the instanton term.

7. A reasonable looking guess for the explicit realization of the quantum classical correspondence between quantum numbers and space-time geometry is that the deformation of the preferred extremal due to the addition of the measurement interaction term is induced by a $U(1)$ gauge transformation induced by a transformation of $\delta CD \times CP_2$ generating the gauge transformation represented by $\phi$. This interpretation makes sense if the fluxes defined by $Q_\phi^m$ and corresponding Hamiltonians affect only zero modes rather than quantum fluctuating degrees of freedom.

To sum up, one could understand the basic properties of WCW metric in this framework. Effective 2-dimensionality would result from the existence of an infinite number of conserved charges in two different time directions (genuine conservation laws plus gauge fixing). The infinite-dimensional symmetric space for given values of zero modes corresponds to the Cartesian product of the WCWs associated with the partonic 2-surfaces at both ends of $CD$ and the generalized Chern-Simons term decomposes into a sum of terms from the ends giving single particle Kähler functions and to the terms from light-like wormhole throats giving interaction term between positive and negative energy parts of the state. Hence Kähler function could be calculated without any knowledge about the interior of the space-time sheets and TGD would reduce to almost topological QFT as speculated earlier. Needless to say this would have immense boost to the program of constructing WCW Kähler geometry.

### 5.6.4 Kähler action for Euclidian regions as Kähler function and Kähler action for Minkowskian regions as Morse function?

One of the nasty questions about the interpretation of Kähler action relates to the square root of the metric determinant. If one proceeds completely straightforwardly, the only reason conclusion is that the square root is imaginary in Minkowskian space-time regions so that Kähler action would be complex. The Euclidian contribution would have a natural interpretation as positive definite Kähler function but how should one interpret the imaginary Minkowskian contribution? Certainly the path integral approach to quantum field theories supports its presence. For some mysterious reason I was able to forget this nasty question and serious consideration of the obvious answer to it. Only when I worked betweeen possibile connections between TGD and Floer homology [K64] I realized that the Minkowskian contribution is an excellent candidate for Morse function whose critical points give information about WCW homology. This would fit nicely with the vision about TGD as almost topological QFT.

Euclidian regions would guarantee the convergence of the functional integral and one would have a mathematically well-defined theory. Minkowskian contribution would give the quantal interference effects and stationary phase approximation. The analog of Floer homology would represent quantum superpositions of critical points identifiable as ground states defined by the extrema of Kähler action for Minkowskian regions. Perturbative approach to quantum TGD would rely on functional integrals around the extrema of Kähler function. One would have maxima also for the Kähler function but only in the zero modes not contributing to the WCW metric.

There is a further question related to almost topological QFT character of TGD. Should one assume that the reduction to Chern-Simons terms occurs for the preferred extremals in *both* Minkowskian and Euclidian regions or only in Minkowskian regions?

1. All arguments for this have been represented for Minkowskian regions [K19] involve local light-like momentum direction which does not make sense in the Euclidian regions. This does not however kill the argument: one can have non-trivial solutions of Laplacian equation in the region of $CP_2$ bounded by wormhole throats: for $CP_2$ itself only covariantly constant right-handed neutrino represents this kind of solution and at the same time supersymmetry. In the general case solutions of Laplacian represent broken super-symmetries and should be in one-one correspondences with the solutions of the modified Dirac equation. The interpretation for the counterparts of momentum and polarization would be in terms of classical representation of color quantum numbers.

2. If the reduction occurs in Euclidian regions, it gives in the case of $CP_2$ two 3-D terms corresponding to two 3-D gluing regions for three coordinate patches needed to define coordinates and spinor connection for $CP_2$ so that one would have two Chern-Simons terms. I have earlier claimed that without any other contributions the first term would be identical with that from Minkowskian region apart from imaginary unit and different coefficient. This statement is wrong since the space-like parts of the corresponding 3-surfaces are discjoint for Euclidian and Minkowskian regions.

3. There is also another very delicate issue involved. Quantum classical correspondence requires that the quantum numbers of partonic states must be coded to the space-time geometry, and this is achieved by adding to the action a measurement interaction term which reduces to what is almost a gauge term present only in Chern-Simons-Dirac equation but not at space-time interior [K19]. This term would represent a coupling to Poincare quantum numbers at the Minkowskian side and to color and electro-weak quantum numbers at $CP_2$ side. Therefore the net Chern-Simons contributions would be different.

4. There is also a very beautiful argument stating that Dirac determinant for Chern-Simons-Dirac action equals to Kähler function, which would be lost if Euclidian regions would not obey holography. The argument obviously generalizes and applies to both Morse and Kähler function which are definitely not proportional to each other.

The Minkowskian contribution of Kähler action is imaginary due to the negative of the metric determinant and gives a phase factor to vacuum functional reducing to Chern-Simons terms at wormhole throats. Ground state degeneracy due to the possibility of having both signs for Minkowskian contribution to the exponent of vacuum functional provides a general view about the description of CP breaking in TGD framework.

1. In TGD framework path integral is replaced by inner product involving integral over WCV. The vacuum functional and its conjugate are associated with the states in the inner product so that the phases of vacuum functionals cancel if only one sign for the phase is allowed. Minkowskian contribution would have no physical significance. This of course cannot be the case. The ground state is actually degenerate corresponding to the phase factor and its complex conjugate since $\sqrt{g}$ can have two signs in Minkowskian regions. Therefore the inner products between states associated with the two ground states define $2 \times 2$ matrix and non-diagonal elements contain interference terms due to the presence of the phase factor. At the limit of full $CP_2$ type vacuum extremal the two ground states would reduce to each other and the determinant of the matrix would vanish.

2. A small mixing of the two ground states would give rise to CP breaking and the first principle description of CP breaking in systems like $K - \overline{K}$ and of CKM matrix should reduce to this mixing. $K^0$ mesons would be CP even and odd states in the first approximation and correspond to the sum and difference of the ground states. Small mixing would be present having exponential sensitivity to the actions of $CP_2$ type extremals representing wormhole throats. This might allow to understand qualitatively why the mixing is about 50 times larger than expected for $B^0$ mesons.

3. There is a strong temptation to assign the two ground states with two possible arrows of geometric time. At the level of M-matrix the two arrows would correspond to state preparation at either upper or lower boundary of CD. Do long- and shortlived neutral K mesons correspond to almost fifty-fifty orthogonal superpositions for the two arrow of geometric time or almost completely to a fixed arrow of time induced by environment? Is the dominant part of the arrow same for both or is it opposite for long and short-lived neutral measons? Different lifetimes would suggest that the arrow must be the same and apart from small leakage that induced by environment. CP breaking would be induced by the fact that CP is performed only $K^0$ but not for the environment in the construction of states. One can probably imagine also alternative interpretations.

### 5.6.5 A general solution ansatz based on almost topological QFT property

The basic vision behind the ansatz is the reduction of quantum TGD to almost topological field theory. This requires that the flow parameters associated with the flow lines of isometry currents and Kähler current extend to global coordinates. This leads to integrability conditions implying generalized Beltrami flow and Kähler action for the preferred extremals reduces to Chern-Simons action when weak electro-weak duality is applied as boundary conditions. The strongest form of the hydrodynamical interpretation requires that all conserved currents are parallel to Kähler current. In the more general case one would have several hydrodynamic flows. Also the braidings (several of them for the most general ansatz) assigned with the light-like 3-surfaces are naturally defined by the flow lines of conserved currents. The independent behavior of particles at different flow lines can be seen as a realization of the complete integrability of the theory. In free quantum field theories on mass shell Fourier components are in a similar role but the geometric interpretation in terms of flow is of course lacking. This picture should generalize also to the solution of the modified Dirac equation.

#### Basic field equations

Consider first the equations at general level.

1. The breaking of the Poincare symmetry due to the presence of monopole field occurs and leads to the isometry group $T\times SO(3)\times SU(3)$ corresponding to time translations, rotations, and color group. The Cartan algebra is four-dimensional and field equations reduce to the conservation laws of energy $E$, angular momentum $J$, color isospin $I_3$, and color hypercharge $Y$.

2. Quite generally, one can write the field equations as conservation laws for $I, J, I_3$, and $Y$.

$$D_\alpha\left[D_\beta(J^{\alpha\beta}H_A) - j^\alpha_K H^A + T^{\alpha\beta}j^l_A h_{kl}\partial_\beta h^l\right] \quad = \quad 0 \ . \tag{5.6.15}$$

   The first term gives a contraction of the symmetric Ricci tensor with antisymmetric Kähler form and vanishes so that one has

$$D_\alpha\left[j^\alpha_K H^A - T^{\alpha\beta}j^k_A h_{kl}\partial_\beta h^l\right] \quad = \quad 0 \ . \tag{5.6.16}$$

   For energy one has $H_A=1$ and energy current associated with the flow lines is proportional to the Kähler current. Its divergence vanishes identically.

3. One can express the divergence of the term involving energy momentum tensor as as sum of terms involving $j^\alpha_K J_{\alpha\beta}$ and contraction of second fundamental form with energy momentum tensor so that one obtains

$$j^\alpha_K D_\alpha H^A \quad = \quad j^\alpha_K J_\alpha{}^\beta j^A_\beta + T^{\alpha\beta}H^k_{\alpha\beta}j^A_k \ . \tag{5.6.17}$$

### Hydrodynamical solution ansatz

The characteristic feature of the solution ansatz would be the reduction of the dynamics to hydrodynamics analogous to that for a continuous distribution of particles initially at the end of $X^3$ of the light-like 3-surface moving along flow lines defined by currents $j_A$ satisfying the integrability condition $j_A \wedge dj_A = 0$. Field theory would reduce effectively to particle mechanics along flow lines with conserved charges defined by various isometry currents. The strongest condition is that all isometry currents $j_A$ and also Kähler current $j_K$ are proportional to the same current $j$. The more general option corresponds to multi-hydrodynamics.

Conserved currents are analogous to hydrodynamical currents in the sense that the flow parameter along flow lines extends to a global space-time coordinate. The conserved current is proportional to the gradient $\nabla\Phi$ of the coordinate varying along the flow lines: $J = \Psi\nabla\Phi$ and by a proper choice of $\Psi$ one can allow to have conservation. The initial values of $\Psi$ and $\Phi$ can be selected freely along the flow lines beginning from either the end of the space-time surface or from wormhole throats.

If one requires hydrodynamics also for Chern-Simons action (effective 2-dimensionality is required for preferred extremals), the initial values of scalar functions can be chosen freely only at the partonic 2-surfaces. The freedom to chose the initial values of the charges conserved along flow lines at the partonic 2-surfaces means the existence of an infinite number of conserved charges so that the theory would be integrable and even in two different coordinate directions. The basic difference as compared to ordinary conservation laws is that the conserved currents are parallel and their flow parameter extends to a global coordinate.

1. The most general assumption is that the conserved isometry currents

$$J_A^{\alpha} = j_K^{\alpha} H^A - T^{\alpha\beta} j_A^k h_{kl} \partial_\beta h^l \qquad (5.6.18)$$

and Kähler current are integrable in the sense that $J_A \wedge J_A = 0$ and $j_K \wedge j_K = 0$ hold true. One could imagine the possibility that the currents are not parallel.

2. The integrability condition $dJ_A \wedge J_A = 0$ is satisfied if one one has

$$J_A = \Psi_A d\Phi_A \ . \qquad (5.6.19)$$

The conservation of $J_A$ gives

$$d * (\Psi_A d\Phi_A) = 0 \ . \qquad (5.6.20)$$

This would mean separate hydrodynamics for each of the currents involved. In principle there is not need to assume any further conditions and one can imagine infinite basis of scalar function pairs $(\Psi_A, \Phi_A)$ since criticality implies infinite number deformations implying conserved Noether currents.

3. The conservation condition reduces to d'Alembert equation in the induced metric if one assumes that $\nabla\Psi_A$ is orthogonal with every $d\Phi_A$.

$$d * d\Phi_A = 0 \ , \ d\Psi_A \cdot d\Phi_A = 0 \ . \qquad (5.6.21)$$

Taking $x = \Phi_A$ as a coordinate the orthogonality condition states $g^{xj}\partial_j\Psi_A = 0$ and in the general case one cannot solve the condition by simply assuming that $\Psi_A$ depends on the coordinates transversal to $\Phi_A$ only. These conditions bring in mind $p \cdot p = 0$ and $p \cdot e$ condition for massless modes of Maxwell field having fixed momentum and polarization. $d\Phi_A$ would correspond to $p$

and $d\Psi_A$ to polarization. The condition that each isometry current corresponds its own pair $(\Psi_A, \Phi_A)$ would mean that each isometry current corresponds to independent light-like momentum and polarization. Ordinary free quantum field theory would support this view whereas hydrodynamics and QFT limit of TGD would support single flow.

These are the most general hydrodynamical conditions that one can assume. One can consider also more restricted scenarios.

1. The strongest ansatz is inspired by the hydrodynamical picture in which all conserved isometry charges flow along same flow lines so that one would have

$$J_A = \Psi_A d\Phi \ . \tag{5.6.22}$$

   In this case same $\Phi$ would satisfy simultaneously the d'Alembert type equations.

$$d*d\Phi = 0 \ , \ d\Psi_A \cdot d\Phi = 0. \tag{5.6.23}$$

   This would mean that the massless modes associated with isometry currents move in parallel manner but can have different polarizations. The spinor modes associated with light-light like 3-surfaces carry parallel four-momenta, which suggest that this option is correct. This allows a very general family of solutions and one can have a complete 3-dimensional basis of functions $\Psi_A$ with gradient orthogonal to $d\Phi$.

2. Isometry invariance under $T \times SO(3) \times SU(3)$ allows to consider the possibility that one has

$$J_A = k_A \Psi_A d\Phi_{G(A)} \ , \ d*(d\Phi_G(A)) = 0 \ , \ d\Psi_A \cdot d\Phi_G(A)) = 0 \ . \tag{5.6.24}$$

   where $G(A)$ is $T$ for energy current, $SO(3)$ for angular momentum currents and $SU(3)$ for color currents. Energy would thus flow along its own flux lines, angular momentum along its own flow lines, and color quantum numbers along their own flow lines. For instance, color currents would differ from each other only by a numerical constant. The replacement of $\Psi_A$ with $\Psi_{G(A)}$ would be too strong a condition since Killing vector fields are not related by a constant factor.

To sum up, the most general option is that each conserved current $J_A$ defines its own integrable flow lines defined by the scalar function pair $(\Psi_A, \Phi_A)$. A complete basis of scalar functions satisfying the d'Alembert type equation guaranteeing current conservation could be imagined with restrictions coming from the effective 2-dimensionality reducing the scalar function basis effectively to the partonic 2-surface. The diametrically opposite option corresponds to the basis obtained by assuming that only single $\Phi$ is involved.

The proposed solution ansatz can be compared to the earlier ansatz [K24] stating that Kähler current is topologized in the sense that for $D(CP_2) = 3$ it is proportional to the identically conserved instanton current (so that 4-D Lorentz force vanishes) and vanishes for $D(CP_2) = 4$ (Maxwell phase). This hypothesis requires that instanton current is Beltrami field for $D(CP_2) = 3$. In the recent case the assumption that also instanton current satisfies the Beltrami hypothesis in strong sense (single function $\Phi$) generalizes the topologization hypothesis for $D(CP_2) = 3$. As a matter fact, the topologization hypothesis applies to isometry currents also for $D(CP_2) = 4$ although instanton current is not conserved anymore.

**Can one require the extremal property in the case of Chern-Simons action?**

Effective 2-dimensionality is achieved if the ends and wormhole throats are extremals of Chern-Simons action. The strongest condition would be that space-time surfaces allow orthogonal slicings by 3-surfaces which are extremals of Chern-Simons action.

Also in this case one can require that the flow parameter associated with the flow lines of the isometry currents extends to a global coordinate. Kähler magnetic field $B = *J$ defines a conserved current so that all conserved currents would flow along the field lines of $B$ and one would have 3-D Beltrami flow. Note that in magnetohydrodynamics the standard assumption is that currents flow along the field lines of the magnetic field.

For wormhole throats light-likeness causes some complications since the induced metric is degenerate and the contravariant metric must be restricted to the complement of the light-like direction. This means that d'Alembert equation reduces to 2-dimensional Laplace equation. For space-like 3-surfaces one obtains the counterpart of Laplace equation with partonic 2-surfaces serving as sources. The interpretation in terms of analogs of Coulomb potentials created by 2-D charge distributions would be natural.

### 5.6.6 Hydrodynamic picture in fermionic sector

Super-symmetry inspires the conjecture that the hydrodynamical picture applies also to the solutions of the modified Dirac equation.

**4-dimensional modified Dirac equation and hydrodynamical picture**

Consider first the solutions of of the induced spinor field in the interior of space-time surface.

1. The local inner products of the modes of the induced spinor fields define conserved currents

$$\begin{aligned} D_\alpha J^\alpha_{mn} &= 0 \ , \\ J^\alpha_{mn} &= \overline{u}_m \hat{\Gamma}^\alpha u_n \ , \\ \hat{\Gamma}^\alpha &= \frac{\partial L_K}{\partial(\partial_\alpha h^k)}\Gamma_k \ . \end{aligned} \tag{5.6.25}$$

   The conjecture is that the flow parameters of also these currents extend to a global coordinate so that one would have in the completely general case the condition

$$\begin{aligned} J^\alpha_{mn} &= \Phi_{mn} d\Psi_{mn} \ , \\ d*(d\Phi_{mn}) &= 0 \ , \ \nabla\Psi_{mn}\cdot\Phi_{mn} = 0 \ . \end{aligned} \tag{5.6.26}$$

   The condition $\Phi_{mn} = \Phi$ would mean that the massless modes propagate in parallel manner and along the flow lines of Kähler current. The conservation condition along the flow line implies tht the current component $J_{mn}$ is constant along it. Everything would reduce to initial values at the ends of the space-time sheet boundaries of $CD$ and 3-D modified Dirac equation would reduce everything to initial values at partonic 2-surfaces.

2. One might hope that the conservation of these super currents for all modes is equivalent with the modified Dirac equation. The modes $u_n$ appearing in $\Psi$ in quantized theory would be kind of "square roots" of the basis $\Phi_{mn}$ and the challenge would be to deduce the modes from the conservation laws.

3. The quantization of the induced spinor field in 4-D sense would be fixed by those at 3-D space-like ends by the fact that the oscillator operators are carried along the flow lines as such so that the anti-commutator of the induced spinor field at the opposite ends of the flow lines at the light-like boundaries of $CD$ is in principle fixed by the anti-commutations at the either end. The anti-commutations at 3-D surfaces cannot be fixed freely since one has 3-D Chern-Simons flow reducing the anti-commutations to those at partonic 2-surfaces.

The following argument suggests that induced spinor fields are in a suitable gauge simply constant along the flow lines of the Kähler current just as massless spinor modes are constant along the geodesic in the direction of momentum.

1. The modified gamma matrices are of form $T^{\alpha}_k\Gamma^k$, $T^{\alpha}_k = \partial L_K/\partial(\partial_\alpha h^k)$. The H-vectors $T^{\alpha}_k$ can be expressed as linear combinations of a subset of Killing vector fields $j^k_A$ spanning the tangent space of $H$. For $CP_2$ the natural choice are the 4 Lie-algebra generators in the complement of $U(2)$ sub-algebra. For $CD$ one can used generator time translation and three generators of rotation group SO(3). The completeness of the basis defined by the subset of Killing vector fields gives completeness relation $h^k_l = j^{Ak}j_{Ak}$. This implies $T^{\alpha k} = T^{\alpha k}j^A_k j^k_A = T^{\alpha A}j^k_A$. One can defined gamma matrices $\Gamma_A$ as $\Gamma_k j^k_A$ to get $T^{\alpha}_k\Gamma^k = T^{\alpha A}\Gamma_A$.

2. This together with the condition that all isometry currents are proportional to the Kähler current (or if this vanishes to same conserved current- say energy current) satisfying Beltrami flow property implies that one can reduce the modified Dirac equation to an ordinary differential equation along flow lines. The quantities $T^{tA}$ are constant along the flow lines and one obtains

$$T^{tA}j_A D_t\Psi = 0 . \tag{5.6.27}$$

By choosing the gauge suitably the spinors are just constant along flow lines so that the spinor basis reduces by effective 2-dimensionality to a complete spinor basis at partonic 2-surfaces.

**Generalized eigen modes for the modified Chern-Simons Dirac equation and hydrodynamical picture**

Hydrodynamical picture helps to understand also the construction of generalized eigen modes of 3-D Chern-Simons Dirac equation.

*The general form of generalized eigenvalue equation for Chern-Simons Dirac action*

Consider first the the general form and interpretation of the generalized eigenvalue equation assigned with the modified Dirac equation for Chern-Simons action [K9] . This is of course only an approximation since an additional contribution to the modified gamma matrices from the Lagrangian multiplier term guaranteing the weak form of electric-magnetic duality must be included.

1. The modified Dirac equation for $\Psi$ is consistent with that for its conjugate if the coefficient of the instanton term is real and one uses the Dirac action $\overline{\Psi}(D^{\rightarrow} - D^{\leftarrow})\Psi$ giving modified Dirac equation as

$$D_{C-S}\Psi + \frac{1}{2}(D_\alpha\hat{\Gamma}^{\alpha}_{C-S})\Psi = 0 . \tag{5.6.28}$$

As noticed, the divergence $D_\alpha\hat{\Gamma}^{\alpha}_{C-S}$ does not contain second derivatives in the case of Chern-Simons action. In the case of Kähler action they occur unless field equations equivalent with the vanishing of the divergence term are satisfied. The extremals of Chern-Simons action provide a natural manner to define effective 2-dimensionality.

Also the fermionic current is conserved in this case, which conforms with the idea that fermions flow along the light-like 3-surfaces. If one uses the action $\overline{\Psi}D^{\rightarrow}\Psi$, $\overline{\Psi}$ does not satisfy the Dirac equation following from the variational principle and fermion current is not conserved.

2. The generalized eigen modes of $D_{C-S}$ should be such that one obtains the counterpart of Dirac propagator which is purely algebraic and does not therefore depend on the coordinates of the throat. This is satisfied if the generalized eigenvalues are expressible in terms of covariantly constant combinations of gamma matrices and here only $M^4$ gamma matrices are possible. Therefore the eigenvalue equation would read as

$$D\Psi = \lambda^k\gamma_k\Psi \ , \ D = D_{C-S} + \frac{1}{2}D_\alpha\hat{\Gamma}^\alpha_{C-S} \ , \ D_{C-S} = \hat{\Gamma}^\alpha_{C-S}D_\alpha \ . \tag{5.6.29}$$

Here the covariant derivatives $D_\alpha$ contain the measurement interaction term as an apparent gauge term. For extremals one has

$$D = D_{C-S} \ . \tag{5.6.30}$$

Covariant constancy allows to take the square of this equation and one has

$$(D^2 + [D, \lambda^k\gamma_k])\Psi = \lambda^k\lambda_k\Psi \ . \tag{5.6.31}$$

The commutator term is analogous to magnetic moment interaction.

3. The generalized eigenvalues correspond to $\lambda = \sqrt{\lambda^k\lambda_k}$ and Dirac determinant is defined as a product of the eigenvalues and conjecture to give the exponent of Kähler action reducing to Chern-Simons term. $\lambda$ is completely analogous to mass. $\lambda_k$ cannot be however interpreted as ordinary four-momentum: for instance, number theoretic arguments suggest that $\lambda_k$ must be restricted to the preferred plane $M^2 \subset M^4$ interpreted as a commuting hyper-complex plane of complexified quaternions. For incoming lines this mass would vanish so that all incoming particles irrespective their actual quantum numbers would be massless in this sense and the propagator is indeed that for a massless particle. Note that the eigen-modes define the boundary values for the solutions of $D_K\Psi = 0$ so that the values of $\lambda$ indeed define the counterpart of the momentum space.

   This transmutation of massive particles to effectively massless ones might make possible the application of the twistor formalism as such in TGD framework [K61] . $N = 4$ SUSY is one of the very few gauge theory which might be UV finite but it is definitely unphysical due to the masslessness of the basic quanta. Could the resolution of the interpretational problems be that the four-momenta appearing in this theory do not directly correspond to the observed four-momenta?

*2. Inclusion of the constraint term*

As already noticed one must include also the constraint term due to the weak form of electric-magnetic duality and this changes somewhat the above simple picture.

1. At the 3-dimensional ends of the space-time sheet and at wormhole throats the 3-dimensionality allows to introduce a coordinate varying along the flow lines of Kähler magnetic field $B = *J$. In this case the integrability conditions state that the flow is Beltrami flow. Note that the value of $B^\alpha$ along the flow line defining magnetic flux appearing in anti-commutation relations is constant. This suggests that the generalized eigenvalue equation for the Chern-Simons action reduces to a collection of ordinary apparently independent differential equations associated with the flow lines beginning from the partonic 2-surface. This indeed happens when the $CP_2$ projection is 2-dimensional. In this case it however seems that the basis $u_n$ is not of much help.

2. The conclusion is wrong: the variations of Chern-Simons action are subject to the constraint that electric-magnetic duality holds true expressible in terms of Lagrange multiplier term

$$\int \Lambda_\alpha(J^{n\alpha} - K\epsilon^{n\alpha\beta\gamma}J_{\beta\gamma})\sqrt{g_4}d^3x \ . \tag{5.6.32}$$

This gives a constraint force to the field equations and also a dependence on the induced 4-metric so that one has only almost topological QFT. This term also guarantees the $M^4$ part of WCW Kähler metric is non-trivial. The condition that the ends of space-time sheet and wormhole throats are extrema of Chern-Simons action subject to the electric-magnetic duality constraint is strongly suggested by the effective 2-dimensionality. Without the constraint term Chern-Simons action would vanish for its extremals so that Kähler function would be identically zero.

This term implies also an additional contribution to the modified gamma matrices besides the contribution coming from Chern-Simons action so tht the first guess for the modified Dirac operator would not be quite correct. This contribution is of exactly of the same general form as the contribution for any general general coordinate invariant action. The dependence of the induced metric on $M^4$ degrees of freedom guarantees that also $M^4$ gamma matrices are present. In the following this term will not be considered.

3. When the contribution of the constraint term to the modifield gamma matrices is neglected, the explicit expression of the modified Dirac operator $D_{C-S}$ associated with the Chern-Simons term is given by

$$\begin{aligned}
D &= \hat{\Gamma}^\mu D_\mu + \frac{1}{2} D_\mu \hat{\Gamma}^\mu \ , \\
\hat{\Gamma}^\mu &= \frac{\partial L_{C-S}}{\partial_\mu h^k}\Gamma_k = \epsilon^{\mu\alpha\beta}\left[2J_{kl}\partial_\alpha h^l A_\beta + J_{\alpha\beta}A_k\right]\Gamma^k D_\mu \ , \\
D_\mu \hat{\Gamma}^\mu &= B^\alpha_K (J_{k\alpha} + \partial_\alpha A_k) \ , \\
B^\alpha_K &= \epsilon^{\alpha\beta\gamma} J_{\beta\gamma} \ , \ J_{k\alpha} = J_{kl}\partial_\alpha s^l \ , \ \hat{\epsilon}^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma}\sqrt{g_3} \ .
\end{aligned} \tag{5.6.33}$$

For the extremals of Chern-Simons action one has $D_\alpha \hat{\Gamma}^\alpha = 0$. Analogous condition holds true when the constraing contriabution to the modified gamma matrices is added.

*3. Generalized eigenvalue equation for Chern-Simons Dirac action*

Consider now the Chern-Simons Dirac equation in more detail assuming that the inclusion of the constraint contribution to the modified gamma matrices does not induce any complications. Assume also extremal property for Chern-Simons action with constraint term and Beltrami flow property.

1. For the extremals the Chern-Simons Dirac operator (constraint term not included) reduces to a one-dimensional Dirac operator

$$D_{C-S} = \hat{\epsilon}^{r\alpha\beta}\left[2J_{k\alpha}A_\beta + J_{\alpha\beta}A_k\right]\Gamma^k D_r \ . \tag{5.6.34}$$

Constraint term implies only a modification of the modified gamma matrices but the form of the operator remains otherwise same when extrema are in question so that one has $D_\alpha \hat{\Gamma^\alpha} = 0$.

2. For the extremals of Chern-Simons action the general solution of the modified Chern-Simons Dirac equation ($\lambda^k = 0$) is covariantly constant with respect to the coordinate $r$:

$$D_r \Psi = 0 \ . \tag{5.6.35}$$

The solution to this condition can be written immediately in terms of a non-integrable phase factor $Pexp(i\int A_r dr)$, where integration is along curve with constant transversal coordinates. If $\hat{\Gamma^v}$ is light-like vector field also $\hat{\Gamma^v}\Psi_0$ defines a solution of $D_{C-S}$. This solution corresponds to a zero mode for $D_{C-S}$ and does not contribute to the Dirac determinant (suggested to give rise to the exponent of Kähler function identified as Kähler action). Note that the dependence of these solutions on transversal coordinates of $X_l^3$ is arbitrary which conforms with the hydrodynamic picture. The solutions of Chern-Simons-Dirac are obtained by similar integration procedure also when extremals are not in question.

The formal solution associated with a general eigenvalue $\lambda$ can be constructed by integrating the eigenvalue equation separately along all coordinate curves. This makes sense if $r$ indeed assigned to possibly light-like flow lines of $B^\alpha$ or more general Beltrami field possible induced by the constraint term. There are very strong consistency conditions coming from the conditions that $\Psi$ in the interior is constant along the flow lines of Kähler current and continuous at the ends and throats (call them collectively boundaries), where $\Psi$ has a non-trivial variation along the flow lines of $B^\alpha$.

1. This makes sense only if the flow lines of the Kähler current are transversal to the boundaries so that the spinor modes at boundaries dictate the modes of the spinor field in the interior. Effective 2-dimensionality means that the spinor modes in the interior can be calculated either by starting from the throats or from the ends so that the data at either upper of lower partonic 2-surfaces dictates everything in accordance with zero energy ontology.

2. This gives an infinite number of commuting diagrams stating that the flow-line time evolution along flow lines along wormhole throats from lower partonic 2-surface to the upper one is equivalent with the flow-line time evolution along the lower end of space-time surface to interior, then along interior to the upper end of the space-time surface and then back to the upper partonic 2-surface. If the space-time surface allows a slicing by partonic 2-surfaces these conditions can be assumed for any pair of partonic 2-surfaces connected by Chern-Simons flow evolution.

3. Since the time evolution along interior keeps the spinor field as constant in the proper gauge and since the flow evolutions at the lower and upper ends are in a reverse direction, there is a strong atemptation to assume that the spinor field at the ends of the of the flow lines of Kähler magnetic field are identical apart from a gauge transformation. This leads to a particle-in-box quantizaton of the values of the pseudo-mass (periodic boundary conditions). These conditions will be assumed in the sequel.

These assumptions lead to the following picture about the generalized eigen modes.

1. By choosing the gauge so that covariant derivative reduces to ordinary derivative and using the constancy of $\hat{\Gamma}^r$, the solution of the generalized eigenvalue equation can be written as

$$\begin{aligned} \Psi &= exp(iL(r)\hat{\Gamma^r}\lambda^k\Gamma_k)\Psi_0 \ , \\ L(r) &= \int_0^r \frac{1}{\sqrt{\hat{g}^{rr}}}dr \ . \end{aligned} \tag{5.6.36}$$

   $L(r)$ can be regarded as the along flux line as defined by the effective metric defined by modified gamma matrices. If $\lambda_k$ is linear combination of $\Gamma^0$ and $\Gamma^{r_M}$ it anti-commutes with $\Gamma^r$ which contains only $CP_2$ gamma matrices so that the pseudo-momentum is a priori arbitrary.

2. When the constraint term taking care of the electric-magnetric duality is included, also $M^4$ gamma matrices are present. If they are in the orthogonal complement of a preferred plane $M^2 \subset M^4$, anti-commutativity is achieved. This assumption cannot be fully justified yet but conforms with the general physical vision. There is an obvious analogy with the condition that polarizations are in a plane orthogonal to $M^2$. The condition indeed states that only transversal deformations define quantum fluctuating WCW degrees of freedom contributing to the WCW Kähler metric. In $M^8-H$ duality the preferred plane $M^2$ is interpreted as a hyper-complex plane belonging to the tangent space of the space-time surface and defines the plane of non-physical polarizations. Also a generalization of this plane to an integrable distribution of planes $M^2(x)$ has been proposed and one must consider also now the possibility of a varying plane $M^2(x)$ for the pseudo-momenta. The scalar function $\Phi$ appearing in the general solution ansatz for the field equations satisfies massless d'Alembert equation and its gradient defines a local light-like

direction at space-time-level and hence a 2-D plane of the tangent space. Maybe the projection of this plane to $M^4$ could define the preferred $M^2$. The minimum condition is that these planes are defined only at the ends of space-time surface and at wormhole throats.

3. If one accepts this hypothesis, one can write

$$\begin{aligned} \Psi &= \left[cos(L(r)\lambda) + isin(\lambda(r))\hat{\Gamma^r}\lambda^k\Gamma_k)\right]\Psi_0 \ , \\ \lambda &= \sqrt{\lambda_k\lambda^k} \ . \end{aligned} \tag{5.6.37}$$

4. Boundary conditions should fix the spectrum of masses. If the the flow lines of Kähler current coincide with the flow lines of Kähler magnetic field or more general Beltrami current at wormhole throats one ends up with difficulties since the induced spinor fields must be constant along flow lines and only trivial eigenvalues are possible. Hence it seems that the two Beltrami fields must be transversal. This requires that at the partonic 2-surfaces the value of the induced spinor mode in the interior coincides with its value at the throat. Since the induced spinor fields in interior are constant along flow lines, one must have

$$exp(i\lambda L(_{max})) = 1 \ . \tag{5.6.38}$$

This implies that one has essentially particle in a box with size defined by the effective metric

$$\lambda_n = \frac{n2\pi}{L(r_{max})} \ . \tag{5.6.39}$$

5. This condition cannot however hold true simultaneously for all points of the partonic 2-surfaces since $L(r_{max})$ depends on the point of the surface. In the most general case one can consider only a subset consisting of the points for which the values of $L(r_{max})$ are rational multiples of the value of $L(r_{max})$ at one of the points -call it $L_0$. This implies the notion of number theoretical braid. Induced spinor fields are localized to the points of the braid defined by the flow lines of the Kähler magnetic field (or equivalently, any conserved current- this resolves the longstanding issue about the identification of number theoretical braids). The number of the included points depends on measurement resolution characterized somehow by the number rationals which are allowed. Only finite number of harmonics and sub-harmonics of $L_0$ are possible so that for integer multiples the number of points is finite. If $n_{max}L_0$ and $L_0/n_{min}$ are the largest and smallest lengths involved, one can argue that the rationals $n_{max}/n$, $n = 1, ..., n_{max}$ and $n/n_{min}$, $n = 1, ..., n_{min}$ are the natural ones.

6. One can consider also algebraic extensions for which $L_0$ is scaled from its reference value by an algebraic number so that the mass scale $m$ must be scaled up in similar manner. The spectrum comes also now in integer multiples. p-Adic mass calculations predicts mass scales to the inverses of square roots of prime and this raises the expectation that $\sqrt{n}$ harmonics and sub-harmonics of $L_0$ might be necessary. Notice however that pseudo-momentum spectrum is in question so that this argument is on shaky grounds.

There is also the question about the allowed values of $(\lambda_0, \lambda_3)$ for a given value of $\lambda$. This issue will be discussed in the next section devoted to the attempt to calculate the Dirac determinant assignable to this spectrum: suffice it to say that integer valued spectrum is the first guess implying that the pseudo-momenta satisfy $n_0^2 - n_3^2 = n^2$ and therefore correspond to Pythagorean triangles. What is remarkable that the notion of number theoretic braid pops up automatically from the Beltrami flow hypothesis.

## 5.7 How to define Dirac determinant?

The basic challenge is to define Dirac determinant hoped to give rise to the exponent of Kähler action associated with the preferred extremal. The reduction to almost topological QFT gives this kind of expression in terms of Chern-Simons action and one might hope of obtaining even more concrete expression from the Chern-Simons Dirac determinant. The calculation of the previous section allowed to calculate the most general spectrum of the modified Dirac operator. If the number of the eigenvalues is infinite as the naive expectation is then Dirac determinant diverges if calculated as the product of the eigenvalues and one must calculate it by using some kind of regularization procedure. Zeta function regularization is the natural manner to do this.

The following arguments however lead to a concrete vision how the regularization could be avoided and a connection with infinite primes. In fact, the manifestly finite option and the option involving zeta function regularization give Kähler functions differing only by a scaling factor and only the manifestly finite option satisfies number theoretical constraints coming from p-adicization. An explicit expression for the Dirac determinant in terms of geometric data of the orbit of the partonic 2-surface emerges.

Arithmetic quantum field theory defined by infinite emerges naturally. The lines of the generalized Feynman graphs are characterized by infinite primes and the selection rules correlating the geometries of the lines of the generalized Feynman graphs corresponds to the conservation of the sum of number theoretic momenta $log(p_i)$ assignable to sub-braids corresponding to different primes $p_i$ assignable to the orbit of parton. This conforms with the vision that infinite primes indeed characterize the geometry of light-like 3-surfaces and therefore also of space-time sheets. The eigenvalues of the modified Dirac operator are proportional $1/\sqrt{p_i}$ where $p_i$ are the primes appearing in the definition of the p-adic prime and the interpretation as analogs of Higgs vacuum expectation values makes sense and is consistent with p-adic length scale hypothesis and p-adic mass calculations. It must be emphasized that all this is essentially due to single basic hypothesis, namely the reduction of quantum TGD to almost topological QFT guaranteed by the Beltrami ansatz for field equations and by the weak form of electric-magnetic duality.

### 5.7.1 Dirac determinant when the number of eigenvalues is infinite

At first sight the general spectrum looks the only reasonable possibility but if the eigenvalues correlate with the geometry of the partonic surface as quantum classical correspondence suggests, this conclusion might be wrong. The original hope was the number of eigenvalues would be finite so that also determinant would be finite automatically. There were some justifications for this hope in the definition of Dirac determinant based on the dimensional reduction of $D_K$ as $D_K = D_{K,3} + D_1$ and the identification of the generalized eigenvalues as those assigned to $D_{K,3}$ as analogs of energy eigenvalues assignable to the light-like 3-surface. It will be found that number theoretic input could allow to achieve a manifest finiteness in the case of $D_{C-S}$ and that this option is the only possible one if number theoretic universality is required.

If there are no constraints on the eigenvalue spectrum of $D_{C-S}$ for a given partonic orbit, the naive definition of the determinant gives an infinite result and one must define Dirac determinant using $\zeta$ function regularization implying that Kähler function reduces to the derivative of the zeta function $\zeta_D(s)$ -call it Dirac Zeta- associated with the eigenvalue spectrum.

Consider now the situation when the number of eigenvalues is infinite.

1. In this kind of situation zeta function regularization is the standard manner to define the Dirac determinant. What one does is to assign zeta function to the spectrum- let us call it Dirac zeta function and denote by $\zeta_D(s)$- as

$$\zeta_D(s) \quad = \quad \sum_k \lambda_k^{-s} \ . \tag{5.7.1}$$

   If the eigenvalue $\lambda_k$ has degeneracy $g_k$ it appears $g_k$ times in the sum. In the case of harmonic oscillator one obtains Riemann zeta for which sum representation converges only for $Re(s) \geq 1$. Riemann zeta can be however analytically continued to the entire complex plane and the idea is that this can be done also in the more general case.

2. By the basic conjecture Kähler function corresponds to the logarithm of the Dirac determinant and equals to the sum of the logarithms of the eigenvalues

$$K = log(\prod \lambda_k) = -\frac{d\zeta_D}{ds}_{|s=0} . \quad (5.7.2)$$

The expression on the left hand side diverges if taken as such but the expression on the right had side based on the analytical continuation of the zeta function is completely well-defined and finite quantity. Note that the replacement of eigenvalues $\lambda_k$ by their powers $\lambda_k^n$ -or equivalently the increase of the degeneracy by a factor $n$ - brings in only a factor $n$ to $K$: $K \rightarrow nK$.

3. Dirac determinant involves in the minimal situation only the integer multiples of pseudo-mass scale $\lambda = 2\pi/L_{min}$. One can consider also rational and even algebraic multiples $qL_{min} < L_{max}$, $q \geq 1$, of $L_{min}$ so that one would have several integer spectra simultaneously corresponding to different braids. Here $L_{min}$ and $L_{max}$ are the extrema of the braid strand length determined in terms of the effective metric as $L = \int (\hat{g}^{rr})^{-1/2}dr$. The question what multiples are involved will be needed later.

4. Each rational or algebraic multiple of $L_{min}$ gives to the zeta function a contribution which is of same form so that one has

$$\zeta_D = \sum_q \zeta(log(qx)s) \ , \ x = \frac{L_{min}}{R} \ , \ 1 \leq q < \frac{L_{max}}{L_{min}} . \quad (5.7.3)$$

Kähler function can be expressed as

$$K = \sum_n log(\lambda_n) = -\frac{d\zeta_D(s)}{ds} = -\sum_q log(qx)\frac{d\zeta(s)}{ds}_{|s=0} \ , \ x = \frac{L_{min}}{R} . \quad (5.7.4)$$

What is remarkable that the number theoretical details of $\zeta_D$ determine only the overall scaling factor of Kähler function and thus the value of Kähler coupling strength, which would be purely number theoretically determined if the hypothesis about the role of infinite primes is correct. Also the value of $R$ is irrelevant since it does not affect the Kähler metric.

5. The dependence of Kähler function on WCW degrees of freedom would be coded completely by the dependence of the length scales $qL_{min}$ on the complex coordinates of WCW: note that this dependence is different for each scale. This is reminiscent of the coding of the shape of the drum (or more generally - manifold) by the spectrum of its eigen frequencies. Now Kähler geometry would code for the dependence of the spectrum on the shape of the drum defined by the partonic 2-surface and the 4-D tangent space distribution associated with it.

What happens at the limit of vacuum extremals serves as a test for the identification of Kähler function as Dirac determinant. The weak form of electric magnetic duality implies that all components of the induced Kähler field vanish simultaneously if Kähler magnetic field cancels. In the modified Chern-Simons Dirac equation one obtains $L = \int (\hat{g}^{rr})^{-1/2}dr$. The modified gamma matrix $\hat{\Gamma}^r$ approaches a finite limit when Kähler magnetic field vanishes

$$\hat{\Gamma}^r = \epsilon^{r\beta\gamma}(2J_{\beta k}A_\gamma + J_{\beta\gamma}A_k)\Gamma^k \rightarrow 2\epsilon^{r\beta\gamma}J_{\beta k}\Gamma^k . \quad (5.7.5)$$

The relevant component of the effective metric is $\hat{g}^{rr}$ and is given by

$$\hat{g}^{rr} = (\hat{\Gamma}^r)^2 = 4\epsilon^{r\beta\gamma}\epsilon^{r\mu\nu}J_{\beta k}J_\mu{}^k A_\gamma A_\nu . \quad (5.7.6)$$

The limit is non-vanishing in general and therefore the eigenvalues remain finite also at this limit as also the parameter $L_{min} = \int (\hat{g}^{rr})^{-1/2} dr$ defining the minimum of the length of the braid strand defined by Kähler magnetic flux line in the effective metric unless $\hat{g}^{rr}$ goes to zero everywhere inside the partonic surface. Chern-Simons action and Kähler action vanish for vacuum extremals so that in this case one could require that Dirac determinant approaches to unity in a properly chosen gauge. Dirac determinant should approach to unit for vacuum extremals indeed approaches to unity since there are no finite eigenvalues at the limit $\hat{g}^{rr} = 0$.

### 5.7.2 Hyper-octonionic primes

Before detailed discussion of the hyper-octonionic option it is good to consider the basic properties of hyper-octonionic primes.

1. Hyper-octonionic primes are of form

$$\Pi_p = (n_0, n_3, n_1, n_2, ..., n_7) \ , \ \Pi_p^2 = n_0^2 - \sum_i n_i^2 = p \ \text{or} \ p^2 \ . \tag{5.7.7}$$

2. Hyper-octonionic primes have a standard representation as hyper-complex primes. The Minkowski norm squared factorizes into a product as

$$n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3) \ . \tag{5.7.8}$$

If one has $n_3 \neq 0$, the prime property implies $n_0 - n_3 = 1$ so that one obtains $n_0 = n_3 + 1$ and $2n_3 + 1 = p$ giving

$$(n_0, n_3) = ((p+1)/2, (p-1)/2) \ . \tag{5.7.9}$$

Note that one has $(p+1)/2$ odd for $p \ mod \ 4 = 1)$ and $(p+1)/2$ even for $p \ mod \ 4 = 3)$. The difference $n_0 - n_3 = 1$ characterizes prime property.

If $n_3$ vanishes the prime prime property implies equivalence with ordinary prime and one has $n_3^2 = p^2$. These hyper-octonionic primes represent particles at rest.

3. The action of a discrete subgroup $G(p)$ of the octonionic automorphism group $G_2$ generates form hyper-complex primes with $n_3 \neq 0$ further hyper-octonionic primes $\Pi(p, k)$ corresponding to the same value of $n_0$ and $p$ and for these the integer valued projection to $M^2$ satisfies $n_0^2 - n_3^2 = n > p$. It is also possible to have a state representing the system at rest with $(n_0, n_3) = ((p+1)/2, 0)$ so that the pseudo-mass varies in the range $[\sqrt{p}, (p+1)/2]$. The subgroup $G(n_0, n_3) \subset SU(3)$ leaving invariant the projection $(n_0, n_3)$ generates the hyper-octonionic primes corresponding to the same value of mass for hyper-octonionic primes with same Minkowskian length $p$ and pseudo-mass $\lambda = n \geq \sqrt{p}$.

4. One obtains two kinds of primes corresponding to the lengths of pseudo-momenta equal to $p$ or $\sqrt{p}$. The first kind of particles are always at rest whereas the second kind of particles can be brought at rest only if one interprets the pseudo-momentum as $M^2$ projection. This brings in mind the secondary p-adic length scales assigned to causal diamonds ($CD$s) and the primary p-adic lengths scales assigned to particles.

If the $M^2$ projections of hyper-octonionic primes with length $\sqrt{p}$ characterize the allowed basic momenta, $\zeta_D$ is sum of zeta functions associated with various projections which must be in the limits dictated by the geometry of the orbit of the partonic surface giving upper and lower bounds $L_{max}$ and $L_{min}$ on the length $L$. $L_{min}$ is scaled up to $\sqrt{n_0^2 - n_3^2 L_{min}}$ for a given projection $(n_0, n_3)$. In general a given $M^2$ projection $(n_0, n_3)$ corresponds to several hyper-octonionic primes since $SU(3)$ rotations give a new hyper-octonionic prime with the same $M^2$ projection. This leads to an inconsistency unless one has a good explanation for why some basic momentum can appear several times. One might argue that the spinor mode is degenerate due to the possibility to perform discrete color rotations of the state. For hyper complex representatives there is no such problem and it seems favored. In any case, one can look how the degeneracy factors for given projection can be calculated.

1. To calculate the degeneracy factor $D(n$ associated with given pseudo-mass value $\lambda = n$ one must find all hyper-octonionic primes $\Pi$, which can have projection in $M^2$ with length $n$ and sum up the degeneracy factors $D(n, p)$ associated with them:

$$\begin{aligned} D(n) &= \sum_p D(n,p) \ , \\ D(n,p) &= \sum_{n_0^2 - n_3^2 = p} D(p, n_0, n_3) \ , \\ n_0^2 - n_3^2 &= n \ , \ \Pi_p^2(n_0, n_3) = n_0^2 - n_3^2 - \sum_i n_i^2 = n - \sum_i n_i^2 = p \ . \end{aligned} \tag{5.7.10}$$

2. The condition $n_0^2 - n_3^2 = n$ allows only Pythagorean triangles and one must find the discrete subgroup $G(n_0, n_3) \subset SU(3)$ producing hyper-octonions with integer valued components with length $p$ and components $(n_0, n_3)$. The points at the orbit satisfy the condition

$$\sum n_i^2 = p - n \ . \tag{5.7.11}$$

The degeneracy factor $D(p, n_0, n_3)$ associated with given mass value $n$ is the number of elements of in the coset space $G(n_0, n_3, p)/H(n_0, n_3, p)$, where $H(n_0, n_3, p)$ is the isotropy group of given hyper-octonionic prime obtained in this manner. For $n_0^2 - n_3^2 = p^2$ $D(n_0, n_3, p)$ obviously equals to unity.

### 5.7.3 Three basic options for the pseudo-momentum spectrum

The calculation of the scaling factor of the Kähler function requires the knowledge of the degeneracies of the mass squared eigen values. There are three options to consider.

#### First option: all pseudo-momenta are allowed

If the degeneracy for pseudo-momenta in $M^2$ is same for all mass values- and formally characterizable by a number $N$ telling how many 2-D pseudo-momenta reside on mass shell $n_0^2 - n_3^2 = m^2$. In this case zeta function would be proportional to a sum of Riemann Zetas with scaled arguments corresponding to scalings of the basic mass $m$ to $m/q$.

$$\zeta_D(s) = N \sum_q \zeta(log(qx)s) \ , \ x = \frac{L_{min}}{R} \ . \tag{5.7.12}$$

This option provides no idea about the possible values of $1 \leq q \leq L_{max}/L_{min}$. The number $N$ is given by the integral of relativistic density of states $\int dk/2\sqrt{k^2 + m^2}$ over the hyperbola and is logarithmically divergent so that the normalization factor $N$ of the Kähler function would be infinite.

#### Second option: All integer valued pseudomomenta are allowed

Second option is inspired by number theoretic vision and assumes integer valued components for the momenta using $m_{max} = 2\pi/L_{min}$ as mass unit. p-Adicization motivates also the assumption that momentum components using $m_{max}$ as mass scale are integers. This would restrict the choice of the number theoretical braids.

Integer valuedness together with masses coming as integer multiples of $m_{max}$ implies $(\lambda_0, \lambda_3) = (n_0, n_3)$ with on mass shell condition $n_0^2 - n_3^2 = n^2$. Note that the condition is invariant under scaling. These integers correspond to Pythagorean triangles plus the degenerate situation with $n_3 = 0$. There exists a finite number of pairs $(n_0, n_3)$ satisfying this condition as one finds by expressing $n_0$ as $n_0 = n_3 + k$ giving $2n_3k + k^2 = p^2$ giving $n_3 < n^2/2, n_0 < n^2/2 + 1$. This would be enough to have a finite degeneracy $D(n) \geq 1$ for a given value of mass squared and $\zeta_D$ would be well defined. $\zeta_D$ would be a modification of Riemann zeta given by

$$\begin{aligned}
\zeta_D &= \sum_q \zeta_1(log(qx)s) \ , \ x = \frac{L_{min}}{R} \ , \\
\zeta_1(s) &= \sum g_n n^{-s} \ , \ g_n \geq 1 \ .
\end{aligned} \tag{5.7.13}$$

For generalized Feynman diagrams this option allows conservation of pseudo-momentum and for loops no divergences are possible since the integral over two-dimensional virtual momenta is replaced with a sum over discrete mass shells containing only a finite number of points. This option looks thus attractive but requires a regularization. On the other hand, the appearance of a zeta function having a strong resemblance with Riemann zeta could explain the finding that Riemann zeta is closely related to the description of critical systems. This point will be discussed later.

#### Third option: Infinite primes code for the allowed mass scales

According to the proposal of [K52] , [L7] the hyper-complex parts of hyper-octonionic primes appearing in their infinite counterparts correspond to the $M^2$ projections of real four-momenta. This hypothesis suggests a very detailed map between infinite primes and standard model quantum numbers and predicts a universal mass spectrum [K52] . Since pseudo-momenta are automatically restricted to the plane $M^2$, one cannot avoid the question whether they could actually correspond to the hyper-octonionic primes defining the infinite prime. These interpretations need not of course exclude each other. This option allows several variants and at this stage it is not possible to exclude any of these options.

1. One must choose between two alternatives for which pseudo-momentum corresponds to hyper-complex prime serving as a canonical representative of a hyper-octonionic prime or a projection of hyper-octonionic prime to $M^2$.

2. One must decide whether one allows a) only the momenta corresponding to hyper-complex primes, b) also their powers (p-adic fractality), or c) all their integer multiples ("Riemann option").

One must also decide what hyper-octonionic primes are allowed.

1. The first guess is that all hyper-complex/hyper-octonionic primes defining length scale $\sqrt{p}L_{min} \leq L_{max}$ or $pL_{min} \leq L_{max}$ are allowed. p-Adic fractality suggests that also the higher p-adic length scales $p^{n/2}L_{min} < L_{max}$ and $p^n L_{min} < L_{max}$, $n \geq 1$, are possible.

   It can however happen that no primes are allowed by this criterion. This would mean vanishing Kähler function which is of course also possible since Kähler action can vanish (for instance, for massless extremals). It seems therefore safer to allow also the scale corresponding to the trivial prime $(n_0, n_3) = (1, 0)$ (1 is formally prime because it is not divisible by any prime different from 1) so that at least $L_{min}$ is possible. This option also allows only rather small primes unless the partonic 2-surface contains vacuum regions in which case $L_{max}$ is infinite: in this case all primes would be allowed and the exponent of Kähler function would vanish.

2. The hypothesis that only the hyper-complex or hyper-octonionic primes appearing in the infinite hyper-octonionic prime are possible looks more reasonable since large values of $p$ would be possible and could be identified in terms of the p-adic length scale hypothesis. All hyper-octonionic primes appearing in infinite prime would be possible and the geometry of the orbit of the partonic 2-surface would define an infinite prime. This would also give a concrete physical interpretation for the earlier hypothesis that hyper-octonionic primes appearing in the infinite prime characterize partonic 2-surfaces geometrically. One can also identify the fermionic and purely bosonic primes appearing in the infinite prime as braid strands carrying fermion number and purely bosonic quantum numbers. This option will be assumed in the following.

### 5.7.4 Expression for the Dirac determinant for various options

The expressions for the Dirac determinant for various options can be deduced in a straightforward manner. Numerically Riemann option and manifestly finite option do not differ much but their number theoretic properties are totally different.

#### Riemann option

All integer multiples of these basic pseudo-momenta would be allowed for Riemann option so that $\zeta_D$ would be sum of Riemann zetas with arguments scaled by the basic pseudo-masses coming as inverses of the basic length scales for braid strands. For the option involving only hyper-complex primes the formula for $\zeta_D$ reads as

$$\begin{gathered}\zeta_D=\zeta(log(x_{min}s))+\textstyle\sum_{i,n}\zeta(log(x_{i,n}s))+\sum_{i,n}\zeta(log(y_{i,n}s)) \ , \\ x_{i,n}=p_i^{n/2}x_{min}\leq x_{max} \ , \ \ p_i\geq 3 \ , \ \ y_{i,n}=p_i^n x_{min}\leq x_{max} \ . \ \ p_i\geq 2 \ ,\end{gathered} \tag{5.7.14}$$

$L_{max}$ *resp.* $L_{min}$ is the maximal *resp.* minimal length $L=\int(\hat{g}^{rr})^{-1/2}dr$ for the braid strand defined by the flux line of the Kähler magnetic field in the effective metric. The contributions correspond to the effective hyper-complex prime $p_1=(1,0)$ and hyper-complex primes with Minkowski lengths $\sqrt{p}$ $(p\geq 3)$ and $p$, $p\geq 2$. If also higher p-adic length scales $L_n=p^{n/2}L_{min}<L_{max}$ and $L_n=p^nL_{min}<L_{max}$, $n>1$, are allowed there is no further restriction on the summation. For the restricted option only $L_n$, $n=0,2$ is allowed.

The expressions for the Kähler function and its exponent reads as

$$\begin{aligned} K &= k(log(x_{min})+\sum_i log(x_i)+\sum_i log(y_i) \ , \\ exp(K) &= (\frac{1}{x_{min}})^k\times\prod_i(\frac{1}{x_i})^k\times\prod_i(\frac{1}{y_i})^k \ , \\ x_i &\leq x_{max} \ , \ \ y_i\leq x_{max} \ , \ \ k=-\frac{d\zeta(s)}{ds}_{|s=0}=\frac{1}{2}log(2\pi)\simeq .9184 \ . \end{aligned} \tag{5.7.15}$$

From the point of view of p-adicization program the appearance of strongly transcendental numbers in the normalization factor of $\zeta_D$ is not a well-come property.

If the scaling of the WCW Kähler metric by $1/k$ is a legitimate procedure it would allow to get rid of the transcendental scaling factor $k$ and this scaling would cancel also the transcendental from the exponent of Kähler function. The scaling is not however consistent with the view that Kähler coupling strength determines the normalization of the WCW metric.

This formula generalizes in a rather obvious manner to the cases when one allows $M^2$ projections of hyper-octonionic primes.

### Manifestly finite options

The options for which one does not allow summation over all integer multiples of the basic momenta characterized by the canonical representatives of hyper-complex primes or their projections to $M^2$ are manifestly finite. They differ from the Riemann option only in that the normalization factor $k =\simeq .9184$ defined by the derivative Riemann Zeta at origin is replaced with $k = 1$. This would mean manifest finiteness of $\zeta_D$. Kähler function and its exponent are given by

$$\begin{aligned} K &= k(log(x_{min}) + \sum_i log(x_i) + \sum_i log(y_i) \ , \ x_i \leq x_{max} \ , \ y_i \leq x_{max} \ , \\ exp(K) &= \frac{1}{x_{min}} \times \prod_i \frac{1}{x_i} \times \prod_i \frac{1}{y_i} \ . \end{aligned} \tag{5.7.16}$$

Numerically the Kähler functions do not differ much since their ratio is .9184. Number theoretically these functions are however completely different. The resulting dependence involves only square roots of primes and is an algebraic function of the lengths $p_i$ and rational function of $x_{min}$. p-Adicization program would require rational values of the lengths $x_{min}$ in the intersection of the real and p-adic worlds if one allows algebraic extension containing the square roots of the primes involved. Note that in p-adic context this algebraic extension involves two additional square roots for $p > 2$ if one does not want square root of $p$. Whether one should allow for $R_p$ also extension based on $\sqrt{p}$ is not quite clear. This would give 8-D extension.

For the more general option allowing all projections of hyper-complex primes to $M^2$ the general form of Kähler function is same. Instead of pseudo-masses coming as primes and their square roots one has pseudomasses coming as square roots of some integers $n \leq p$ or $n \leq p^2$ for each $p$. In this case the conservation laws are not so strong.

Note that in the case of vacuum extremals $x_{min} = \infty$ holds true so that there are no primes satisfying the condition and Kähler function vanishes as it indeed should.

### More concrete picture about the option based on infinite primes

The identification of pseudo-momenta in terms of infinite primes suggests a rather concrete connection between number theory and physics.

1. One could assign the finite hyper-octonionic primes $\Pi_i$ making the infinite prime to the sub-braids identified as Kähler magnetic flux lines with the same length $L$ in the effective metric. The primes assigned to the finite part of the infinite prime correspond to single fermion and some number of bosons. The primes assigned to the infinite part correspond to purely bosonic states assignable to the purely bosonic braid strands. Purely bosonic state would correspond to the action of a WCW Hamiltonian to the state.

   This correspondence can be expanded to include all quantum numbers by using the pair of infinite primes corresponding to the "vacuum primes" $X \pm 1$, where $X$ is the product of all finite primes [K52] . The only difference with respect to the earlier proposal is that physical momenta would be replaced by pseudo-momenta.

2. Different primes $p_i$ appearing in the infinite prime would correspond to their own sub-braids. For each sub-braid there is a $N$-fold degeneracy of the generalized eigen modes corresponding to the number $N$ of braid strands so that many particle states are possible as required by the braid picture.

3. The correspondence of infinite primes with the hierarchy of Planck constants could allow to understand the fermion-many boson states and many boson states assigned with a given finite prime in terms of many-particle states assigned to $n_a$ and $n_b$-sheeted singular covering spaces of $CD$ and $CP_2$ assignable to the two infinite primes. This interpretation requires that only single p-adic prime $p_i$ is realized as quantum state meaning that quantum measurement always selects a particular p-adic prime $p_i$ (and corresponding sub-braid) characterizing the p-adicity of the quantum state. This selection of number field behind p-adic physics responsible for cognition looks very plausible.

4. The correspondence between pairs of infinite primes and quantum states [K52] allows to interpret color quantum numbers in terms of the states associated with the representations of a finite subgroup of $SU(3)$ transforming hyper-octonionic primes to each other and preserving the $M^2$ pseudo-momentum. Same applies to $SO(3)$. The most natural interpretation is in terms of wave functions in the space of discrete $SU(3)$ and $SO(3)$ transforms of the partonic 2-surface. The dependence of the pseudo-masses on these quantum numbers is natural so that the projection hypothesis finds support from this interpretation.

5. The infinite prime characterizing the orbit of the partonic 2-surface would thus code which multiples of the basic mass $2\pi/L_{min}$ are possible. Either the $M^2$ projections of hyper-octonionic primes or their hyper-complex canonical representatives would fix the basic $M^2$ pseudo-momenta for the corresponding number theoretic braid associated. In the reverse direction the knowledge of the light-like 3-surface, the $CD$ and $CP_2$ coverings, and the number of the allowed discrete $SU(3)$ and $SU(2)$ rotations of the partonic 2-surface would dictate the infinite prime assignable to the orbit of the partonic 2-surface.

One would also like to understand whether there is some kind of conservation laws associated with the pseudo-momenta at vertices. The arithmetic QFT assignable to infinite primes would indeed predict this kind of conservation laws.

1. For the manifestly finite option the ordinary conservation of pseudo-momentum conservation at vertices is not possible since the addition of pseudo-momenta does not respect the condition $n_0 - n_3 = 1$. In fact, this difference in the sum of hyper-complex prime momenta tells how many momenta are present. If one applies the conservation law to the sum of the pseudo-momenta corresponding to different primes and corresponding braids, one can have reactions in which the number of primes involved is conserved. This would give the selection rule $\sum_1^N p_i = \sum_1^N p_f$. These reactions have interpretation in terms of the geometry of the 3-surface representing the line of the generalized Feynman diagram.

2. Infinite primes define an arithmetic quantum field theory in which the total momentum defined as $\sum n_i log(p_i)$ is a conserved quantity. As matter fact, each prime $p_i$ would define a separately conserved momentum so that there would be an infinite number of conservation laws. If the sum $\sum_i log(p_i)$ is conserved in the vertex , the primes $p_i$ associated with the incoming particle are shared with the outgoing particles so that also the total momentum is conserved. This looks the most plausible option and would give very powerful number theoretical selection rules at vertices since the collection of primes associated with incoming line would be union of the collections associated with the outgoing lines and also total pseudo-momentum would be conserved.

3. For the both Riemann zeta option and manifestly finite options the arithmetic QFT associated with infinite primes would be realized at the level of pseudo-momenta meaning very strong selection rules at vertices coding for how the geometries of the partonic lines entering the vertex correlate. WCW integration would reduce for the lines of Feynman diagram to a sum over light-like 3-surfaces characterized by $(x_{min}, x_{max})$ with a suitable weighting factor and the exponent of Kähler function would give an exponential damping as a function of $x_{min}$.

#### Which option to choose?

One should be able to make two choices. One must select between hyper-complex representations and the projections of hyper-octonionic primes and between the manifestly finite options and the one producing Riemann zeta?

Hyper-complex option seems to be slightly favored over the projection option.

1. The appearance of the scales $\sqrt{p_i}x_{min}$ and possibly also their $p^n$ multiples brings in mind p-adic length scales coming as $\sqrt{p^n}$ multiples of $CP_2$ length scale. The scales $p_i x_{min}$ associated with hyper-complex primes reducing to ordinary primes in turn bring in mind the size scales assignable to $CD$s. The hierarchy of Planck constants implies also $\hbar/\hbar_0 = \sqrt{n_a n_b}$ multiples of these length scales but mass scales would not depend on $n_a$ and $n_b$ [K53] . For large values of $p$ the pseudo-momenta are almost light-like for hyper-complex option whereas the projection option allows also states at rest.

2. Hyper-complex option predicts that only the p-adic pseudo-mass scales appear in the partition function and is thus favored by the p-adic length scale hypothesis. Projection option predicts also the possibility of the mass scales (not all of them) coming as $1/\sqrt{n}$. These mass scales are however not predicted by the hierarchy of Planck constants.

3. The same pseudo-mass scale can appear several times for the projection option. This degeneracy corresponds to the orbit of the hyper-complex prime under the subgroup of $SU(3)$ respecting integer property. Similar statement holds true in the case of $SO(3)$: these groups are assigned to the two infinite primes characterizing parton. The natural assignment of this degeneracy is to the discrete color rotational and rotational degrees associated with the partonic 2-surface itself rather than spinor modes at fixed partonic 2-surface. That the pseudo-mass would depend on color and angular momentum quantum numbers would make sense.

Consider next the arguments in favor of the manifestly finite option.

1. The manifestly finite option is admittedly more elegant than the one based on Riemann zeta and also guarantees that no additional loop summations over pseudo-momenta are present. The strongest support for the manifestly finite option comes from number theoretical universality.

2. One could however argue that the restriction of the pseudo-momenta to a finite number is not consistent with the modified Dirac-Chern-Simons equation. Quantum classical correspondence however implies correlation between the geometry of the partonic orbits and the pseudo-momenta and the summation over all prime valued pseudo-momenta is present but with a weighting factor coming from Kähler function implying exponential suppression.

The Riemann zeta option could be also defended.

1. The numerical difference of the normalization factors of the Kähler function is however only about 8 per cent and quantum field theorists might interpret the replacement the length scales $x_i$ and $y_i$ with $x_i^d$ and $y_i^d$, $d \simeq .9184$, in terms of an anomalous dimension of these length scales. Could one say that radiative corrections mean the scaling of the original preferred coordinates so that one could still have consistency with number theoretic universality?

2. Riemann zeta with a non-vanishing argument could have also other applications in quantum TGD. Riemann zeta has interpretation as a partition function and the zeros of partition functions have interpretation in terms of phase transitions. The quantum criticality of TGD indeed corresponds to a phase transition point. There is also experimental evidence that the distribution of zeros of zeta corresponds to the distribution of energies of quantum critical systems in the sense that the energies correspond to the imaginary parts of the zeros of zeta [A27] .

   The first explanation would be in terms of the analogs of the harmonic oscillator coherent states with integer multiple of the basic momentum taking the role of occupation number of harmonic oscillator and the zeros $s = 1/2 + iy$ of $\zeta$ defining the values of the complex coherence parameters. TGD inspired strategy for the proof of Riemann hypothesis indeed leads to the identification of the zeros as coherence parameters rather than energies as in the case of Hilbert-Polya hypothesis [K46] and the vanishing of the zeta at zero has interpretation as orthogonality of the state with respect to the state defined by a vanishing coherence parameter interpreted as a tachyon. One should demonstrate that the energies of quantum states can correspond to the imaginary parts of the coherence parameters.

   Second interpretation could be in terms of quantum critical zero energy states for which the "complex square root of density matrix" defines time-like entanglement coefficients of $M$-matrix. The complex square roots of the probabilities defined by the coefficient of harmonic oscillator states (perhaps identifiable in terms of the multiples of pseudo-momentum) in the coherent state defined by the zero of $\zeta$ would define the $M$-matrix in this situation. Energy would correspond also now to the imaginary part of the coherence parameter. The norm of the state would be completely well-defined.

**Representation of configuration Kähler metric in terms of eigenvalues of $D_{C-S}$**

A surprisingly concrete connection of the configuration space metric in terms of generalized eigenvalue spectrum of $D_{C-S}$ results. From the general expression of Kähler metric in terms of Kähler function

$$G_{k\bar{l}} = \partial_k \partial_{\bar{l}} K = \frac{\partial_k \partial_{\bar{l}} exp(K)}{exp(K)} - \frac{\partial_k exp(K)}{exp(K)} \frac{\partial_{\bar{l}} exp(K)}{exp(K)} \ , \tag{5.7.17}$$

and from the expression of $exp(K) = \prod_i \lambda_i$ as the product of of finite number of eigenvalues of $D_{C-S}$ , the expression

$$G_{k\bar{l}} = \sum_i \frac{\partial_k \partial_{\bar{l}} \lambda_i}{\lambda_i} - \frac{\partial_k \lambda_i}{\lambda_i} \frac{\partial_{\bar{l}} \lambda_i}{\lambda_i} \tag{5.7.18}$$

for the configuration space metric follows. Here complex coordinates refer to the complex coordinates of configuration space. Hence the knowledge of the eigenvalue spectrum of $D_{C-S}(X^3)$ as function of some complex coordinates of configuration space allows to deduce the metric to arbitrary accuracy. If the above arguments are correct the calculation reduces to the calculation of the derivatives of $log(\sqrt{p}L_{min}/R)$, where $L_{min}$ is the length of the Kähler magnetic flux line between partonic 2-surfaces with respect to the effective metric defined by the anti-commutators of the modified gamma matrices. Note that these length scales have different dependence on WCW coordinates so that one cannot reduce everything to $L_{min}$. Therefore one would have explicit representation of the basic building brick of WCW Kähler metric in terms of the geometric data associated with the orbit of the partonic 2-surface.

**The formula for the Kähler action of $CP_2$ type vacuum extremals is consistent with the Dirac determinant formula**

The first killer test for the formula of Kähler function in terms of the Dirac determinant based on infinite prime hypothesis is provided by the action of $CP_2$ type vacuum extremals. One of the first attempts to make quantitative predictions in TGD framework was the prediction for the gravitational constant. The argument went as follows.

1. For dimensional reasons gravitational constant must be proportional to p-adic length scale squared, where $p$ characterizes the space-time sheet of the graviton. It must be also proportional to the square of the vacuum function for the graviton representing a line of generalized Feynman diagram and thus to the exponent $exp(-2K)$ of Kähler action for topologically condensed $CP_2$ type vacuum extremals with very long projection. If topological condensation does not reduce much of the volume of $CP_2$ type vacuum extremal, the action is just Kähler action for $CP_2$ itself. This gives

$$\hbar_0 G = L_p^2 exp(2L_K(CP_2) = pR^2 exp(2L_K(CP_2) \ . \tag{5.7.19}$$

2. Using as input the constraint $\alpha_K \simeq \alpha_{em} \sim 1/137$ for Kähler coupling strengths coming from the comparison of the TGD prediction for the rotation velocity of distant galaxies around galactic nucleus and the p-adic mass calculation for the electron mass, one obtained the result

$$exp(2L_K(CP_2) = \frac{1}{p \times \prod_{p_i \leq 23} p_i} \ . \tag{5.7.20}$$

The product contains the product of all primes smaller than 24 ($p_i \in \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$). The expression for the Kähler function would be just of the form predicted by the Dirac determinant formula with $L_{min}$ replaced with $CP_2$ length scale. As a matter fact, this was the first indication that particles are characterized by several p-adic primes but that only one of them is "active". As explained, the number theoretical state function reduction explains this.

3. The same formula for the gravitational constant would result for any prime $p$ but the value of Kähler coupling strength would depend on prime $p$ logarithmically for this option. I indeed proposed that this formula fixes the discrete evolution of the Kähler coupling strength as function of p-adic prime from the condition that gravitational constant is renormalization group invariant quantity but gave up this hypothesis later. It is wisest to keep an agnostic attitude to this issue.

4. I also made numerous brave attempts to deduce an explicit formula for Kähler coupling strength. The general form of the formula is

$$\frac{1}{\alpha_K} = klog(K^2), \quad K^2 = p \times 2 \times 3 \times 5.. \times 23 \ . \tag{5.7.21}$$

The problem is the exact value of $k$ cannot be known precisely and the guesses for is value depend on what one means with number theoretical universality. Should Kähler action be a rational number? Or is it Kähler function which is rational number (it is for the Dirac determinant option in this particular case). Is Kähler coupling strength $g_K^2/4\pi$ or $g_K^2$ a rational number? Some of the guesses were $k = \pi/4$ and $k = 137/107$. The facts that the value of Kähler action for the line of a generalized diagram is not exactly $CP_2$ action and the value of $\alpha_K$ is not known precisely makes these kind of attempts hopeless in absence of additional ideas.

Also other elementary particles -in particular exchanged bosons- should involve the exponent of Kähler action for $CP_2$ type vacuum extremal. Since the values of gauge couplings are gigantic as compared to the expression of the gravitational constant the value of Kähler action must be rather small form them. $CP_2$ type vacuum extremals must be short in the sense that $L_{min}$ in the effective metric is very short. Note however that the p-adic prime characterizing the particle according to p-adic mass calculations would be large also now. One can of course ask whether this p-adic prime characterizes the gravitational space-time sheets associated with the particle and not the particle itself. The assignment of p-adic mass calculations with thermodynamics at gravitational space-time sheets of the particle would be indeed natural. The value of $\alpha_K$ would depend on $p$ in logarithmic manner for this option. The topological condensation of could also eat a lot of $CP_2$ volume for them.

**Eigenvalues of $D_{C-S}$ as vacuum expectations of Higgs field?**

Infinite prime hypothesis implies the analog of p-adic length scale hypothesis but since pseudo-momenta are in question, this need not correspond to the p-adic length scale hypothesis for the actual masses justified by p-adic thermodynamics. Note also that $L_{min}$ does not correspond to $CP_2$ length scale. This is actually not a problem since the effective metric is not $M^4$ metric and one can quite well consider the possibility that $L_{min}$ corresponds to $CP_2$ length scale in the the induced metric. The reason is that light-like 3- surface is in question the distance along the Kähler magnetic flux line reduces essentially to a distance along the partonic 2-surface having size scale of order $CP_2$ length for the partonic 2-surfaces identified as wormhole throats. Therefore infinite prime can code for genuine p-adic length scales associated with the light-like 3-surface and quantum states would correspond by number theoretical state function reduction hypothesis to single ordinary prime.

Support for this identification comes also from the expression of gravitational constant deduced from p-adic length scale hypothesis. The result is that gravitational constant is assumed to be proportional to have the expression $G = L_p^2 exp(-2S_K(CP_2))$, where $p$ characterizes graviton or the space-time sheet mediating gravitational interaction and exponent gives Kähler action for $CP_2$ type vacuum extremal representing graviton. The argument allows to identify the p-adic prime $p = M_{127}$ associated with electron (largest Mersenne prime which does not correspond to super-astronomical length scale) as the p-adic prime characterizing also graviton. The exponent of Kähler action is proportional to $1/p$ which conforms with the general expression for Kähler function. I have considered several identifications of the numerical factor and one of them has been as product of primes $2 \leq p \leq 23$ assuming that somehow the primes $\{2, ..., 23, p\}$ characterize graviton. This guess is indeed consistent with the prediction of the infinite-prime hypothesis.

The first guess inspired by the p-adic mass calculations is that the squares $\lambda_i^2$ of the eigenvalues of $D_{C-S}$ could correspond to the conformal weights of ground states. Another natural physical interpretation of $\lambda$ is as an analog of the Higgs vacuum expectation. The instability of the Higgs=0 phase would corresponds to the fact that $\lambda = 0$ mode is not localized to any region in which ew magnetic field or induced Kähler field is non-vanishing. By the previous argument one would have order of magnitude estimate $h_0 = \sqrt{2\pi/L_{min}}$.

1. The vacuum expectation value of Higgs is only proportional to the scale of $\lambda$. Indeed, Higgs and gauge bosons as elementary particles correspond to wormhole contacts carrying fermion and anti-fermion at the two wormhole throats and must be distinguished from the space-time correlate of its vacuum expectation as something proportional to $\lambda$. For free fermions the vacuum expectation value of Higgs does not seem to be even possible since free fermions do not correspond to wormhole contacts between two space-time sheets but possess only single wormhole throat (p-adic mass calculations are consistent with this). If fermion suffers topological condensation as indeed assumed to do in interaction region, a wormhole contact is generated and makes possible the generation of Higgs vacuum expectation value.

2. Physical considerations suggest that the vacuum expectation of Higgs field corresponds to a particular eigenvalue $\lambda_i$ of modified Chern-Simons Dirac operator so that the eigenvalues $\lambda_i$ would define TGD counterparts for the minima of Higgs potential. For the minimal option one has only a finite number of pseudo-mass eigenvalues inversely proportional $\sqrt{p}$ so that the identification as a Higgs vacuum expectation is consistent with the p-adic length scale hypothesis. Since the vacuum expectation of Higgs corresponds to a condensate of wormhole contacts giving rise to a coherent state, the vacuum expectation cannot be present for topologically condensed $CP_2$ type vacuum extremals representing fermions since only single wormhole throat is involved. This raises a hen-egg question about whether Higgs contributes to the mass or whether Higgs is only a correlate for massivation having description using more profound concepts. From TGD point of view the most elegant option is that Higgs does not give rise to mass but Higgs vacuum expectation value accompanies bosonic states and is naturally proportional to $\lambda_i$. With this interpretation $\lambda_i$ could give a contribution to both fermionic and bosonic masses.

3. If the coset construction for super-symplectic and super Kac-Moody algebra implying Equivalence Principle is accepted, one encounters what looks like a problem. p-Adic mass calculations require negative ground state conformal weight compensated by Super Virasoro generators in order to obtain massless states. The tachyonicity of the ground states would mean a close analogy with both string models and Higgs mechanism. $\lambda_i^2$ is very natural candidate for the ground state conformal weights identified but would have wrong sign. Therefore it seems that $\lambda_i^2$ can define only a deviation of the ground state conformal weight from negative value and is positive.

4. In accordance with this $\lambda_i^2$ would give constant contribution to the ground state conformal weight. What contributes to the thermal mass squared is the deviation of the ground state conformal weight from half-odd integer since the negative integer part of the total conformal weight can be compensated by applying Virasoro generators to the ground state. The first guess motivated by cyclotron energy analogy is that the lowest conformal weights are of form $h_c = -n/2+\lambda_i^2$ where the negative contribution comes from Super Virasoro representation. The negative integer part of the net conformal weight can be canceled using Super Virasoro generators but $\Delta h_c$ would give to mass squared a contribution analogous to Higgs contribution. The mapping of the real ground state conformal weight to a p-adic number by canonical identification involves some delicacies.

5. p-Adic mass calculations are consistent with the assumption that Higgs type contribution is vanishing (that is small) for fermions and dominates for gauge bosons. This requires that the deviation of $\lambda_i^2$ with smallest magnitude from half-odd integer value in the case of fermions is considerably smaller than in the case of gauge bosons in the scale defined by p-adic mass scale $1/L(k)$ in question. Somehow this difference could relate to the fact that bosons correspond to pairs of wormhole throats.

### Is there a connection between p-adic thermodynamics, hierarchy of Planck constants, and infinite primes

The following observations suggest that there might be an intrinsic connection between p-adic thermodynamics, hierarchy of Planck constants, and infinite primes.

1. p-Adic thermodynamics [K29] is based on string mass formula in which mass squared is proportional to conformal weight having values which are integers apart from the contribution of the conformal weight of vacuum which can be non-integer valued. The thermal expectation in p-adic thermodynamics is obtained by replacing the Boltzman weight $exp(-E/T)$ of ordinary thermodynamics with p-adic conformal weight $p^{n/T_p}$, where $n$ is the value of conformal weight and $1/T_p = m$ is integer values inverse p-adic temperature. Apart from the ground state contribution and scale factor p-adic mass squared is essentially the expectation value

$$\langle n \rangle = \frac{\sum_n g(n) n p^{\frac{n}{T_p}}}{\sum_n g(n) p^{\frac{n}{T_p}}} . \tag{5.7.22}$$

   $g(n)$ denotes the degeneracy of a state with given conformal weight and depends only on the number of tensor factors in the representations of Virasoro or Super-Virasoro algebra. p-Adic mass squared is mapped to its real counterpart by canonical identification $\sum x_n p^n \rightarrow \sum x_n p^{-n}$. The real counterpart of p-adic thermodynamics is obtained by the replacement $p^{-\frac{n}{T_p}}$ and gives under certain additional assumptions in an excellent accuracy the same results as the p-adic thermodynamics.

2. An intriguing observation is that one could interpret p-adic and real thermodynamics for mass squared also in terms of number theoretic thermodynamics for the number theoretic momentum $log(p^n) = nlog(p)$. The expectation value for this differs from the expression for $\langle n \rangle$ only by the factor $log(p)$.

3. In the proposed characterization of the partonic orbits in terms of infinite primes the primes appearing in infinite prime are identified as p-adic primes. For minimal option the p-adic prime characterizes $\sqrt{p}$- or $p$- multiple of the minimum length $L_{min}$ of braid strand in the effective metric defined by modified Chern-Simons gamma matrice. One can consider also $(\sqrt{p})^n$ and $p^n$ (p-adic fractality)- and even integer multiples of $L_{min}$ if they are below $L_{max}$. If light-like 3-surface contains vacuum regions arbitrary large $p$:s are possible since for these one has $L_{min} \rightarrow \infty$. Number theoretic state function reduction implies that only single $p$ can be realized -one might say "is active"- for a given quantum state. The powers $p_i^n$ appearing in the infinite prime have interpretation as many particle states with total number theoretic momentum $n_i log(p)_i$. For the finite part of infinite prime one has one fermion and $n_i - 1$ bosons and for the bosonic part $n_i$ bosons. The arithmetic QFT associated with infinite primes - in particular the conservation of the number theoretic momentum $\sum n_i log(p_i)$ - would naturally describe the correlations between the geometries of light-like 3-surfaces representing the incoming lines of the vertex of generalized Feynman diagram. As a matter fact, the momenta associated with different primes are separately conserved so that one has infinite number of conservation laws.

4. One must assign two infinite primes to given partonic two surface so that one has for a given prime $p$ two integers $n_+$ and $n_-$. Also the hierarchy of Planck constants assigns to a given page of the Big Book two integers and one has $\hbar = n_a n_b \hbar_0$. If one has $n_a = n_+$ and $n_b = n_-$ then the reactions in which given initial number theoretic momenta $n_{\pm,i} log(p_i)$ is shared between final states would have concrete interpretation in terms of the integers $n_a, n_b$ characterizing the coverings of incoming and outgoing lines.

   Note that one can also consider the possibility that the hierarchy of Planck constants emerges from the basic quantum TGD. Basically due to the vacuum degeneracy of Kähler action the canonical momentum densities correspond to several values of the time derivatives of the imbedding space oordinates so that for a given partonic 2-surface there are several space-time sheets with same conserved quantities defined by isometry currents and Kähler current. This forces the introduction of $N$-fold covering of $CD \times CP_2$ in order to describe the situation. The splitting of the partonic 2-surface into $N$ pieces implies a charge fractionization during its travel to the upper end of $CD$. One can also develop an argument suggesting that the coverings factorize to coverings of $CD$ and $CP_2$ so that the number of the sheets of the covering is $N = n_a n_b$ [K24] .

These observations make one wonder whether there could be a connection between p-adic thermodynamics, hierarchy of Planck constants, and infinite primes.

1. Suppose that one accepts the identification $n_a = n_+$ and $n_b = n_-$. Could one perform a further identification of these integers as non-negative conformal weights characterizing physical states so that conservation of the number theoretic momentum for a given p-adic prime would correspond to the conservation of conformal weight. In p-adic thermodynamics this conformal weight is sum of conformal weights of 5 tensor factors of Super-Virasoro algebra. The number must be indeed five and one could assign them to the factors of the symmetry group. One factor for color symmetries and two factors of electro-weak $SU(2)_L \times U(1)$ are certainly present. The remaining two factors could correspond to transversal degrees of freedom assignable to string like objects but one can imagine also other identifications [K29] .

2. If this interpretation is correct, a given conformal weight $n = n_a = n_+$ (say) would correspond to all possible distributions of five conformal weights $n_i$, $i = 1, ..., 5$ between the $n_a$ sheets of covering of $CD$ satisfying $\sum_{i=1}^{5} n_i = n_a = n_+$. Single sheet of covering would carry only unit conformal weight so that one would have the analog of fractionization also now and a possible interpretation would be in terms of the instability of states with conformal weight $n > 1$. Conformal thermodynamics would also mean thermodynamics in the space of states determined by infinite primes and in the space of coverings.

3. The conformal weight assignable to the $CD$ would naturally correspond to mass squared but there is also the conformal weight assignable to $CP_2$ and one can wonder what its interpretation might be. Could it correspond to the expectation of pseudo mass squared characterizing the generalized eigenstates of the modified Dirac operator? Note that one should allow in the spectrum also the powers of hyper-complex primes up to some maximum power $p^{n_{max}/2} \leq L_{max}/L_{min}$ so that Dirac determinant would be non-vanishing and Kähler function finite. From the point of conformal invariance this is indeed natural.

## 5.8 Number theoretic braids and global view about anti-commutations of induced spinor fields

The anti-commutations of the induced spinor fields are reasonably well understood locally. The basic objects are 3-dimensional light-like 3-surfaces. These surfaces can be however seen as random light-like orbits of partonic 2-D partonic surface and the effective 2-dimensionality means that partonic 2-surfaces plus there 4-D tangent space take the role of fundamental dynamical objects. This is expressed concretely by the condition that the ends of the space-time surface and wormhole throats are extremals of Chern-Simons action. For $J + J_1$ option this allows also 3-D $CP_2$ projections. Conformal invariance would in turn make the 2-D partons 1-D objects (analogous to Euclidian strings) and braids, which can be regarded as the ends of string world sheets with Minkowskian signature, in turn would discretize these Euclidian strings. It must be however noticed that the status of Euclidian strings is uncertain.

Somehow these views should be unifiable into a more global view about the situation allowing to understand the reduction of effective dimension of the system as one goes to short scales.

1. The notions of measurement resolution and braid concept indeed provides the needed physical insights in this respect. The precise definition of the notion of braid and its number theoretic counterpart remains however open and one can imagine several alternatives.

2. Electric-magnetic duality and the ideas stimulated by it led to a further progress. The braid concept emerges automatically from the reduction of Chern-Simons Dirac equation to separate ordinary differential equations at the flux lines of the Kähler magnetic field. Boundary conditions at the ends of the light-like 3-surface allow only solutions which are concentrated on braids for which the strands have same length in the effective metric defined by the modified gamma matrices and a connection with p-adic length scale hypothesis, hierarchy of Planck constants, and infinite primes emerges. Number theoretic braids correspond to braids for which the length is rational or at most algebraic number. Possible additional conditions on the coordinates of

$X^2$ can be of course considered but already the quantization of lengths is enough to guarantee that the exponent of Kähler function identified as Dirac determinant rational function and for rational braid lengths a simple algebraic number involving only square roots of primes making sense also p-adically.

The identification of flows defining the braids has been one of the key issues since quite a large variety of candidates can be imagined. The integrability of this flow by Beltrami condition implies for that there is huge variety of topologically equivalent flows and might resolve also the identification issue: all isometry currents and Kähler current and perhaps also the instanton current define one and same flow.

3. A physically attractive realization of the braids - and more generally- of slicings of space-time surface by 3-surfaces and string world sheets, is discussed in [K25] by starting from the observation that TGD defines an almost topological QFT of braids, braid cobordisms, and 2-knots. The boundaries of the string world sheets at the space-like 3-surfaces at boundaries of $CD$s and wormhole throats would define space-like and time-like braids uniquely.

   The idea relies on a rather direct translation of the notions of singular surfaces and surface operators used in gauge theory approach to knots [A73] to TGD framework. It leads to the identification of slicing by three-surfaces as that induced by the inverse images of $r = constant$ surfaces of $CP_2$, where $r$ is $U(2)$ invariant radial coordinate of $CP_2$ playing the role of Higgs field vacuum expectation value in gauge theories. $r = \infty$ surfaces correspond to geodesic spheres and define analogs of fractionally magnetically charged Dirac strings identifiable as preferred string world sheets. The union of these sheets labelled by subgroups $U(2) \subset SU(3)$ would define the slicing of space-time surface by string world sheets. The choice of $U(2)$ relates directly to the choice of quantization axes for color quantum numbers characterizing $CD$ and would have the choice of braids and string world sheets as a space-time correlate.

### 5.8.1 Quantization of the modified Dirac action and configuration space geometry

The quantization of the modified Dirac action involves a fusion of various number theoretical ideas. The naive approach would be based on standard canonical quantization of induced spinor fields by posing anti-commutation relations between $\Psi$ and canonical momentum density $\partial L/\partial(\partial_t\Psi)$.

One can imagine two alternative forms of the anti-commutation relations.

1. The standard canonical anti-commutation relations for the induced the spinor fields would be given by

$$\{\overline{\Psi}\hat{\Gamma}^0(x), \Psi(y)\} = \delta^2_{x,y} \ . \tag{5.8.1}$$

   The factor that $\hat{\Gamma}^0(x)$ corresponds to the canonical momentum density associated with Kähler action. The discrete variant of the anti-commutation relations applying in the case of non-stringy space-time sheets is

$$\{\overline{\Psi}\hat{\Gamma}^0(x_i), \Psi(x_j)\} = \delta_{i,j} \ . \tag{5.8.2}$$

   where $x_i$ and $x_j$ label the points of the number theoretic braid. These anticommutations are are inconsistent at the limit of vacuum extremal and also extremely non-linear in the imbedding space coordinates.

2. The construction of WCW gamma matrices leads to a nonsingular form of anti-commutation relations given by

$$\{\overline{\Psi}(x)\gamma^0, \Psi(x)\} = (1+K)J\delta_{x,y} \ . \tag{5.8.3}$$

Here $J$ denotes the Kähler magnetic flux $J_m$ and Kähler electric flux relates to via the formula $J_e = KJ_m$, where $K$ is symplectic invariant. What is nice that at the limit of vacuum extremals the right hand side vanishes so that spinor fields become non-dynamical. Therefore this optin- actually the original one- seems to be the only reasonable choice.

For the latter option the super counterparts of local flux Hamiltonians can be written in the form

$$\begin{aligned}
H_{A,+,n} &= H_{A,+,q,n} + H_{A,+,L,n} \ , \ H_{A,-,n} = H_{A,-,q,n} + H_{A,-,L,n} \ , \\
H_{A,+,q,n} &= \oint \overline{\Psi} J_+^A q_n d^2x \ , \\
H_{A,-,q,n} &= \oint \overline{q}_n J_-^A \Psi d^2x \ , \\
H_{A,-,L,n} &= \oint \overline{\Psi} J_+^A L_n d^2x \ , \\
H_{A,+,L,n} &= \oint \overline{L}_n J_-^A \Psi d^2x \ , \\
J_+^A &= j^{Ak}\Gamma_k \ , \ J_-^A = j^{A\overline{k}}\Gamma_{\overline{k}} \ .
\end{aligned} \tag{5.8.4}$$

Suppose that there is a one-one correspondence between quark modes and leptonic modes is satisfied and the label $n$ decomposes as $n = (m, i)$, where $n$ labels a scalar function basis and $i$ labels spinor components. This would give

$$\begin{aligned}
q_n = q_{m,i} &= \Phi_m q_i \ , \\
L_n = L_{m,i} &= \Phi_m L_i \ , \\
\overline{q}_i \gamma^0 q_j &= \overline{L}_i \gamma^0 L_j = g_{ij} \ .
\end{aligned} \tag{5.8.5}$$

Suppose that the inner products $g_{ij}$ are constant. The simplest possibility is $g_{ij} = \delta_{ij}$ Under these assumptions the anticommutators of the super-symmetric flux Hamiltonians give flux Hamiltonians.

$$\{H_{A,+,n}, H_{A,-,n}\} = g_{ij} \oint (1+K)\overline{\Phi}_m \Phi_n H_A J d^2x \ . \tag{5.8.6}$$

The product of scalar functions can be expressed as

$$\overline{\Phi}_m \Phi_n = c_{mn}^k \Phi_k \ . \tag{5.8.7}$$

Note that the notion of symplectic QFT led to a scalar function algebra of similar kind consisting of phase factors and there excellent reasons to consider the possibility that there is a deep connection with this approach.

One expects that the symplectic algebra is restricted to a direct sum of symplectic algebras localized to the regions where the induced Kähler form is non-vanishing implying that the algebras associated with different region form to a direct sum. Also the contributions to configuration space metric are direct sums. The symplectic algebras associated with different region can be truncated to finite-dimensional spaces of symplectic algebras $S^2 \times S$ associated with the regions in question. As far as coordinatization of the reduced configuration space is considered, these symplectic sub-spaces are enough. These truncated algebras naturally correspond to the hyper-finite factor property of the Clifford algebra of configuration space.

### 5.8.2 Expressions for configuration space super-symplectic generators in finite measurement resolution

The expressions of configuration space Hamiltonians and their super counterparts just discussed were based on 2-dimensional integrals. This is problematic for several reasons.

1. In p-adic context integrals do not makes sense so that this representation fails in p-adic context. Sums would be more appropriate if one wants number theoretic universality at the level of basic formulas.

2. The use of sums would also conform with the notion of finite measurement resolution having discretization in terms of intersections of $X^2$ with number theoretic braids as a space-time correlate.

3. Number theoretic duality suggests a unique realization of the discretization in the sense that only the points of partonic 2-surface $X^2$ whose $\delta M^4_{\pm}$ projections commute in hyper-octonionic sense and thus belong to the intersections of the projection $P_{M^4}(X^2)$ with radial light-like geodesics $M_{\pm}$ representing intersections of $M^2 \subset M^4 \subset M^8$ with $\delta M^4_{\pm} \times CP_2$ contribute to the configuration space Hamiltonians and super Hamiltonians and therefore to the configuration space metric.

Clearly, finite measurement resolution seems to be an unavoidable aspect of the geometrization of the configuration space as one can expect on basis of the fact that configuration space Clifford algebra provides representation for hyper-finite factors of type $II_1$ whose inclusions provide a representation for the finite measurement resolution. This means that the infinite-dimensional configuration space can be represented as a finite-dimensional space in arbitrary precise approximation so that also also configuration Clifford algebra and configuration space spinor fields becomes finite-dimensional.

The modification of anti-commutation relations to this case is

$$\{\overline{\Psi}(x_m)\gamma^0, \Psi(x_n)\} = (1+K)J\delta_{x_m,x_n} \quad . \tag{5.8.8}$$

Note that the constancy of $\gamma^0$ implies a complete symmetry between the two points. The number of points must be the maximal one consistent with the Kronecker delta type anti-commutation relations so that information is not lost.

The question arises about the choice of the points $x_m$. This choice should ge general coordinate invariant. The following ideas have been considered.

1. The number theoretic vision leads to the notion of number theoretic braid defined as the set of points common to real and p-adic variant of $X^2$. The points of the number theoretic braid are excellent candidates for points $x_n$. The p-adic variant of $X^2$ exists only if $X^2$ is defined by rational functions with coefficients which are possibly algebraic and thus make sense both in real and p-adic sense. These points belong to the algebraic extension of rational numbers appearing in the representation of $X^2$ as an algebraic surface but one can consider quite generally the possibility that the points of the number theoretic braid are rational or in a finite algebraic extension of rationals. If one restricts the consideration to rational points this criterion makes sense even if $X^2$ is not algebraic. In the generic case one can expect that the number of these points is finite. The objection is that this definition is not general coordinate invariant. One can however identify preferred coordinates since $CP_2$ and the sphere $S^2$ associated with light-cone boundary are symmetric spaces.

2. An alternative identification emerged from the solution of the Chern-Simons Dirac equation. Since the number of generalized eigenmodes of $D_{C-S}$ is finite for a given braid, the local anti-commutation relations cannot be satisfied unless they are restricted to a finite subset of points of $X^2$ and the condition that the lengths of the braid strands in the effective metric are same fixes the braid uniquely. Number theoretic braids are obtained when the length is rational or algebraic number. This identification of number theoretic braid is enough for p-adicization since it guarateens that the exponent of Kähler function is rational function of braid lengths labelled by a finite number of primes. The interpretation of the braids as orbits of the ends of string world sheets is possible and the braiding of the string ends brings in topological QFT aspect. Of utmost importance is that this identification is also general coordinate invariant and guarantees that WCW Kähler function is rational function of general coordinate invariant variables and thus allows p-adicization.

   This option leaves no room for the choice of the ends of braid strands unless one allows the length scale identified as the minimum length of the braid strand to vary in certain limits. This variation might only induce a gauge transformation $K \rightarrow K+f+\overline{f}$ of the WCW Kähler function. One can of consider the possibility that the braid points as points of the imbedding space are rational or algebraic numbers in some preferred coordinates of $H$ in the intersection of the real and p-adic worlds so that the original hypothesis would make sense in this intersection only.

For the latter option the number of solutions of the Chern-Simons Dirac equation for given spinorial quantum numbers is the number of braid points so that the number of fermionic oscillator operators for a given mode is same as the number of braid points and the anticommutation relations have a unique solution.

Symplectic fusion algebra [K8] might also be important element in quantization. The relationship between symplectic fusion algebra and its conjugate has not been characterized and one can consider the possibility that the algebra generators satisfy the conditions $e_m\overline{e}_n = \delta_{m,n}$. If induced spinor field at points of number theoretic braid defining the symplectic fusion algebra is multiplied by $e_m$ then the anti-commutation relations reduce automatically to a form in which anti-commutators at same point are involved. This would reduce the number of conditions to $8N_B$ from $8N_B^2$. The notion of finite measurement resolution could be used to defend this option.

### 5.8.3 QFT description of particle reactions at the level of braids

The overall view conforms with zero energy ontology in which hierarchy of causal diamonds ($CD$s) within $CD$s gives rise to a hierarchy of generalized Feynman diagrams and geometric description of the radiative corrections. Each sub-CD gives also rise to to zero energy states and thus particle reactions in its own time scale so that improvement of the time resolution brings in also new physics as it does also in reality.

The natural question is what happens to the braids at vertices.

1. The vision based on infinite primes led to the conclusion that the selection rules of arithmetic quantum field theory based on the conservation of the total number theoretic momentum $P = \sum n_i log(p_i)$ dictate the selection rules at the vertixes. For given $p_i$ the momentum $n_i log(p_i)$ can be shared between the outgoing lines and this allows several combinations of infinite primes in outgoing lines having interpretations in terms of singular coverings of $CD$ and $CP_2$.

2. What happens then to the braid strands? If the bosons and fermions with given $p_i$ are shared between several outgoing particles, does this require that the braid strands replicate? Or is their number preserved if one regards each braid strand as having $n_a$ *resp.* $n_b$ copies at the sheets of the corresponding coverings? This is required by the conservation of number theoretic momentum if one accepts the connection between the hierarchy of Planck constants and infinite primes.

3. The question raised already earlier is whether DNA replication could have a counterpart at the level of fundamental physics. The interpretation of the incoming lines of generalized Feynman diagram as representations of topological quantum computations and the virtual particle lines as representations of quantum communications would support this picture. The no-cloning theorem [B9] would hold true since exact copies of quantum states would not be possible by the conservation of the number theoretical momentum. One could however say that the bosonic occupation number $n_i$ means the presence of $n_i$-fold copy of same piece of information so that the sharing of information by sharing the pages of the singular covering associated with $n_i$ would be possible in the limits posed by the values of $n_i$. Note again that the identification $n_i = n_a$ or $n_i = n_b$ (two infinite primes characterize the quantum state) makes sense only if only one of the p-adic primes associated with the 3-surface is realized as a physical state since the identification forces the selection of the covering. The quantum model for DNA based on hierarchy of Planck constants [K59] inspires the question whether DNA replication could be actually accompanied by its proposed counterpart at the fundamental level defining the fundamental information transfer process.

4. The localization of the quantum numbers to braid strands suggests that braid ends of a given braid continue to one particular line or more generally, are shared between several lines. This condition is quite strong since without additional quantization conditions the ends of the braids

of outgoing particles do not co-incide with the ends of the incoming braid. These kind of quantization conditions would conform with the generalized Bohr orbit property of light-like 3-surfaces.

5. Without these quantization conditions one meets the challenge of calculating the anticommutators of fermionic oscillator operators associated with non-co-inciding points of the incoming and outgoing braids. This raises the question whether one should regard the quantizations of induced spinor fields based on the $L_{min}$ as one possible gauge only and allow the variation of $L_{min}$ in some limits. If these quantizations are equivalent, the fermionic oscillator operators would be unitarily related. How to deduce this unitary transformation would be the non-trivial problem and it seems that the simpler picture is much more attractive.

This picture means that particle reactions occur at several levels which brings in mind a kind of universal mimicry inspired by Universe as a Universal Computer hypothesis. Particle reactions in QFT sense correspond to the reactions for the number theoretic braids inside partons. This level seems to be the simplest one to describe mathematically. At parton level particle reactions correspond to generalized Feynman diagrams obtained by gluing partonic 3-surfaces along their ends at vertices. Particle reactions are realized also at the level of 4-D space-time surfaces. One might hope that this multiple realization could code the dynamics already at the simple level of single partonic 3-surface.

### 5.8.4 How do generalized braid diagrams relate to the perturbation theory?

The association of generalized braid diagrams characterized by infinite primes to the incoming and outgoing partonic legs and internal lines of the generalized Feynman diagrams forces to ask whether the generalized braid diagrams could give rise to a counterpart of perturbation theoretical formalism via the functional integral over configuration space degrees of freedom.

The basic question is how the functional integral over configuration space degrees of freedom relates to the generalized braid diagrams.

1. If one believes in perturbation theoretic approach, the basic conjecture motivated also number theoretically is that radiative corrections in this sense sum up to zero for critical values of Kähler coupling strength and Kähler function codes radiative corrections to classical physics via the dependence of the scale of $M^4$ metric on Planck constant. Cancelation could occur only for critical values of Kähler coupling strength $\alpha_K$: for general values of $\alpha_K$ the cancellation would require separate vanishing of each term in the sum and does not occur.

   In perturbative approach the expression of Kähler function as Chern-Simons action could be used and propagator would correspond to the inverse of the 1-1 part of the second variation of the Chern-Simons action with respect to complex WCW coordinates evaluated allowing only the extrema of Chern-Simons action for the ends of space-time surface and for wormhole throats. One would have perturbation theory for a sum over maxima of Kähler function. From the expression of the Kähler function as Dirac determinant the maxima would correspond to the local minima of $L_p = \sqrt{p}L_{min}$ for a given infinite prime. The connection between Chern-Simons representation and Dirac determinant representation of Kähler function would be obviously highly desirable.

2. The possibility to define WCW functional integral in terms of harmonic analysis for infinite-dimensional spaces leads to a non-perturbative approach to functional integration allowing also a generalization the p-adic context [K54] . In this approach there is no need to make additional assumptions.

For both cases the assignment of the collection of braids characterized by pairs of infinite primes allows to organize the generalized Feynman diagrams into a sum of generalized Feynman diagrams and for each diagram type the exponent of Kähler function - if given by the Dirac determinant- would be simply the product $\prod_i L_{p_i}^{-1}$, $L_p = \sqrt{p}L_{min}$. One should perform a sum over different infinite primes in the internal lines subject to the conservation of the total number theoretic momenta. The conservation of the incoming number theoretic momentum would allow only a finite number of configurations for the intermediate lines. For the approach based on harmonic analysis the expression of the Kähler function in terms of the Dirac determinant would be optimal since it is manifestly algebraic function.

Both approaches involve a perturbative summation in the sense of introducing sub-$CD$s with time scales coming as $2^{-n}$ powers of the time scale of $CD$ defining the infrared cutoff.

1. The addition of zero energy insertions corresponding to sub-$CD$s as radiative corrections allows to improve measurement resolution. Hence a connection with QFT type Feyman diagram expansion would be obtained and Connes tensor product would have a practical computational realization.

2. The time scale resolution defined by the temporal distance between the tips of the causal diamond defined by the future and past light-cones applies to the addition of zero energy sub-states and one obtains a direct connection with p-adic length scale evolution of coupling constants since the time scales in question naturally come as negative powers of two. More precisely, p-adic primes near power of two are very natural since the coupling constant evolution comes in powers of two of fundamental 2-adic length scale.

### 5.8.5 How p-adic coupling constant evolution and p-adic length scale hypothesis emerge?

The condition $T_n = 2^n T_0$ would assign to the hierarchy of $CD$s as hierarchy of time scales coming as octaves. A weaker condition would be $T_p = pT_0$, $p$ prime, and would assign all secondary p-adic time scales to the size scale hierarchy of $CD$s.

One can wonder how this picture relates to the earlier hypothesis that p-adic length coupling constant evolution. Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^n T_0$ induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, $R$ $CP_2$ length scale? This looks like an attractive idea but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of $k$ are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time $t$ satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces $X^2$ are as 2-D dynamical systems random apart from light-likeness of their orbit. For $CP_2$ type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in $M^4$. The orbits of Brownian particle would now correspond to light-like geodesics $\gamma_3$ at $X^3$. The projection of $\gamma_3$ to a time=constant section $X^2 \subset X^3$ would define the 2-D path $\gamma_2$ of the Brownian particle. The $M^4$ distance $r$ between the end points of $\gamma_2$ would be given $r^2 = Dt$. The favored values of $t$ would correspond to $T_n = 2^n T_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^k T_0$ for $D = R^2/T_0$. Since only $CP_2$ scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance, in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around .1 eV would correspond to $L(169) \simeq 5$ $\mu$m (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of $X^3$ so that p-adic prime $p$ would indeed be an inherent property of $X^3$. For $T_p = pT_0$ the above argument is not enough for p-adic length scale hypothesis and p-adic length scale hypothesis might be seen as an outcome of a process analogous to natural selection. Resonance like effect favoring octaves of a fundamental frequency might be in question. In this case, $p$ would a property of $CD$ and all light-like 3-surfaces inside it and also that corresponding sector of configuration space.

## 5.9 How to define generalized Feynman diagrams?

S-matrix codes to a high degree the predictions of quantum theories. The longstanding challenge of TGD has been to construct or at least demonstrate the mathematical existence of S-matrix- or actually M-matrix which generalizes this notion in zero energy ontology (ZEO) [K44] . This work has led to the notion of generalized Feynman diagram and the challenge is to give a precise mathematical meaning for this object. The attempt to understand the counterpart of twistors in TGD framework [K61] has inspired several key ideas in this respect but it turned out that twistors themselves need not be absolutely necessary in TGD framework.

1. The notion of generalized Feyman diagram defined by replacing lines of ordinary Feynman diagram with light-like 3-surfaces (elementary particle sized wormhole contacts with throats carrying quantum numbers) and vertices identified as their 2-D ends - I call them partonic 2-surfaces is central. Speaking somewhat loosely, generalized Feynman diagrams (plus background space-time sheets) define the "world of classical worlds" (WCW). These diagrams involve the analogs of stringy diagrams but the interpretation is different: the analogs of stringy loop diagrams have interpretation in terms of particle propagating via two different routes simultaneously (as in the classical double slit experiment) rather than as a decay of particle to two particles. For stringy diagrams the counterparts of vertices are singular as manifolds whereas the entire diagrams are smooth. For generalized Feynman diagrams vertices are smooth but entire diagrams represent singular manifolds just like ordinary Feynman diagrams do. String like objects however emerge in TGD and even ordinary elementary particles are predicted to be magnetic flux tubes of length of order weak gauge boson Compton length with monopoles at their ends as shown in accompanying article. This stringy character should become visible at LHC energies.

2. Zero energy ontology (ZEO) and causal diamonds (intersections of future and past directed lightcones) is second key ingredient. The crucial observation is that in ZEO it is possible to identify off mass shell particles as pairs of on mass shell particles at throats of wormhole contact since both positive and negative signs of energy are possible. The propagator defined by modified Dirac action does not diverge (except for incoming lines) although the fermions at throats are on mass shell. In other words, the generalized eigenvalue of the modified Dirac operator containing a term linear in momentum is non-vanishing and propagator reduces to $G = i/\lambda\gamma$, where $\gamma$ is so called modified gamma matrix in the direction of stringy coordinate [K9] . This means opening of the black box of the off mass shell particle-something which for some reason has not occurred to anyone fighting with the divergences of quantum field theories.

3. A powerful constraint is number theoretic universality requiring the existence of Feynman amplitudes in all number fields when one allows suitable algebraic extensions: roots of unity are certainly required in order to realize p-adic counter parts of plane waves. Also imbedding space, partonic 2-surfaces and WCW must exist in all number fields and their extensions. These constraints are enormously powerful and the attempts to realize this vision have dominated quantum TGD for last two decades.

4. Representation of 8-D gamma matrices in terms of octonionic units and 2-D sigma matrices is a further important element as far as twistors are considered [K61] . Modified gamma matrices at space-time surfaces are quaternionic/associative and allow a genuine matrix representation. As a matter fact, TGD and WCW can be formulated as study of associative local sub-algebras of the local Clifford algebra of 8-D imbedding space parameterized by quaternionic space-time surfaces. Central conjecture is that quaternionic 4-surfaces correspond to preferred extremals of Kähler action [K9] identified as critical ones (second variation of Kähler action vanishes for infinite number of deformations defining super-conformal algebra) and allow a slicing to string worldsheets parametrized by points of partonic 2-surfaces.

5. As far as twistors are considered, the first key element is the reduction of the octonionic twistor structure to quaternionic one at space-time surfaces and giving effectively 4-D spinor and twistor structure for quaternionic surfaces.

Quite recently quite a dramatic progress took place in this approach [K61] .

1. The progress was stimulated by the simple observation that on mass shell property puts enormously strong kinematic restrictions on the loop integrations. With mild restrictions on the number of parallel fermion lines appearing in vertices (there can be several since fermionic oscillator operator algebra defining SUSY algebra generates the parton states)- all loops are manifestly finite and if particles has always mass -say small p-adic thermal mass also in case of massless particles and due to IR cutoff due to the presence largest CD- the number of diagrams is finite. Unitarity reduces to Cutkosky rules [B18] automatically satisfied as in the case of ordinary Feynman diagrams.

2. Ironically, twistors which stimulated all these development do not seem to be absolutely necessary in this approach although they are of course possible. Situation changes if one does not assume small p-adically thermal mass due to the presence of massless particles and one must sum infinite number of diagrams. Here a potential problem is whether the infinite sum respects the algebraic extension in question.

This is about fermionic and momentum space aspects of Feynman diagrams but not yet about the functional (not path-) integral over small deformations of the partonic 2-surfaces. The basic challenges are following.

1. One should perform the functional integral over WCW degrees of freedom for fixed values of on mass shell momenta appearing in the internal lines. After this one must perform integral or summation over loop momenta. Note that the order is important since the space-time surface assigned to the line carries information about the quantum numbers associated with the line by quantum classical correspondence realized in terms of modified Dirac operator.

2. One must define the functional integral also in the p-adic context. p-Adic Fourier analysis relying on algebraic continuation raises hopes in this respect. p-Adicity suggests strongly that the loop momenta are discretized and ZEO predicts this kind of discretization naturally.

It indeed seems that the functional integrals over WCW could be carried out at general level both in real and p-adic context. This is due to the symmetric space property (maximal number of isometries) of WCW required by the mere mathematical existence of Kähler geometry [K24] in infinite-dimensional context already in the case of much simpler loop spaces [A44] .

1. The p-adic generalization of Fourier analysis allows to algebraize integration- the horrible looking technical challenge of p-adic physics- for symmetric spaces for functions allowing the analog of discrete Fourier decomposion. Symmetric space property is indeed essential also for the existence of Kähler geometry for infinite-D spaces as was learned already from the case of loop spaces. Plane waves and exponential functions expressible as roots of unity and powers of p multiplied by the direct analogs of corresponding exponent functions are the basic building bricks and key functions in harmonic analysis in symmetric spaces. The physically unavoidable finite measurement resolution corresponds to algebraically unavoidable finite algebraic dimension of algebraic extension of p-adics (at least some roots of unity are needed). The cutoff in roots of unity is very reminiscent to that occurring for the representations of quantum groups and is certainly very closely related to these as also to the inclusions of hyper-finite factors of type II¡sub¿1¡/sub¿ defining the finite measurement resolution.

2. WCW geometrization reduces to that for a single line of the generalized Feynman diagram defining the basic building brick for WCW. Kähler function decomposes to a sum of "kinetic" terms associated with its ends and interaction term associated with the line itself. p-Adicization boils down to the condition that Kähler function, matrix elements of Kähler form, WCW Hamiltonians and their super counterparts, are rational functions of complex WCW coordinates just as they are for those symmetric spaces that I know of. This allows straightforward continuation to p-adic context.

3. As far as diagrams are considered, everything is manifestly finite as the general arguments (non-locality of Kähler function as functional of 3-surface) developed two decades ago indeed allow to expect. General conditions on the holomorphy properties of the generalized eigenvalues $\lambda$ of the modified Dirac operator can be deduced from the conditions that propagator decomposes to a

sum of products of harmonics associated with the ends of the line and that similar decomposition takes place for exponent of Kähler action identified as Dirac determinant. This guarantees that the convolutions of propagators and vertices give rise to products of harmonic functions which can be Glebsch-Gordanized to harmonics and only the singlet contributes to the WCW integral in given vertex. The still unproven central conjecture is that Dirac determinant equals the exponent of Kähler function.

In the following this vision about generalized Feynman diagrams is discussed in more detail.

### 5.9.1 Questions

The goal is a proposal for how to perform the integral over WCW for generalized Feynman digrams and the best manner to proceed to to this goal is by making questions.

#### What does finite measurement resolution mean?

The first question is what finite measurement resolution means.

1. One expects that the algebraic continuation makes sense only for a finite measurement resolution in which case one obtains only finite sums of what one might hope to be algebraic functions. The finiteness of the algebraic extension would be in fact equivalent with the finite measurement resolution.

2. Finite measurement resolution means a discretization in terms of number theoretic braids. p-Adicization condition suggests that that one must allow only the number theoretic braids. For these the ends of braid at boundary of $CD$ are algebraic points of the imbedding space. This would be true at least in the intersection of real and p-adic worlds.

3. The question is whether one can localize the points of the braid. The necessity to use momentum eigenstates to achieve quantum classical correspondence in the modified Dirac action [K9] suggests however a delocalization of braid points, that is wave function in space of braid points. In real context one could allow all possible choices for braid points but in p-adic context only algebraic points are possible if one wants to replace integrals with sums. This implies finite measurement resolution analogous to that in lattice. This is also the only possibility in the intersection of real and p-adic worlds.

   A non-trivial prediction giving a strong correlation between the geometry of the partonic 2-surface and quantum numbers is that the total number $n_F + n_{\overline{F}}$ of fermions and antifermions is bounded above by the number $n_{alg}$ of algebraic points for a given partonic 2-surface: $n_F + n_{\overline{F}} \leq n_{alg}$. Outside the intersection of real and p-adic worlds the problematic aspect of this definition is that small deformations of the partonic 2-surface can radically change the number of algebraic points unless one assumes that the finite measurement resolution means restriction of WCW to a sub-space of algebraic partonic surfaces.

4. One has also a discretization of loop momenta if one assumes that virtual particle momentum corresponds to ZEO defining rest frame for it and from the discretization of the relative position of the second tip of CD at the hyperboloid isometric with mass shell. Only the number of braid points and their momenta would matter, not their positions. The measurement interaction term in the modified Dirac action gives coupling to the space-time geometry and Kähler function through generalized eigenvalues of the modified Dirac operator with measurement interaction term linear in momentum and in the color quantum numbers assignable to fermions [K9] .

#### How to define integration in WCW degrees of freedom?

The basic question is how to define the integration over WCW degrees of freedom.

1. What comes mind first is Gaussian perturbation theory around the maxima of Kähler function. Gaussian and metric determinants cancel each other and only algebraic expressions remain. Finiteness is not a problem since the Kähler function is non-local functional of 3-surface so that no local interaction vertices are present. One should however assume the vanishing of loops

required also by algebraic universality and this assumption look unrealistic when one considers more general functional integrals than that of vacuum functional since free field theory is not in question. The construction of the inverse of the WCW metric defining the propagator is also a very difficult challenge. Duistermaat-Hecke theorem states that something like this known as localization might be possible and one can also argue that something analogous to localization results from a generalization of mean value theorem.

2. Symmetric space property is more promising since it might reduce the integrations to group theory using the generalization of Fourier analysis for group representations so that there would be no need for perturbation theory in the proposed sense. In finite measurement resolution the symmetric spaces involved would be finite-dimensional. Symmetric space structure of WCW could also allow to define p-adic integration in terms of p-adic Fourier analysis for symmetric spaces. Essentially algebraic continuation of the integration from the real case would be in question with additional constraints coming from the fact that only phase factors corresponding to finite algebraic extensions of rationals are used. Cutoff would emerge automatically from the cutoff for the dimension of the algebraic extension.

**How to define generalized Feynman diagrams?**

Integration in symmetric spaces could serve as a model at the level of WCW and allow both the understanding of WCW integration and p-adicization as algebraic continuation. In order to get a more realistic view about the problem one must define more precisely what the calculation of the generalized Feynman diagrams means.

1. WCW integration must be carried out separately for all values of the momenta associated with the internal lines. The reason is that the spectrum of eigenvalues $\lambda_i$ of the modified Dirac operator $D$ depends on the momentum of line and momentum conservation in vertices translates to a correlation of the spectra of $D$ at internal lines.

2. For tree diagrams algebraic continuation to the p-adic context if the expression involves only the replacement of the generalized eigenvalues of $D$ as functions of momenta with their p-adic counterparts besides vertices. If these functions are algebraically universal and expressible in terms of harmonics of symmetric space , there should be no problems.

3. If loops are involved, one must integrate/sum over loop momenta. In p-adic context difficulties are encountered if the spectrum of the momenta is continuous. The integration over on mass shell loop momenta is analogous to the integration over sub-CDs, which suggests that internal line corresponds to a $sub - CD$ in which it is at rest. There are excellent reasons to believe that the moduli space for the positions of the upper tip is a discrete subset of hyperboloid of future light-cone. If this is the case, the loop integration indeed reduces to a sum over discrete positions of the tip. p-Adizication would thus give a further good reason why for zero energy ontology.

4. Propagator is expressible in terms of the inverse of generalized eigenvalue and there is a sum over these for each propagator line. At vertices one has products of WCW harmonics assignable to the incoming lines. The product must have vanishing quantum numbers associated with the phase angle variables of WCW. Non-trivial quantum numbers of the WCW harmonic correspond to WCW quantum numbers assignable to excitations of ordinary elementary particles. WCW harmonics are products of functions depending on the "radial" coordinates and phase factors and the integral over the angles leaves the product of the first ones analogous to Legendre polynomials $P_{l,m}$, These functions are expected to be rational functions or at least algebraic functions involving only square roots.

5. In ordinary QFT incoming and outgoing lines correspond to propagator poles. In the recent case this would mean that the generalized eigenvalues $\lambda = 0$ characterize them. Internal lines coming as pairs of throats of wormhole contacts would be on mass shell with respect to momentum but off shell with respect to $\lambda$.

### 5.9.2 Generalized Feynman diagrams at fermionic and momentum space level

Negative energy ontology has already led to the idea of interpreting the virtual particles as pairs of positive and negative energy wormhole throats. Hitherto I have taken it as granted that ordinary Feynman diagrammatics generalizes more or less as such. It is however far from clear what really happens in the verties of the generalized Feynmann diagrams. The safest approach relies on the requirement that unitarity realized in terms of Cutkosky rules in ordinary Feynman diagrammatics allows a generalization. This requires loop diagrams. In particular, photon-photon scattering can take place only via a fermionic square loop so that it seems that loops must be present at least in the topological sense.

One must be however ready for the possibility that something unexpectedly simple might emerge. For instance, the vision about algebraic physics allows naturally only finite sums for diagrams and does not favor infinite perturbative expansions. Hence the true believer on algebraic physics might dream about finite number of diagrams for a given reaction type. For simplicity generalized Feynman diagrams without the complications brought by the magnetic confinement since by the previous arguments the generalization need not bring in anything essentially new.

The basic idea of duality in early hadronic models was that the lines of the dual diagram representing particles are only re-arranged in the vertices. This however does not allow to get rid of off mass shell momenta. Zero energy ontology encourages to consider a stronger form of this principle in the sense that the virtual momenta of particles could correspond to pairs of on mass shell momenta of particles. If also interacting fermions are pairs of positive and negative energy throats in the interaction region the idea about reducing the construction of Feynman diagrams to some kind of lego rules might work.

#### Virtual particles as pairs of on mass shell particles in ZEO

The first thing is to try to define more precisely what generalized Feynman diagrams are. The direct generalization of Feynman diagrams implies that both wormhole throats and wormhole contacts join at vertices.

1. A simple intuitive picture about what happens is provided by diagrams obtained by replacing the points of Feynman diagrams (wormhole contacts) with short lines and imagining that the throats correspond to the ends of the line. At vertices where the lines meet the incoming on mass shell quantum numbers would sum up to zero. This approach leads to a straightforward generalization of Feynman diagrams with virtual particles replaced with pairs of on mass shell throat states of type $++$, $--$, and $+-$. Incoming lines correspond to $++$ type lines and outgoing ones to $--$ type lines. The first two line pairs allow only time like net momenta whereas $+-$ line pairs allow also space-like virtual momenta. The sign assigned to a given throat is dictated by the the sign of the on mass shell momentum on the line. The condition that Cutkosky rules generalize as such requires $++$ and $--$ type virtual lines since the cut of the diagram in Cutkosky rules corresponds to on mass shell outgoing or incoming states and must therefore correspond to $++$ or $--$ type lines.

2. The basic difference as compared to the ordinary Feynman diagrammatics is that loop integrals are integrals over mass shell momenta and that all throats carry on mass shell momenta. In each vertex of the loop mass incoming on mass shell momenta must sum up to on mass shell momentum. These constraints improve the behavior of loop integrals dramatically and give excellent hopes about finiteness. It does not however seem that only a finite number of diagrams contribute to the scattering amplitude besides tree diagrams. The point is that if a the reactions $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$,, where $N_i$ denote particle numbers, are possible in a common kinematical region for $N_2$-particle states then also the diagrams $N_1 \rightarrow N_2 \rightarrow N_2 \rightarrow N_3$ are possible. The virtual states $N_2$ include all all states in the intersection of kinematically allow regions for $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$. Hence the dream about finite number possible diagrams is not fulfilled if one allows massless particles. If all particles are massive then the particle number $N_2$ for given $N_1$ is limited from above and the dream is realized.

3. For instance, loops are not possible in the massless case or are highly singular (bringing in mind twistor diagrams) since the conservation laws at vertices imply that the momenta are parallel.

In the massive case and allowing mass spectrum the situation is not so simple. As a first example one can consider a loop with three vertices and thus three internal lines. Three on mass shell conditions are present so that the four-momentum can vary in 1-D subspace only. For a loop involving four vertices there are four internal lines and four mass shell conditions so that loop integrals would reduce to discrete sums. Loops involving more than four vertices are expected to be impossible.

4. The proposed replacement of the elementary fermions with bound states of elementary fermions and monopoles $X_{\pm}$ brings in the analog of stringy diagrammatics. The 2-particle wave functions in the momentum degrees of freedom of fermiona and $X_{\pm}$ migh allow more flexibility and allow more loops. Note however that there are excellent hopes about the finiteness of the theory also in this case.

**Loop integrals are manifestly finite**

One can make also more detailed observations about loops.

1. The simplest situation is obtained if only 3-vertices are allowed. In this case conservation of momentum however allows only collinear momenta although the signs of energy need not be the same. Particle creation and annihilation is possible and momentum exchange is possible but is always light-like in the massless case. The scattering matrices of supersymmetric YM theories would suggest something less trivial and this raises the question whether something is missing. Magnetic monopoles are an essential element of also these theories as also massivation and symmetry breaking and this encourages to think that the formation of massive states as fermion $X_{\pm}$ pairs is needed. Of course, in TGD framework one has also high mass excitations of the massless states making the scattering matrix non-trivial.

2. In YM theories on mass shell lines would be singular. In TGD framework this is not the case since the propagator is defined as the inverse of the 3-D dimensional reduction of the modified Dirac operator $D$ containing also coupling to four-momentum (this is required by quantum classical correspondence and guarantees stringy propagators),

$$\begin{aligned} D &= i\hat{\Gamma}^{\alpha}p_{\alpha} + \hat{\Gamma}^{\alpha}D_{\alpha} \ , \\ p_{\alpha} &= p_k\partial_{\alpha}h^k \ . \end{aligned} \tag{5.9.1}$$

The propagator does not diverge for on mass shell massless momenta and the propagator lines are well-defined. This is of course of essential importance also in general case. Only for the incoming lines one can consider the possibility that 3-D Dirac operator annihilates the induced spinor fields. All lines correspond to generalized eigenstates of the propagator in the sense that one has $D_3\Psi = \lambda\gamma\Psi$, where $\gamma$ is modified gamma matrix in the direction of the stringy coordinate emanating from light-like surface and $D_3$ is the 3-dimensional dimensional reduction of the 4-D modified Dirac operator. The eigenvalue $\lambda$ is analogous to energy. Note that the eigenvalue spectrum depends on 4-momentum as a parameter.

3. Massless incoming momenta can decay to massless momenta with both signs of energy. The integration measure $d^2k/2E$ reduces to $dx/x$ where $x \geq 0$ is the scaling factor of massless momentum. Only light-like momentum exchanges are however possible and scattering matrix is essentially trivial. The loop integrals are finite apart from the possible delicacies related to poles since the loop integrands for given massless wormhole contact are proportional to $dx/x^3$ for large values of $x$.

4. Irrrespective of whether the particles are massless or not, the divergences are obtained only if one allows too high vertices as self energy loops for which the number of momentum degrees of freedom is $3N-4$ for $N$-vertex. The construction of SUSY limit of TGD in [K20] led to the conclusion that the parallelly propagating $N$ fermions for given wormhole throat correspond to a product of N fermion propagators with same four-momentum so that for fermions and ordinary bosons one has the standard behavior but for $N > 2$ non-standard so that these excitations are

not seen as ordinary particles. Higher vertices are finite only if the total number $N_F$ of fermions propagating in the loop satisfies $N_F > 3N-4$. For instance, a 4-vertex from which $N=2$ states emanate is finite.

**Taking into account magnetic confinement**

What has been said above is not quite enough. The weak form of electric-magnetic duality [B8] leads to the picture about elementary particles as pairs of magnetic monopoles inspiring the notions of weak confinement based on magnetic monopole force. Also color confinement would have magnetic counterpart. This means that elementary particles would behave like string like objects in weak boson length scale. Therefore one must also consider the stringy case with wormhole throats replaced with fermion-$X_\pm$ pairs ($X_\pm$ is electromagnetically neutral and $\pm$ refers to the sign of the weak isospin opposite to that of fermion) and their super partners.

1. The simplest assumption in the stringy case is that fermion-$X_\pm$ pairs behave as coherent objects, that is scatter elastically. In more general case only their higher excitations identifiable in terms of stringy degrees of freedom would be created in vertices. The massivation of these states makes possible non-collinear vertices. An open question is how the massivation fermion-$X_\pm$ pairs relates to the existing TGD based description of massivation in terms of Higgs mechanism and modified Dirac operator.

2. Mass renormalization could come from self energy loops with negative energy lines as also vertex normalization. By very general arguments supersymmetry implies the cancellation of the self energy loops but would allow non-trivial vertex renormalization [K20] .

3. If only 3-vertices are allowed, the loops containing only positive energy lines are possible if on mass shell fermion-$X_\pm$ pair (or its superpartner) can decay to a pair of positive energy pair particles of same kind. Whether this is possible depends on the masses involved. For ordinary particles these decays are not kinematically possible below intermediate boson mass scale (the decays $F_1 \rightarrow F_2 + \gamma$ are forbidden kinematically or by the absence of flavor changing neutral currents whereas intermediate gauge bosons can decay to on mass shell fermion-antifermion pair).

4. The introduction of IR cutoff for 3-momentum in the rest system associated with the largest $CD$ (causal diamond) looks natural as scale parameter of coupling constant evolution and p-adic length scale hypothesis favors the inverse of the size scale of $CD$ coming in powers of two. This parameter would define the momentum resolution as a discrete parameter of the p-adic coupling constant evolution. This scale does not have any counterpart in standard physics. For electron, $d$ quark, and $u$ quark the proper time distance between the tips of $CD$ corresponds to frequency of 10 Hz, 1280 Hz, and 160 Hz: all these frequencies define fundamental bio-rhythms [K16] .

These considerations have left completely untouched one important aspect of generalized Feynman diagrams: the necessity to perform a functional integral over the deformations of the partonic 2-surfaces at the ends of the lines- that is integration over WCW. Number theoretical universality requires that WCW and these integrals make sense also p-adically and in the following these aspects of generalized Feynman diagrams are discussed.

### 5.9.3 How to define integration and p-adic Fourier analysis, integral calculus, and p-adic counterparts of geometric objects?

p-Adic differential calculus exists and obeys essentially the same rules as ordinary differental calculus. The only difference from real context is the existence of p-adic pseudoconstants: any function which depends on finite number of pinary digits has vanishing p-adic derivative. This implies non-determinism of p-adic differerential equations. One can defined p-adic integral functions using the fact that indefinite integral is the inverse of differentiation. The basis problem with the definite integrals is that p-adic numbers are not well-ordered so that the crucial ordering of the points of real axis in definite integral is not unique. Also p-adic Fourier analysis is problematic since direct counterparts of ep(ix) and trigonometric functions are not periodic. Also exp(-x) fails to converse exponentially since it has p-adic norm equal to 1. Note also that these functions exists only when the p-adic norm of $x$ is smaller than 1.

The following considerations support the view that the p-adic variant of a geometric objects, integration and p-adic Fourier analysis exists but only when one considers highly symmetric geometric objects such as symmetric spaces. This is wellcome news from the point of view of physics. At the level of space-time surfaces this is problematic. The field equations associated with Kähler action and modified Dirac equation make sense. Kähler action defined as integral over p-adic space-time surface fails to exist. If however the Kähler function identified as Kähler for a preferred extremal of Kähler action is rational or algebraic function of preferred complex coordinates of WCW with ratonal coefficients, its p-adic continuation is expected to exist.

### Circle with rotational symmetries and its hyperbolic counterparts

Consider first circle with emphasis on symmetries and Fourier analysis.

1. In this case angle coordinate $\phi$ is the natural coordinate. It however does not make sense as such p-adically and one must consider either trigonometric functions or the phase $exp(i\phi)$ instead. If one wants to do Fourier analysis on circle one must introduce roots $U_{n,N} = exp(in2\pi/N)$ of unity. This means discretization of the circle. Introducing all roots $U_{n,p} = exp(i2\pi n/p)$, such that $p$ divides $N$, one can represent all $U_{k,n}$ up to $n = N$. Integration is naturally replaced with sum by using discrete Fourier analysis on circle. Note that the roots of unity can be expressed as products of powers of roots of unity $exp(in2\pi/p^k)$, where $p^k$ divides $N$.

2. There is a number theoretical delicacy involved. By Fermat's theorem $a^{p-1}\ mod\ p = 1$ for $a = 1, ...p-1$ for a given p-adic prime so that for any integer $M$ divisible by a factor of $p-1$ the $M$:th roots of unity exist as ordinary p-adic numbers. The problem disappears if these values of $M$ are excluded from the discretization for a given value of the p-adic prime. The manner to achieve this is to assume that $N$ contains no divisors of $p-1$ and is consistent with the notion of finite measurement resolution. For instance, $N = p^n$ is an especially natural choice guaranteing this.

3. The p-adic integral defined as a Fourier sum does not reduce to a mere discretization of the real integral. In the real case the Fourier coefficients must approach to zero as the wave vector $k = n2\pi/N$ increases. In the p-adic case the condition consistent with the notion of finite measurement resolution for angles is that the p-adic valued Fourier coefficients approach to zero as $n$ increases. This guarantees the p-adic convergence of the discrete approximation of the integral for large values of $N$ as $n$ increases. The map of p-adic Fourier coefficients to real ones by canonical identification could be used to relate p-adic and real variants of the function to each other.

This finding would suggests that p-adic geometries -in particular the p-adic counterpart of $CP_2$, are discrete. Variables which have the character of a radial coordinate are in natural manner p-adically continuous whereas phase angles are naturally discrete and described in terms of algebraic extensions. The conclusion is disappoing since one can quite well argue that the discrete structures can be regarded as real. Is there any manner to escape this conclusion?

1. Exponential function $exp(ix)$ exists p-adically for $|x|_p \leq 1/p$ but is not periodic. It provides representation of p-adic variant of circle as group $U(1)$. One obtains actually a hierarchy of groups $U(1)_{p,n}$ corresponding to $|x|_p \leq 1/p^n$. One could consider a generalization of phases as products $Exp_p(N, n2\pi/N + x) = exp(in2\pi n/N)exp(ix)$ of roots of unity and exponent functions with an imaginary exponent. This would assign to each root of unity p-adic continuum interpreted as the analog of the interval between two subsequent roots of unity at circle. The hierarchies of measurement resolutions coming as $2\pi/p^n$ would be naturally accompanied by increasingly smaller p-adic groups $U(1)_{p,n}$.

2. p-Adic integration would involve summation plus possibly also an integration over each p-adic variant of discretization interval. The summation over the roots of unity implies that the integral of $\int exp(inx)dx$ would appear for $n = 0$. Whatever the value of this integral is, it is compensated by a normalization factor guaranteing orthonormality.

3. If one interprets the p-adic coordinate as p-adic integer without the identification of points differing by a multiple of $n$ as different points the question whether one should require p-adic continuity arises. Continuity is obtained if $U_n(x + mp^m) = U_n(x)$ for large values of $m$. This is obtained if one has $n = p^k$. In the spherical geometry this condition is not needed and would mean quantization of angular momentum as $L = p^k$, which does not look natural. If representations of translation group are considered the condition is natural and conforms with the spirit of the p-adic length scale hypothesis.

The hyperbolic counterpart of circle corresponds to the orbit of point under Lorentz group in two 2-D Minkowski space. Plane waves are replaced with exponentially decaying functions of the coordinate $\eta$ replacing phase angle. Ordinary exponent function $exp(x)$ has unit p-adic norm when it exists so that it is not a suitable choice. The powers $p^n$ existing for p-adic integers however approach to zero for large values of $x = n$. This forces discretization of $\eta$ or rather the hyperbolic phase as powers of $p^x$, $x = n$. Also now one could introduce products of $Exp_p(nlog(p) + z) = p^n exp(x)$ to achieve a p-adic continuum. Also now the integral over the discretization interval is compensated by orthonormalization and can be forgotten. The integral of exponential function would reduce to a sum $\int Exp_p dx = \sum_k p^k = 1/(1-p)$. One can also introduce finite-dimensional but non-algebraic extensions of p-adic numbers allowing $e$ and its roots $e^{1/n}$ since $e^p$ exists p-adically.

**Plane with translational and rotational symmetries**

Consider first the situation by taking translational symmetries as a starting point. In this case Cartesian coordinates are natural and Fourier analysis based on plane waves is what one wants to define. As in the previous case, this can be done using roots of unity and one can also introduce p-adic continuum by using the p-adic variant of the exponent function. This would effectively reduce the plane to a box. As already noticed, in this case the quantization of wave vectors as multiples of $1/p^k$ is required by continuity.

One can take also rotational symmetries as a starting point. In this case cylindrical coordinates $(\rho, \phi)$ are natural.

1. Radial coordinate can have arbitrary values. If one wants to keep the connection $\rho = \sqrt{x^2 + y^2}$ with the Cartesian picture square root allowing extension is natural. Also the values of radial coordinate proportional to odd power of $p$ are problematic since one should introduce $\sqrt{p}$: is this extension internally consistent? Does this mean that the points $\rho \propto p^{2n+1}$ are excluded so that the plane decomposes to annuli?

2. As already found, angular momentum eigen states can be described in terms of roots of unity and one could obtain continuum by allowing also phases defined by p-adic exponent functions.

3. In radial direction one should define the p-adic variants for the integrals of Bessel functions and they indeed might make sense by algebraic continuation if one consistently defines all functions as Fourier expansions. Delta-function renormalization causes technical problems for a continuum of radial wave vectors. One could avoid the problem by using expontentially decaying variants of Bessel function in the regions far from origin, and here the already proposed description of the hyperbolic counterparts of plane waves is suggestive.

4. One could try to understand the situation also using Cartesian coordinates. In the case of sphere this is achieved by introducing two coordinate patches with Cartesian coordinates. Pythagorean phases are rational phases (orthogonal triangles for which all sides are integer valued) and form a dense set on circle. Complex rationals (orthogonal triangles with integer valued short sides) define a more general dense subset of circle. In both cases it is difficult to imagine a discretized version of integration over angles since discretization with constant angle incrrement is not possible.

**The case of sphere and more general symmetric space**

In the case of sphere spherical coordinates are favored by symmetry considerations. For spherical coordinates $sin(\theta)$ is analogous to the radial coordinate of plane. Legedre polynomials expressible as polynomials of $sin(\theta)$ and $cos(\theta)$ are expressible in terms of phases and the integration measure

$sin^2(\theta)d\theta d\phi$ reduces the integral of $S^2$ to summation. As before one can introduce also p-adic continuum. Algebraic cutoffs in both angular momentum $l$ and $m$ appear naturally. Similar cutoffs appear in the representations of quantum groups and there are good reasons to expect that these phenomena are correlated.

Exponent of Kähler function appears in the integration over configuration space. From the expression of Kähler gauge potential given by $A_\alpha = J_\alpha{}^\theta \partial_\theta K$ one obtains using $A_\alpha = cos(\theta)\delta_{\alpha,\phi}$ and $J_{\theta\phi} = sin(\theta)$ the expression $exp(K) = sin(\theta)$. Hence the exponent of Kähler function is expressible in terms of spherical harmonics.

The completion of the discretized sphere to a p-adic continuum- and in fact any symmetric space- could be performed purely group theoretically.

1. Exponential map maps the elements of the Lie-algebra to elements of Lie-group. This recipe generalizes to arbitrary symmetric space $G/H$ by using the Cartan decomposition $g = t + h$, $[h,h] \subset h$,$[h,t] \subset t$,$[t,t] \subset h$. The exponentiation of $t$ maps $t$ to $G/H$ in this case. The exponential map has a p-adic generalization obtained by considering Lie algebra with coefficients with p-adic norm smaller than one so that the p-adic exponent function exists. As a matter fact, one obtains a hierarchy of Lie-algebras corresponding to the upper bounds of the p-adic norm coming as $p^{-k}$ and this hierarchy naturally corresponds to the hierarchy of angle resolutions coming as $2\pi/p^k$. By introducing finite-dimensional transcendental extensions containing roots of $e$ one obtains also a hierarchy of p-adic Lie-algebras associated with transcendental extensions.

2. In particular, one can exponentiate the complement of the $SO(2)$ sub-algebra of $SO(3)$ Lie-algebra in p-adic sense to obtain a p-adic completion of the discrete sphere. Each point of the discretized sphere would correspond to a p-adic continuous variant of sphere as a symmetric space. Similar construction applies in the case of $CP_2$. Quite generally, a kind of fractal or holographic symmetric space is obtained from a discrete variant of the symmetric space by replacing its points with the p-adic symmetric space.

3. In the N-fold discretization of the coordinates of M-dimensional space $t$ one $(N-1)^M$ discretization volumes which is the number of points with non-vanishing $t$-coordinates. It would be nice if one could map the p-adic discretization volumes with non-vanishing $t$-coordinates to their positive valued real counterparts by applying canonical identification. By group invariance it is enough to show that this works for a discretization volume assignable to the origin. Since the p-adic numbers with norm smaller than one are mapped to the real unit interval, the p-adic Lie algebra is mapped to the unit cell of the discretization lattice of the real variant of $t$. Hence by a proper normalization this mapping is possible.

The above considerations suggest that the hierarchies of measurement resolutions coming as $\Delta\phi = 2\pi/p^n$ are in a preferred role. One must be however cautious in order to avoid too strong assumptions. The following arguments however support this identification.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where $M$ and $N$ are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of $M$ unless one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of $M$ correspond to different numbers of Fourier components. Otherwise themeasurement ersolution is just $\Delta\phi = 2\pi/p^n$. If one regards $N$ as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret $N$ as a p-adic integer and assume that state function reduction selects one particular prime (no superposition of quantum states with different p-adic topologies). For $N = p^n M$, where $M$ is not divisible by $p$, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M$ mod $p^n$. The phase would non-trivial only for p-adic primes appearing as factors in $N$. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$. One could assign to a given measurement resolution all the p-adic primes appearing as factors in $N$ so that the notion of

multi-p p-adicity would make sense. One can also consider the identification of the measurement resolution as $\Delta\phi = |N/M|_p = 2\pi/p^k$. This interpretation is supported by the approach based on infinite primes [K52] .

### What about integrals over partonic 2-surfaces and space-time surface?

One can of course ask whether also the integrals over partonic 2-surfaces and space-time surface could be p-adicized by using the proposed method of discretization. Consider first the p-adic counterparts of the integrals over the partonic 2-surface $X^2$.

1. WCW Hamiltonians and Kähler form are expressible using flux Hamiltonians defined in terms of $X^2$ integrals of $JH_A$, where $H_A$ is $\delta CD \times CP_2$ Hamiltonian, which is a rational function of the preferred coordinates defined by the exponentials of the coordinates of the sub-space $t$ in the appropriate Cartan algebra decomposition. The flux factor $J = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$ is scalar and does not actually depend on the induced metric.

2. The notion of finite measurement resolution would suggest that the discretization of $X^2$ is somehow induced by the discretization of $\delta CD \times CP_2$. The coordinates of $X^2$ could be taken to be the coordinates of the projection of $X^2$ to the sphere $S^2$ associated with $\delta M^4_\pm$ or to the homologically non-trivial geodesic sphere of $CP_2$ so that the discretization of the integral would reduce to that for $S^2$ and to a sum over points of $S^2$.

3. To obtain an algebraic number as an outcome of the summation, one must pose additional conditions guaranteing that both $H_A$ and $J$ are algebraic numbers at the points of discretization (recall that roots of unity are involved). Assume for definiteness that $S^2$ is $r_M = constant$ sphere. If the remaining preferred coordinates are functions of the preferred $S^2$ coordinates mapping phases to phases at discretization points, one obtains the desired outcome. These conditions are rather strong and mean that the various angles defining $CP_2$ coordinates -at least the two cyclic angle coordinates- are integer multiples of those assignable to $S^2$ at the points of discretization. This would be achieved if the preferred complex coordinates of $CP_2$ are powers of the preferred complex coordinate of $S^2$ at these points. One could say that $X^2$ is algebraically continued from a rational surface in the discretized variant of $\delta CD \times CP_2$. Furthermore, if the measurement resolutions come as $2\pi/p^n$ as p-adic continuity actually requires and if they correspond to the p-adic group $G_{p,n}$ for which group parameters satisfy $|t|_p \leq p^{-n}$, one can precisely characterize how a p-adic prime characterizes the real partonic 2-surface. This would be a fulfilment of one of the oldest dreams related to the p-adic vision.

A even more ambitious dream would be that even the integral of the Kähler action for preferred extremals could be defined using a similar procedure. The conjectured slicing of Minkowskian space-time sheets by string world sheets and partonic 2-surfaces encourages these hopes.

1. One could introduce local coordinates of $H$ at both ends of $CD$ by introducing a continuous slicing of $M^4 \times CP_2$ by the translates of $\delta M^4_\pm \times CP_2$ in the direction of the time-like vector connecting the tips of $CD$. As space-time coordinates one could select four of the eight coordinates defining this slicing. For instance, for the regions of the space-time sheet representable as maps $M^4 \to CP_2$ one could use the preferred $M^4$ time coordinate, the radial coordinate of $\delta M^4_+$, and the angle coordinates of $r_M = constant$ sphere.

2. Kähler action density should have algebraic values and this would require the strengthening of the proposed conditions for $X^2$ to apply to the entire slicing meaning that the discretized space-time surface is a rational surface in the discretized $CD \times CP_2$. If this condition applies to the entire space-time surface it would effectively mean the discretization of the classical physics to the level of finite geometries. This seems quite strong implication but is consistent with the preferred extremal property implying the generalized Bohr rules. The reduction of Kähler action to 3-dimensional boundary terms is implied by rather general arguments. In this case only the effective algebraization of the 3-surfaces at the ends of $CD$ and of wormhole throats is needed [K24] . By effective 2-dimensionality these surfaces cannot be chosen freely.

3. If Kähler function and WCW Hamiltonians are rational functions, this kind of additional conditions are not necessary. It could be that the integrals of defining Kähler action flux Hamiltonians make sense only in the intersection of real and p-adic worlds assumed to be relevant for the physics of living systems.

**Tentative conclusions**

These findings suggest following conclusions.

1. Exponent functions play a key role in the proposed p-adicization. This is not an accident since exponent functions play a fundamental role in group theory and p-adic variants of real geometries exist only under symmetries- possibly maximal possible symmetries- since otherwise the notion of Fourier analysis making possible integration does not exist. The inner product defined in terms of integration reduce for functions representable in Fourier basis to sums and can be carried out by using orthogonality conditions. Convolution involving integration reduces to a product for Fourier components. In the case of imbedding space and WCW these conditions are satisfied but for space-time surfaces this is not possible.

2. There are several manners to choose the Cartan algebra already in the case of sphere. In the case of plane one can consider either translations or rotations and this leads to different p-adic variants of plane. Also the realization of the hierarchy of Planck constants leads to the conclusion that the extended imbedding space and therefore also WCW contains sectors corresponding to different choices of quantization axes meaning that quantum measurement has a direct geometric correlate.

3. The above described 2-D examples represent symplectic geometries for which one has natural decomposition of coordinates to canonical pairs of cyclic coordinate (phase angle) and corresponding canonical conjugate coordinate. p-Adicization depends on whether the conjugate corresponds to an angle or noncompact coordinate. In both cases it is however possible to define integration. For instance, in the case of $CP_2$ one would have two canonically conjugate pairs and one can define the p-adic counterparts of $CP_2$ partial waves by generalizing the procedure applied to spherical harmonics. Products of functions expressible using partial waves can be decomposed by tensor product decomposition to spherical harmonics and can be integrated. In particular inner products can be defined as integrals. The Hamiltonians generating isometries are rational functions of phases: this inspires the hope that also WCW Hamiltonians also rational functions of preferred WCW coordinates and thus allow p-adic variants.

4. Discretization by introducing algebraic extensions is unavoidable in the p-adicization of geometrical objects but one can have p-adic continuum as the analog of the discretization interval and in the function basis expressible in terms of phase factors and p-adic counterparts of exponent functions. This would give a precise meaning for the p-adic counterparts of the imbedding space and WCW if the latter is a symmetric space allowing coordinatization in terms of phase angles and conjugate coordinates.

5. The intersection of p-adic and real worlds would be unique and correspond to the points defining the discretization.

### 5.9.4 Harmonic analysis in WCW as a manner to calculate WCW functional integrals

Previous examples suggest that symmetric space property, Kähler and symplectic structure and the use of symplectic coordinates consisting of canonically conjugate pairs of phase angles and corresponding "radial" coordinates are essential for WCW integration and p-adicization. Kähler function, the components of the metric, and therefore also metric determinant and Kähler function depend on the "radial" coordinates only and the possible generalization involves the identification the counterparts of the "radial" coordinates in the case of WCW.

**Conditions guaranteing the reduction to harmonic analysis**

The basic idea is that harmonic analysis in symmetric space allows to calculate the functional integral over WCW.

1. Each propagator line corresponds to a symmetric space defined as a coset space $G/H$ of the symplectic group and Kac-Moody group and one might hope that the proposed p-adicization works for it- at least when one considers the hierarchy of measurement resolutions forced by the finiteness of algebraic extensions. This coset space is as a manifold Cartesian product $(G/H) \times (G/H)$ of symmetric spaces $G/H$ associated with ends of the line. Kähler metric contains also an interaction term between the factors of the Cartesian product so that Kähler function can be said to reduce to a sum of "kinetic" terms and interaction term.

2. Effective 2-dimensionality and ZEO allow to treat the ends of the propagator line independently. This means an enormous simplification. Each line contributes besides propagator a piece to the exponent of Kähler action identifiable as interaction term in action and depending on the propagator momentum. This contribution should be expressible in terms of generalized spherical harmonics. Essentially a sum over the products of pairs of harmonics associated with the ends of the line multiplied by coefficients analogous to $1/(p^2-m^2)$ in the case of the ordinary propagator would be in question. The optimal situation is that the pairs are harmonics and their conjugates appear so that one has invariance under $G$ analogous to momentum conservation for the lines of ordinary Feynman diagrams.

3. Momentum conservation correlates the eigenvalue spectra of the modified Dirac operator $D$ at propagator lines [K9] . $G$-invariance at vertex dictates the vertex as the singlet part of the product of WCW harmonics associated with the vertex and one sums over the harmonics for each internal line. p-Adicization means only the algebraic continuation to real formulas to p-adic context.

4. The exponent of Kähler function depends on both ends of the line and this means that the geometries at the ends are correlated in the sense that that Kähler form contains interaction terms between the line ends. It is however not quite clear whether it contains separate "kinetic" or self interaction terms assignable to the line ends. For Kähler function the kinetic and interaction terms should have the following general expressions as functions of complex WCW coordinates:

$$\begin{aligned} K_{kin,i} &= \sum_n f_{i,n}(Z_i)\overline{f_{i,n}(Z_i)} + c.c \ , \\ K_{int} &= \sum_n g_{1,n}(Z_1)\overline{g_{2,n}(Z_2)} + c.c \ , i = 1,2 \ . \end{aligned} \tag{5.9.2}$$

   Here $K_{kin,i}$ define "kinetic" terms and $K_{int}$ defines interaction term. One would have what might be called holomorphic factorization suggesting a connection with conformal field theories.

   Symmetric space property -that is isometry invariance- suggests that one has

$$f_{i,n} = f_{2,n} \equiv f_n \ , \ g_{1,n} = g_{2,n} \equiv g_n \tag{5.9.3}$$

   such that the products are invariant under the group $H$ appearing in $G/H$ and therefore have opposite $H$ quantum numbers. The exponent of Kähler function does not factorize although the terms in its Taylor expansion factorize to products whose factors are products of holomorphic and antihilomorphic functions.

5. If one assumes that the exponent of Kähler function reduces to a product of eigenvalues of the modified Dirac operator eigenvalues must have the decomposition

$$\lambda_k = \prod_{i=1,2} exp\left[\sum_n c_{k,n} g_n(Z_i)\overline{g_n(Z_i)} + c.c\right] \times exp\left[\sum_n d_{k,n} g_n(Z_1)\overline{g_n(Z_2)} + c.c\right] \quad (5.9.4)$$

Hence also the eigenvalues coming from the Dirac propagators have also expansion in terms of $G/H$ harmonics so that in principle WCW integration would reduce to Fourier analysis in symmetric space.

### Generalization of WCW Hamiltonians

This picture requires a generalization of the view about configuration space Hamiltonians since also the interaction term between the ends of the line is present not taken into account in the previous approach.

1. The proposed representation of WCW Hamiltonians as flux Hamiltonians [K10, K9]

$$\begin{aligned} Q(H_A) &= \int H_A(1+K)Jd^2x \ , \\ J &= \epsilon^{\alpha\beta}J_{\alpha\beta} \ , \ J^{03}\sqrt{g_4} = KJ_{12} \ . \end{aligned} \quad (5.9.5)$$

   works for the kinetic terms only since $J$ cannot be the same at the ends of the line. The formula defining $K$ assumes weak form of self-duality ($^{03}$ refers to the coordinates in the complement of $X^2$ tangent plane in the 4-D tangent plane). $K$ is assumed to be symplectic invariant and constant for given $X^2$. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge $g_K$ gives the condition $K = g_K^2/\hbar$, where $g_K$ is Kähler coupling constant. Within experimental uncertainties one has $\alpha_K = g_K^2 4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where $\alpha_{em}$ is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

   The assumption that Poisson bracket of WCW Hamiltonians reduces to the level of imbedding space - in other words $\{Q(H_A), Q(H_B\} = Q(\{H_A, H_B\})$ - can be justified. One starts from the representation in terms of say flux Hamiltonians $Q(H_A)$ and *defines* $J_{A,B}$ as $J_{A,B} \equiv Q(\{H_A, H_B\})$. One has $\partial H_A/\partial t_B = \{H_B, H_A\}$, where $t_B$ is the parameter associated with the exponentiation of $H_B$. The inverse $J^{AB}$ of $J_{A,B} = \partial H_B/\partial t_A$ is expressible as $J^{A,B} = \partial t_A/\partial H_B$. From these formulas one can deduce by using chain rule that the bracket $\{Q(H_A), Q(H_B\} = \partial t_C Q(H_A) J^{CD} \partial t_D Q(H_B)$ of flux Hamiltonians equals to the flux Hamiltonian $Q(\{H_A, H_B\})$.

2. One should be able to assign to WCW Hamiltonians also a part corresponding to the interaction term. The symplectic conjugation associated with the interaction term permutes the WCW coordinates assignable to the ends of the line. One should reduce this apparently non-local symplectic conjugation (if one thinks the ends of line as separate objects) to a non-local symplectic conjugation for $\delta CD \times CP_2$ by identifying the points of lower and upper end of $CD$ related by time reflection and assuming that conjugation corresponds to time reflection. Formally this gives a well defined generalization of the local Poisson brackets between time reflected points at the boundaries of $CD$. The connection of Hermitian conjugation and time reflection in quantum field theories is is in accordance with this picture.

3. The only manner to proceed is to assign to the flux Hamiltonian also a part obtained by the replacement of the flux integral over $X^2$ with an integral over the projection of $X^2$ to a sphere $S^2$ assignable to the light-cone boundary or to a geodesic sphere of $CP_2$, which come as two varieties corresponding to homologically trivial and non-trivial spheres. The projection is defined as by the geodesic line orthogonal to $S^2$ and going through the point of $X^2$. The hierarchy of Planck constants assigns to $CD$ a preferred geodesic sphere of $CP_2$ as well as a unique sphere $S^2$ as a sphere for which the radial coordinate $r_M$ or the light-cone boundary defined uniquely is constant: this radial coordinate corresponds to spherical coordinate in the rest system defined

by the time-like vector connecting the tips of $CD$. Either spheres or possibly both of them could be relevant.

Recall that also the construction of number theoretic braids and symplectic QFT [K12] led to the proposal that braid diagrams and symplectic triangulations could be defined in terms of projections of braid strands to one of these spheres. One could also consider a weakening for the condition that the points of the number theoretic braid are algebraic by requiring only that the $S^2$ coordinates of the projection are algebraic and that these coordinates correspond to the discretization of $S^2$ in terms of the phase angles associated with $\theta$ and $\phi$.

This gives for the corresponding contribution of the WCW Hamiltonian the expression

$$Q(H_A)_{int} = \int_{S^2_\pm} H_A X \delta^2(s_+,s_-) d^2 s_\pm = \int_{P(X^2_+)\cap P(X^2_-)} \frac{\partial(s^1,s^2)}{\partial(x^1_\pm,x^2_\pm)} d^2 x_\pm \ . \quad (5.9.6)$$

Here the Poisson brackets between ends of the line using the rules involve delta function $\delta^2(s_+,s_-)$ at $S^2$ and the resulting Hamiltonians can be expressed as a similar integral of $H_{[A,B]}$ over the upper or lower end since the integral is over the intersection of $S^2$ projections.

The expression must vanish when the induced Kähler form vanishes for either end. This is achieved by identifying the scalar $X$ in the following manner:

$$\begin{aligned} X &= J_+^{kl} J^-_{kl} \ , \\ J_\pm^{kl} &= (1+K_\pm)\partial_\alpha s^k \partial_\beta s^l J_\pm^{\alpha\beta} \ . \end{aligned} \quad (5.9.7)$$

The tensors are lifts of the induced Kähler form of $X^2_\pm$ to $S^2$ (not $CP_2$).

4. One could of course ask why these Hamiltonians could not contribute also to the kinetic terms and why the brackets with flux Hamiltonians should vanish. This relate to how one *defines* the Kähler form. It was shown above that in case of flux Hamiltonians the definition of Kähler form as brackets gives the basic formula $\{Q(H_A),Q(H_B)\} = Q(\{H_A,H_B\}$ and same should hold true now. In the recent case $J_{A,B}$ would contain an interaction term defined in terms of flux Hamiltonians and the previous argument should go through also now by identifying Hamiltonians as sums of two contributions and by introducing the doubling of the coordinates $t_A$.

5. The quantization of the modified Dirac operator must be reconsidered. It would seem that one must add to the super-Hamiltonian completely analogous term obtained by replacing $(1+K)J$ with $X\partial(s^1,s^2)/\partial(x^1_\pm,x^2_\pm)$. Besides the anticommutation relations defining correct anticommutators to flux Hamiltonians, one should pose anticommutation relations consistent with the anticommutation relations of super Hamiltonians. In these anticommutation relations $(1+K)J\delta^2(x,y)$ would be replaced with $X\delta^2(s^+,s^-)$. This would guarantee that the oscillator operators at the ends of the line are not independent and that the resulting Hamiltonian reduces to integral over either end for $H_{[A,B]}$.

6. In the case of $CP_2$ the Hamiltonians generating isometries are rational functions. This should hold true also now so that p-adic variants of Hamiltonians as functions in WCW would make sense. This in turn would imply that the components of the WCW Kähler form are rational functions. Also the exponentiation of Hamiltonians make sense p-adically if one allows the exponents of group parameters to be functions $Exp_p(t)$.

**Does the expansion in terms of partial harmonics converge?**

The individual terms in the partial wave expansion seem to be finite but it is not at all clear whether the expansion in powers of $K$ actually converges.

1. In the proposed scenario one performs the expansion of the vacuum functional $exp(K)$ in powers of $K$ and therefore in negative powers of $\alpha_K$. In principle an infinite number of terms can be present. This is analogous to the perturbative expansion based on using magnetic monopoles as basic objects whereas the expansion using the contravariant Kähler metric as a propagator would be in positive powers of $\alpha_K$ and analogous to the expansion in terms of magnetically bound states of wormhole throats with vanishing net value of magnetic charge. At this moment one can only suggest various approaches to how one could understand the situation.

2. Weak form of self-duality and magnetic confinement could change the sitution. Performing the perturbation around magnetic flux tubes together with the assumed slicing of the space-time sheet by stringy world sheets and partonic 2-surfaces could mean that the perturbation corresponds to the action assignable to the electric part of Kähler form proportional to $\alpha_K$ by the weak self-duality. Hence by $K = 4\pi\alpha_K$ relating Kähler electric field to Kähler magnetic field the expansion would come in powers of a term containing sum of terms proportional to $\alpha_K^0$ and $\alpha_K$. This would leave to the scattering amplitudes the exponents of Kähler function at the maximum of Kähler function so that the non-analytic dependence on $\alpha_K$ would not disappear.

A further reason to be worried about is that the expansion containing infinite number of terms proportional to $\alpha_K^0$ could fail to converge.

1. This could be also seen as a reason for why magnetic singlets are unavoidable except perhaps for $\hbar < \hbar_0$. By the holomorphic factorization the powers of the interaction part of Kähler action in powers of $1/\alpha_K$ would naturally correspond to increasing and opposite net values of the quantum numbers assignable to the WCW phase coordinates at the ends of the propagator line. The magnetic bound states could have similar expansion in powers of $\alpha_K$ as pairs of states with arbitrarily high but opposite values of quantum numbers. In the functional integral these quantum numbers would compensate each other. The functional integral would leave only an expansion containing powers of $\alpha_K$ starting from some finite possibly negative (unless one assumes the weak form of self-duality) power. Various gauge coupling strengths are expected to be proportional to $\alpha_K$ and these expansions should reduce to those in powers of $\alpha_K$.

2. Since the number of terms in the fermionic propagator expansion is finite, one might hope on basis of super-symmetry that the same is true in the case of the functional integral expansion. By the holomorpic factorization the expansion in powers of $K$ means the appearance of terms with increasingly higher quantum numbers. Quantum number conservation at vertices would leave only a finite number of terms to tree diagrams. In the case of loop diagrams pairs of particles with opposite and arbitrarily high values of quantum numbers could be generated at the vertex and magnetic confinement might be necessary to guarantee the convergence. Also super-symmetry could imply cancellations in loops.

#### Could one do without flux Hamiltonians?

The fact that the Kähler functions associated with the propagator lines can be regarded as interaction terms inspires the question whether the Kähler function could contain only the interaction terms so that Kähler form and Kähler metric would have components only between the ends of the lines.

1. The basic objection is that flux Hamiltonians too beautiful objects to be left without any role in the theory. One could also argue that the WCW metric would not be positive definite if only the non-diagonal interaction term is present. The simplest example is Hermitian $2 \times 2$-matrix with vanishing diagonal for which eigenvalues are real but of opposite sign.

2. One could of course argue that the expansions of $exp(K)$ and $\lambda_k$ give in the general powers $(f_n\overline{f_n})^m$ analogous to diverging tadpole diagrams of quantum field theories due to local interaction vertices. These terms do not produce divergences now but the possibility that the exponential series of this kind of terms could diverge cannot be excluded. The absence of the kinetic terms would allow to get rid of these terms and might be argued to be the symmetric space counterpart for the vanishing of loops in WCW integral.

3. In zero energy ontology this idea does not look completely non-sensical since physical states are pairs of positive and negative energy states. Note also that in quantum theory only creation operators are used to create positive energy states. The manifest non-locality of the interaction terms and absence of the counterparts of kinetic terms would provide a trivial manner to get rid of infinities due to the presence of local interactions. The safest option is however to keep both terms.

**Summary**

The discussion suggests that one must treat the entire Feynman graph as single geometric object with Kähler geometry in which the symmetric space is defined as product of what could be regarded as analogs of symmetric spaces with interaction terms of the metric coming from the propagator lines. The exponent of Kähler function would be the product of exponents associated with all lines and contributions to lines depend on quantum numbers (momentum and color quantum numbers) propagating in line via the coupling to the modified Dirac operator. The conformal factorization would allow the reduction of integrations to Fourier analysis in symmetric space. What is of decisive importance is that the entire Feynman diagrammatics at WCW level would reduce to the construction of WCW geometry for a single propagator line as a function of quantum numbers propagating on the line.

## 5.10 Could the notion of hyperdeterminant be useful in TGD framework?

The vanishing of ordinary determinant tells that a group of linear equations possesses non-trivial solutions. Hyperdeterminant [B4] generalizes this notion to a situation in which one has homogenous multilinear equations. The notion has applications to the description of quantum entanglement and has stimulated interest in physics blogs [B2, B6] . Hyperdeterminant applies to hyper-matrices with $n$ matrix indices defined for an $n$-fold tensor power of vector space - or more generally - for a tensor product of vector spaces with varying dimensions. Hyper determinant is an $n$-linear function of the arguments in the tensor factors with the property that all partial derivatives of the hyper determinant vanish at the point, which corresponds to a non-trivial solution of the equation. A simple example is potential function of $n$ arguments linear in each argument.

### 5.10.1 About the definition of hyperdeterminant

Hyperdeterminant was discovered by Cayley for a tensor power of 2-dimensional vector space $V_2$ ($n$-linear case for $n$-fold tensor power of 2-dimensional linear space) and he gave an explicit formula for the hyperdeterminant in this case. For $n = 3$ the definition is following.

$$A^1_{i_3j_3} = \frac{1}{2}\epsilon^{i_1j_1}\epsilon^{i_2j_2}\epsilon^{i_3j_3}A_{i_1i_2i_3}A_{j_1j_2j_3} \quad .$$

In more general case one must take tensor product of $k = 2$ hyper-matrices and perform the contractions of indices belonging to the two groups in by using $n$ 2-D permutations symbols.

$$det(A) = \frac{1}{2^n}(\prod_{a=1}^{n}\epsilon^{i_k^a j_k^a})A_{i_1^a i_2^a ... i_n^a}A_{j_1^a j_2^a ... j_n^a} \quad .$$

The first guess is that the definition for $V_k$, $k > 2$ is essentially identical: one takes the tensor product of $k$ hyper-matrices and performs the contractions using $k$-dimensional permutation symbols.

Under some conditions one can define hyperdeterminant also when one has a tensor product of linear spaces with different dimensions. The condition is that the largest vector space dimension in the product does not exceed the sum of other dimensions.

### 5.10.2 Could hyperdeterminant be useful in the description of criticality of Kähler action?

Why the notion of hyperdeterminant- or rather its infinite-dimensional generalization- might be interesting in TGD framework relates to the quantum criticality of TGD stating that TGD Universe involves a fractal hierarchy of criticalities: phase transitions inside phase transitions inside... At classical level the lowest order criticality means that the extremal of Kähler action possesses non-trivial second variations for which the action is not affected. The system is critical. In QFT context one speaks about zero modes. The vanishing of the so called Gaussian (of functional) determinant associated with second variations is the condition for the existence of critical deformations. In QFT context this situation corresponds to the presence of zero modes.

The simplest physical model for a critical system is cusp catastrophe defined by a potential function $V(x)$ which is fourth order polynomial. At the edges of cusp two extrema of potential function stable and unstable extrema co-incide and the rank of the matrix defined by the potential function vanishes. This means vanishing of its determinant. At the tip of the cusp the also the third derivative vanishes of potential function vanishes. This situation is however not describable in terms of hyperdeterminant since it is genuinely non-linear rather than only multilinear.

In a complete analogy, one can consider also the vanishing of $n$:th variations in TGD framework as higher order criticality so that the vanishing of hyperdeterminant might serve as a criterion for the higher order critical point and occurrence of phase transition.

1. The field equations are formally multilinear equations for variables which correspond to imbedding space coordinates at different space-time points. The generic form of the variational equations is

$$\int \frac{\delta^n S}{\delta h^{k_1}(x_1)\delta h^{k_2}(x_2)...\delta h^{k_n}(x_n)}\delta h^{k_2}(x_2)...\delta h^{k_n}(x_n)\prod_{i=2}^{n} d^4x_k = 0 \ .$$

   Here the partial derivatives are replaced with functional derivatives. On basis of the formula one has formally an $n$-linear situation. This is however an illusion in the generic case. For a local action the equations reduce to local partial differential equations involving higher order derivatives and field equations involve products of field variables and their various partial derivatives at single point so that one has a genuinely non-linear situation in absence of special symmetries.

2. If one has multi-linearity, the tensor product is formally an infinite tensor power of 8-D (or actually 4-D by General Coordinate Invariance) linear tangent spaces of $H$ associated with the space-time points. A less formal representation is in terms of some discrete basis for the deformations allowing also linear ordering of the basis functions. One might hope in some basis vanishing diagonal terms in all orders and multilinearity.

3. When one uses discretization, the equations stating the vanishing of the second variation couple nearest neighbour points given as infinite-D matrix with non-vanishing elements at diagonal and in a band along diagonal. For higher variations one obtains similar matrix along a diagonal of infinite cube and the width of the band increases by two units as $n$ increases by 1 unit. One might perhaps say that the range of long range correlations increases as $n$ increases. The vanishing of the elements at the diagonal- not necessarily in this representation- is necessary in order to achieve multi-linear situation.

### 5.10.3 Could the field equations for higher variations be multilinear?

The question is whether for some highly symmetric actions- say Kähler action for preferred extremals- the notion of functional (or Gaussian) determinant could have a generalization to hyperdeterminant allowing to concisely express whether the solutions allow deformations for which the action is not affected.

1. In standard field theory framework this notion need not be of much use but in TGD framework, where Kähler action has infinite-dimensional vacuum degeneracy, the situation is quite different. Vacuum degeneracy means that every space-time surface with at most 2-D $CP_2$ projection which

is so called Lagrangian manifold is vacuum extremal. Physically this correspond to Kähler gauge potential, which is pure gauge and implies spin glass degeneracy. This dynamical and local $U(1)$ symmetry of vacua is induced by symplectic transformations of $CP_2$ and has nothing to do with U(1) gauge invariance. For non-vacua it corresponds to isometries of "world of classical worlds". In particular, for $M^4$ imbedded in canonical manner to $M^4 \times CP_2$ fourth order variation is the first non-vanishing variation. The static mechanical analogy is potential function which is fourth order polynomial. Dynamical analogy is action for which both kinetic and potential terms are fourth order polynomials.

2. The vacuum degeneracy is responsible for much of new physics and mathematics related to TGD. Vacuum degeneracy and the consequent complete failure of canonical quantization and path integral approach forced the vision about physics as geometry of "World of Classical Worlds" (WCW) meaning a generalization of Einstein's geometrization of physics program. 4-D spin glass degeneracy is of the physical implications and among other things allows to have a failure of the standard form of classical determinism as a space-time correlate of quantum non-determinism. There are reasons to hope that also the hierarchy of Planck constants reduces to the 1-to-many correspondence between canonical momentum densities and time derivatives of imbedding space-coordinates. Quantum criticality and its classical counterparts is a further implication of the vacuum degeneracy and has provided a lot of insights to the world according to TGD. Therefore it would be nice if the generalization of the hyperdeterminant could provide new insights to quantum criticality.

### 5.10.4 Multilinearity, integrability, and cancellation of infinities

The multilinearity in the general sense would have a very interesting physical interpretation. One can consider the variations of both Kähler action and Kähler function defined as Kähler action for a preferred extremal.

1. Multilinearity would mean multi-linearization of field equations in some discrete basis for deformations- say the one defined by second variations. Dynamics would be only apparently non-linear. One might perhaps say that the theory is integrable- perhaps even in the usual sense. The basic idea behind quantum criticality is indeed the existence of infinite number of conserved currents assignable to the second variations hoped to give rise to an integrable theory. In fact, the possibility -or more or less the fact - that also higher variations can vanish for more restricted configurations would imply further conserved currents.

2. Second implication would be the vanishing of local divergences. These divergences result in QFT from purely local interaction terms with degree higher than two. Even mass insertion which is second order produces divergences. If diagonal terms are absent from Kähler function, also these divergences are absent in the functional integral. The main idea behind the notion of Kähler function is that it is a non-local functional of 3-surface although Kähler action is a local functional of space-time sheet serving as the analog of Bohr orbit through 3-surface. As one varies the 3-surface, one obtains a 3-surface (light-like wormhole throats with degenerate four-metric) which is also an extremal of Chern-Simons action satisfying weak form of electric magnetic duality.

3. The weak of electric magnetic duality together with the Beltrami property for conserved currents associated with isometries and for Kähler current and corresponding instanton current imply that the Coulomb term in Kähler action vanishes and it reduces to Chern-Simons term at 3-D light-like wormhole throats plus Lagrange multiplier term taking care that the weak electric magnetic duality is satisfied. This contributes a constraint force to field equations so that the theory does not reduce to topological QFT but to what could be called almost topological QFT.

4. Chern-Simons term is a local functional of 3-surface and one argue that the dangerous locality creeps in via the electric-magnetic duality after all. By using the so called Darboux coordates $(P_i, Q_i)$ for $CP_2$ Chern-Simons action reduces to a third order polynomial proportional to $\epsilon^{ijk} P_i dP_j dQ_k$ so that one indeed has multilinearity rather than non-linearity. The Lagrangian multiplier term however breaks strict locality and also contributes to higher functional derivatives of Kähler function and is potentially dangerous. It contains information about the preferred

extremals via the normal derivatives associated with the Kähler electric field in normal direction and it higher derivatives.

5. One has however good hopes about multilinearity of higher variations Kähler function and of Kähler action for preferred extremals on basis of general arguments related to the symmetric space property of WCW. As a matter fact, effective two-dimensionality seems to guarantee genuine non-locality. Recall that effective two-dimensionality is implied by the strong form of General Coordinate Invariance stating that the basic geometric objects can be taken to be either light-like 3-surfaces or space-like 3-surfaces at the ends of space-time surface at boundaries of causal diamond. This implies that partonic 2-surfaces defining the intersections of these surfaces plus their 4-D tangent space-data code for physics. By effective 2-dimensionality Chern-Simons action is a non-local functional of data about partonic 2-surface and its tangent space. Hence the $n$:th variation of 3-surface and space-time surface reduces to a non-local functional of $n$:th variation of the partonic 2-surface and its tangent space data. This is just what genuine multilinearity means.

### 5.10.5 Hyperdeterminant and entanglement

A highly interesting application of hyperdeterminants is to the description of quantum entanglement-in particular to the entanglement of $n$ qubits in quantum computation. For pure states the matrix describing entanglement between two systems has minimum rank for pure states and thus vanishing determinant. Hyper-matrix and hyperdeterminant emerge naturally when one speaks about entanglement between $n$ quantum systems. The vanishing of hyperdeterminant means that the state is not maximally non-pure.

For the called hyper-finite factor defined by second quantized induced spinor fields one has very formally infinite tensor product of 8-D H-spinor space. By induced spinor equation the dimension effectively reduces to four. Similar formal $8 \rightarrow 4$ reduction occurs by General Coordinate Invariance for the $n$:th variations. Quantum classical correspondence states that many-fermion states have correlates at the level of space-time geometry. The very naive question inspired also by supersymmetry is whether the vanishing of $n$-particle hyperdeterminant for the fermionic entanglement has as a space-time correlate $n$:th order criticality. If so, one could say that the non-locality with all its beautiful consequences is forced by quantum classical correspondence!

### 5.10.6 Could multilinear Higgs potentials be interesting?

It seems that hyperdeterminant has quite limited applications to finite-dimensional case. The simplest situation corresponds to a potential function $V(x_1, .., x_n)$. In this case one obtains also partial derivatives up to $n$:th order for single variable and one has genuine non-linearity rather than multilinearity. This spoils the possibility to apply the notion of hyperdeterminant to tell whether critical deformations are possible unless the potential function is multilinear function of its arguments. An interesting idea is that Higgs potential of this form. In this case the extrema allow scalings of the coordinates $x_i$. In 3-D case 3-linear function of 6 coordinates coming as doublets $(x_i, y_i)$ , $i = 1, 2, 3$ and characterized by a matrix $A_{i_1 i_2 i_3}$, where $i_k$ is two-valued index, would provide an example of this kind of Higgs potential.

## 5.11 About the interpretation of Kähler Dirac equation

The physical interpretation of Kähler Dirac equation is not at all straightforward. The following arguments inspired by effective 2-dimensionality suggest that the modified gamma matrices and corresponding effective metric could allow dual gravitational description of the physics associated with wormhole throats. This applies in particular to condensed matter physics.

### 5.11.1 Three Dirac equations

To begin with, Dirac equation appears in three forms in TGD.

1. The Dirac equation in world of classical worlds codes for the super Virasoro conditions for the super Kac-Moody and similar representations formed by the states of wormhole contacts forming the counterpart of string like objects (throats correspond to the ends of the string. This Dirac generalizes the Dirac of 8-D imbedding space by bringing in vibrational degrees of freedom. This Dirac equation should gives as its solutions zero energy states and corresponding M-matrices generalizing S-matrix and their collection defining the unitary U-matrix whose natural application appears in consciousness theory as a coder of what Penrose calls U-process.

2. There is generalized eigenvalue equation for Chern-Simons Dirac operator at light-like wormhole throats. The generalized eigenvalue is $p^k\gamma_k$. The interpretation of pseudo-momentum $p^k$ has been a problem but twistor Grassmannian approach suggests strongly that it can be interpreted as the counterpart of equally mysterious region momentum appearing in momentum twistor Grassmannian approach to $\mathcal{N}=4$ SYM. The pseudo-/region momentum p is quantized (this does not spoil the basics of Grasssmannian residues integral approach) and $1/p^k\gamma_k$ defines propagator in lines of generalized Feynman diagrams. The Yangian symmetry discovered generalizes in a very straightforward manner and leads alsoto the realization that TGD could allow also a twistorial formulation in terms of product $CP_3 \times CP_3$ of two twistor spaces [K63] . General arguments lead to a proposal for explicit form for the solutions of field equation represented identified as holomorphic 6-surfaces in this space subject to additional partial different equations for homogenenous functions of projective twistor coordinates suggesting strongly the quantal interpretation as analogs of partial waves. Therefore quantum-classical correspondence would be realize in beatiful manner.

3. There is Kähler Dirac equation in the interior of space-time. In this equation the gamma matrices are replaced with modified gamma matrices defined by the contractions of canonical momentum currents $T^\alpha_k = \partial L/\partial_\alpha h^k$ with imbedding space gamma matrices $\Gamma_k$. This replacement is required by internal consistency and by super-conformal symmetries.

Could Kähler Dirac equation provide a first principle justification for the light-hearted use of effective mass and the analog of Dirac equation in condensed manner physics? This would conform with the holographic philosophy. Partonic 2-surfaces with tangent space data and their light-like orbits would give hologram like representation of physics and the interior of space-time the 4-D representation of physics. Holography would have in the recent situation interpretation also as quantum classical correspondence between representations of physics in terms of quantized spinor fields at the light-like 3-surfaces on one hand and in terms of classical fields on the other hand.

The resulting dispersion relation for the square of the Kähler-Dirac operator assuming that induced like metric, Kähler field, etc. are very slowly varying contains quadratic and linear terms in momentum components plus a term corresponding to magnetic moment coupling. In general massive dispersion relation is obtained as is also clear from the fact that Kähler Dirac gamma matrices are combinations of $M^4$ and $CP_2$ gammas so that modified Dirac mixes different $M^4$ chiralities (basic signal for massivation). If one takes into account the dependence of the induced geometric quantities on space-time point dispersion relations become non-local.

### 5.11.2 Does energy metric provide the gravitational dual for condensed matter systems?

The modified gamma matrices define an effective metric via their anticommutators which are quadratic in components of energy momentum tensor (canonical momentum densities). This effective metric vanishes for vacuum extremals. Note that the use of modified gamma matrices guarantees among other things internal consistency and super-conformal symmetries of the theory. The physical interpretation has remained obscure hitherto although corresponding effective metric for Chern-Simons Dirac action has now a clear physical interpretation.

If the above argument is on the right track, this effective metric should have applications in condensed matter theory. In fact, energy metric has a natural interpretation in terms of effective light velocities which depend on direction of propagation. One can diagonalize the energy metric $g_e^{\alpha\beta}$ (contravariant form results from the anticommutators) and one can denote its eigenvalues by $(v_0, v_i)$ in the case that the signature of the effective metric is $(1,-1,-1,-1)$. The 3-vector $v_i/v_0$has interpretation as components of effective light velocity in various directions as becomes clear by thinking the d'Alembert equation for the energy metric. This velocity field could be interpreted as that of hydrodynamic flow. The study of the extremals of Kähler action shows that if this flow is actually Beltrami flow so that the flow parameter associated with the flow lines extends to global coordinate, Kähler action reduces to a 3-D Chern-Simons action and one obtains effective topological QFT. The conserved fermion current $\overline{\Psi}\Gamma_e^\alpha\Psi$ has interpretation as incompressible hydrodynamical flow.

This would give also a nice analogy with AdS/CFT correspondence allowing to describe various kinds of physical systems in terms of higher-dimensional gravitation and black holes are introduced quite routinely to describe condensed matter systems. In TGD framework one would have an analogous situation but with 10-D space-time replaced with the interior of 4-D space-time and the boundary of AdS representing Minkowski space with the light-like 3-surfaces carrying matter. The effective gravitation would correspond to the "energy metric". One can associate with it curvature tensor, Ricci tensor and Einstein tensor using standard formulas and identify effective energy momentum tensor associated as Einstein tensor with effective Newton's constant appearing as constant of proportionality. Note however that the besides ordinary metric and "energy" metric one would have also the induced classical gauge fields having purely geometric interpretation and action would be Kähler action. This 4-D holography would provide a precise, dramatically simpler, and also a very concrete dual description. This cannot be said about model of graphene based on the introduction of 10-dimensional black holes, branes, and strings chosen in more or less ad hoc manner.

This raises questions. Does this give a general dual gravitational description of dissipative effects in terms of the "energy" metric and induced gauge fields? Does one obtain the counterparts of black holes? Do the general theorems of general relativity about the irreversible evolution leading to black holes generalize to describe analogous fate of condensed matter systems caused by dissipation? Can one describe non-equilibrium thermodynamics and self-organization in this manner?

One might argue that the incompressible Beltrami flow defined by the dynamics of the preferred extremals is dissipationless and viscosity must therefore vanish locally. The failure of complete non-determinism of Kähler action however means generation of entropy since the knowledge about the state decreases gradually. This in turn should have a phenomenological local description in terms of viscosity which characterizes the transfer of energy to shorter scales and eventually to radiation. The deeper description should be non-local and basically topological and might lead to quantization rules. For instance, one can imagine the quantization of the ratio $\eta/s$ of the viscosity to entropy density as multiples of a basic unit defined by its lower bound (note that this would be analogous to Quantum Hall effect). For the first M-theory inspired derivation of the lower bound of $\eta/s$ [D4] . The lower bound for $\eta/s$ is satisfied in good approximation by what should have been QCD plasma but found to be something different (RHIC and the first evidence for new physics from LHC [K32] ).

An encouraring sign comes from the observation that for so called massless extremals representing classically arbitrarily shaped pulses of radiation propagating without dissipation and dispersion along single direction the canonical momentum currents are light-like. The effective contravariant metric vanishes identically so that fermions cannot propate in the interior of massless extremals! This is of course the case also for vacuum extremals. Massless extremals are purely bosonic and represent bosonic radiation. Many-sheeted space-time decomposes into matter containing regions and radiation containing regions. Note that when wormhole contact (particle) is glued to a massless extremal, it is deformed so that $CP_2$ projection becomes 4-D guaranteing that the weak form of electric magnetic duality can be satisfied. Therefore massless extremals can be seen as asymptotic regions. Perhaps one could say that dissipation corresponds to a decoherence process creating space-time sheets consisting of matter and radiation. Those containing matter might be even seen as analogs blackholes as far as energy metric is considered.

### 5.11.3 Preferred extremals as perfect fluids

Almost perfect fluids seems to be abundant in Nature. For instance, QCD plasma was originally thought to behave like gas and therefore have a rather high viscosity to entropy density ratio $x = \eta/s$. Already RHIC found that it however behaves like almost perfect fluid with $x$ near to the minimum predicted by AdS/CFT. The findings from LHC gave additional conform the discovery [C2]. Also Fermi gas is predicted on basis of experimental observations to have at low temperatures a low viscosity roughly 5-6 times the minimal value [D2] . In the following the argument that the preferred extremals of Kähler action are perfect fluids apart from the symmetry breaking to space-time sheets is developed. The argument requires some basic formulas summarized first.

The detailed definition of the viscous part of the stress energy tensor linear in velocity (oddness in velocity relates directly to second law) can be found in [D1] .

1. The symmetric part of the gradient of velocity gives the viscous part of the stress-energy tensor as a tensor linear in velocity. Velocity gardient decomposes to a term traceless tensor term and a term reducing to scalar.

$$\partial_i v_j + \partial_j v_i = \frac{2}{3}\partial_k v^k g_{ij} + (\partial_i v_j + \partial_j v_i - \frac{2}{3}\partial_k v^k g_{ij}) \ . \tag{5.11.1}$$

   The viscous contribution to stress tensor is given in terms of this decomposition as

$$\sigma_{visc;ij} = \zeta \partial_k v^k g_{ij} + \eta(\partial_i v_j + \partial_j v_i - \frac{2}{3}\partial_k v^k g_{ij}) \ . \tag{5.11.2}$$

   From $dF^i = T^{ij}S_j$ it is clear that bulk viscosity $\zeta$ gives to energy momentum tensor a pressure like contribution having interpretation in terms of friction opposing. Shear viscosity $\eta$ corresponds to the traceless part of the velocity gradient often called just viscosity. This contribution to the stress tensor is non-diagonal and corresponds to momentum transfer in directions not parallel to momentum and makes the flow rotational. This termm is essential for the thermal conduction and thermal conductivity vanishes for ideal fluids.

2. The 3-D total stress tensor can be written as

$$\sigma_{ij} = \rho v_i v_j - p g_{ij} + \sigma_{visc;ij} \ . \tag{5.11.3}$$

   The generalization to a 4-D relativistic situation is simple. One just adds terms corresponding to energy density and energy flow to obtain

$$T^{\alpha\beta} = (\rho - p) u^\alpha u^\beta + p g^{\alpha\beta} - \sigma^{\alpha\beta}_{visc} \ . \tag{5.11.4}$$

   Here $u^\alpha$ denotes the local four-velocity satisfying $u^\alpha u_\alpha = 1$. The sign factors relate to the concentions in the definition of Minkowski metric $((1,-1,-1,-1))$.

3. If the flow is such that the flow parameters associated with the flow lines integrate to a global flow parameter one can identify new time coordinate $t$ as this flow parametger. This means a transition to a coordinate system in which fluid is at rest everywhere (comoving coordinates in cosmology) so that energy momentum tensor reduces to a diagonal term plus viscous term.

$$T^{\alpha\beta} = (\rho - p) g^{tt}\delta^\alpha_t \delta^\beta_t + p g^{\alpha\beta} - \sigma^{\alpha\beta}_{visc} \ . \tag{5.11.5}$$

   In this case the vanishing of the viscous term means that one has perfect fluid in strong sense.

   The existence of a global flow parameter means that one has

$$v_i = \Psi \partial_i \Phi \ . \tag{5.11.6}$$

   $\Psi$ and $\Phi$ depend on space-time point. The proportionality to a gradient of scalar $\Phi$ implies that $\Phi$ can be taken as a global time coordinate. If this condition is not satisfied, the perfect fluid property makes sense only locally.

AdS/CFT correspondence allows to deduce a lower limit for the coefficient of shear viscosity as

$$x = \frac{\eta}{s} \geq \frac{\hbar}{4\pi} . \tag{5.11.7}$$

This formula holds true in units in which one has $k_B = 1$ so that temperature has unit of energy.

What makes this interesting from TGD view is that in TGD framework perfect fluid property in approriately generalized sense indeed characterizes locally the preferred extremals of Kähler action defining space-time surface.

1. Kähler action is Maxwell action with U(1) gauge field replaced with the projection of $CP_2$ Kähler form so that the four $CP_2$ coordinates become the dynamical variables at QFT limit. This means enormous reduction in the number of degrees of freedom as compared to the ordinary unifications. The field equations for Kähler action define the dynamics of space-time surfaces and this dynamics reduces to conservation laws for the currents assignable to isometries. This means that the system has a hydrodynamic interpretation. This is a considerable difference to ordinary Maxwell equations. Notice however that the "topological" half of Maxwell's equations (Faraday's induction law and the statement that no non-topological magnetic are possible) is satisfied.

2. Even more, the resulting hydrodynamical system allows an interpretation in terms of a perfect fluid. The general ansatz for the preferred extremals of field equations assumes that various conserved currents are proportional to a vector field characterized by so called Beltrami property. The coefficient of proportionality depends on space-time point and the conserved current in question. Beltrami fields by definition is a vector field such that the time parameters assignable to its flow lines integrate to single global coordinate. This is highly non-trivial and one of the implications is almost topological QFT property due to the fact that Kähler action reduces to a boundary term assignable to wormhole throats which are light-like 3-surfaces at the boundaries of regions of space-time with Euclidian and Minkowskian signatures. The Euclidian regions (or wormhole throats, depends on one's tastes ) define what I identify as generalized Feynman diagrams.

   Beltrami property means that if the time coordinate for a space-time sheet is chosen to be this global flow parameter, all conserved currents have only time component. In TGD framework energy momentum tensor is replaced with a collection of conserved currents assignable to various isometries and the analog of energy momentum tensor complex constructed in this manner has no counterparts of non-diagonal components. Hence the preferred extremals allow an interpretation in terms of perfect fluid without any viscosity.

This argument justifies the expectation that TGD Universe is characterized by the presence of low-viscosity fluids. Real fluids of course have a non-vanishing albeit small value of $x$. What causes the failure of the exact perfect fluid property?

1. Many-sheetedness of the space-time is the underlying reason. Space-time surface decomposes into finite-sized space-time sheets containing topologically condensed smaller space-time sheets containing.... Only within given sheet perfect fluid property holds true and fails at wormhole contacts and because the sheet has a finite size. As a consequence, the global flow parameter exists only in given length and time scale. At imbedding space level and in zero energy ontology the phrasing of the same would be in terms of hierarchy of causal diamonds (CDs).

2. The so called eddy viscosity is caused by eddies (vortices) of the flow. The space-time sheets glued to a larger one are indeed analogous to eddies so that the reduction of viscosity to eddy viscosity could make sense quite generally. Also the phase slippage phenomenon of super-conductivity meaning that the total phase increment of the super-conducting order parameter is reduced by a multiple of $2\pi$ in phase slippage so that the average velocity proportional to the increment of the phase along the channel divided by the length of the channel is reduced by a quantized amount.

   The standard arrangement for measuring viscosity involves a lipid layer flowing along plane. The velocity of flow with respect to the surface increases from $v = 0$ at the lower boundary to

$v_{upper}$ at the upper boundary of the layer: this situation can be regarded as outcome of the dissipation process and prevails as long as energy is feeded into the system. The reduction of the velocity in direction orthogonal to the layer means that the flow becomes rotational during dissipation leading to this stationary situation.

This suggests that the elementary building block of dissipation process corresponds to a generation of vortex identifiable as cylindrical space-time sheets parallel to the plane of the flow and orthogonal to the velocity of flow and carrying quantized angular momentum. One expects that vortices have a spectrum labelled by quantum numbers like energy and angular momentum so that dissipation takes in discrete steps by the generation of vortices which transfer the energy and angular momentum to environment and in this manner generate the velocity gradient.

3. The quantization of the parameter $x$ is suggestive in this framework. If entropy density and viscosity are both proportional to the density $n$ of the eddies, the value of $x$ would equal to the ratio of the quanta of entropy and kinematic viscosity $\eta/n$ for single eddy if all eddies are identical. The quantum would be $\hbar/4\pi$ in the units used and the suggestive interpretation is in terms of the quantization of angular momentum. One of course expects a spectrum of eddies so that this simple prediction should hold true only at temperatures for which the excitation energies of vortices are above the thermal energy. The increase of the temperature would suggest that gradually more and more vortices come into play and that the ratio increases in a stepwise manner bringing in mind quantum Hall effect. In TGD Universe the value of $\hbar$ can be large in some situations so that the quantal character of dissipation could become visible even macroscopically. Whether this a situation with large $\hbar$ is encountered even in the case of QCD plasma is an interesting question.

The following poor man's argument tries to make the idea about quantization a little bit more concrete.

1. The vortices transfer momentum parallel to the plane from the flow. Therefore they must have momentum parallel to the flow given by the total cm momentum of the vortex. Before continuing some notations are needed. Let the densities of vortices and absorbed vortices be $n$ and $n_{abs}$ respectively. Denote by $v_{\parallel}$ *resp.* $v_{\perp}$ the components of cm momenta parallel to the main flow *resp.* perpendicular to the plane boundary plane. Let $m$ be the mass of the vortex. Denote by $S$ are parallel to the boundary plane.

2. The flow of momentum component parallel to the main flow due to the absorbed at $S$ is

$$n_{abs} m v_{\parallel} v_{\perp} S \ .$$

This momentum flow must be equal to the viscous force

$$F_{visc} = \eta \frac{v_{\parallel}}{d} \times S \ .$$

From this one obtains

$$\eta = n_{abs} m v_{\perp} d \ .$$

If the entropy density is due to the vortices, it equals apart from possible numerical factors to

$$s = n$$

so that one has

$$\frac{\eta}{s} = m v_{\perp} d \ .$$

This quantity should have lower bound $x = \hbar/4\pi$ and perhaps even quantized in multiples of $x$, Angular momentum quantization suggests strongly itself as origin of the quantization.

3. Local momentum conservation requires that the comoving vortices are created in pairs with opposite momenta and thus propagating with opposite velocities $v_\perp$. Only one half of vortices is absorbed so that one has $n_{abs} = n/2$. Vortex has quantized angular momentum associated with its internal rotation. Angular momentum is generated to the flow since the vortices flowing downwards are absorbed at the boundary surface.

   Suppose that the distance of their center of mass lines parallel to plane is $D = \epsilon d$, $\epsilon$ a numerical constant not too far from unity. The vortices of the pair moving in opposite direction have same angular momentum $mv\ \ D/2$ relative to their center of mass line between them. Angular momentum conservation requires that the sum these relative angular momenta cancels the sum of the angular momenta associated with the vortices themselves. Quantization for the total angular momentum for the pair of vortices gives

$$\frac{\eta}{s} = \frac{n\hbar}{\epsilon}$$

   Quantization condition would give

$$\epsilon = 4\pi \quad .$$

   One should understand why $D = 4\pi d$ - four times the circumference for the largest circle contained by the boundary layer- should define the minimal distance between the vortices of the pair. This distance is larger than the distance $d$ for maximally sized vortices of radius $d/2$ just touching. This distance obviously increases as the thickness of the boundary layer increasess suggesting that also the radius of the vortices scales like $d$.

4. One cannot of course take this detailed model too literally. What is however remarkable that quantization of angular momentum and dissipation mechanism based on vortices identified as space-time sheets indeed could explain why the lower bound for the ratio $\eta/s$ is so small.

### 5.11.4 Is the effective metric one- or two-dimensional?

The following argument suggests that the effective metric defined by the anti-commutators of the modified gamma matrices is effectively one- or two-dimensional. Effective one-dimensionality would conform with the observation that the solutions of the modified Dirac equations can be localized to one-dimensional world lines in accordance with the vision that finite measurement resolution implies discretization reducing partonic many-particle states to quantum superpositions of braids. This localization to 1-D curves occurs always at the 3-D orbits of the partonic 2-surfaces.

The argument is based on the following assumptions.

1. The modified gamma matrices for Kähler action are contractions of the canonical momentum densities $T^\alpha_k$ with the gamma matrices of $H$.

2. The strongest assumption is that the isometry currents

$$J^{A\alpha} = T^\alpha_k j^{Ak}$$

   for the preferred extremals of Kähler action are of form

$$J^{A\alpha} = \Psi^A(\nabla\Phi)^\alpha \tag{5.11.8}$$

   with a common function $\Phi$ guaranteeing that the flow lines of the currents integrate to coordinate lines of single global coordinate variables (Beltrami property). Index raising is carried out by using the ordinary induced metric.

3. A weaker assumption is that one has two functions $\Phi_1$ and $\Phi_2$ assignable to the isometry currents of $M^4$ and $CP_2$ respectively.:

$$\begin{aligned} J_1^{A\alpha} &= \Psi_1^A(\nabla\Phi_1)^\alpha \ , \\ J_2^{A\alpha} &= \Psi_2^A(\nabla\Phi_2)^\alpha \ . \end{aligned} \tag{5.11.9}$$

The two functions $\Phi_1$ and $\Phi_2$ could define dual light-like curves spanning string world sheet. In this case one would have effective 2-dimensionality and decomposition to string world sheets [K25] . Isometry invariance does not allow more that two independent scalar functions $\Phi_i$.

Consider now the argument.

1. One can multiply both sides of this equation with $j^{Ak}$ and sum over the index $A$ labeling isometry currents for translations of $M^4$ and $SU(3)$ currents for $CP_2$. The tensor quantity $\sum_A j^{Ak}j^{Al}$ is invariant under isometries and must therefore satisfy

$$\sum_A \eta_{AB} j^{Ak} j^{Al} = h^{kl} \ , \tag{5.11.10}$$

where $\eta_{AB}$ denotes the flat tangent space metric of $H$. In $M^4$ degrees of freedom this statement becomes obvious by using linear Minkowski coordinates. In the case of $CP_2$ one can first consider the simpler case $S^2 = CP_1 = SU(2)/U(1)$. The coset space property implies in standard complex coordinate transforming linearly under $U(1)$ that only the the isometry currents belonging to the complement of $U(1)$ in the sum contribute at the origin and the identity holds true at the origin and by the symmetric space property everywhere. Identity can be verified also directly in standard spherical coordinates. The argument generalizes to the case of $CP_2 = SU(3)/U(2)$ in an obvious manner.

2. In the most general case one obtains

$$\begin{aligned} T_1^{\alpha k} &= \sum_A \Psi_1^A j^{Ak} \times (\nabla\Phi_1)^\alpha \equiv f_1^k(\nabla\Phi_1)^\alpha \ , \\ T_2^{\alpha k} &= \sum_A \Psi_1^A j^{Ak} \times (\nabla\Phi_2)^\alpha \equiv f_2^k(\nabla\Phi_2)^\alpha \ . \end{aligned} \tag{5.11.11}$$

3. The effective metric given by the anti-commutator of the modified gamma matrices is in turn is given by

$$G^{\alpha\beta} = m_{kl} f_1^k f_1^l (\nabla\Phi_1)^\alpha (\nabla\Phi_1)^\beta + s_{kl} f_2^k f_2^l (\nabla\Phi_2)^\alpha (\nabla\Phi_2)^\beta \ . \tag{5.11.12}$$

The covariant form of the effective metric is effectively 1-dimensional for $\Phi_1 = \Phi_2$ in the sense that the only non-vanishing component of the covariant metric $G_{\alpha\beta}$ is diagonal component along the coordinate line defined by $\Phi \equiv \Phi_1 = \Phi_2$. Also the contravariant metric is effectively 1-dimensional since the index raising does not affect the rank of the tensor but depends on the other space-time coordinates. This would correspond to an effective reduction to a dynamics of point-like particles for given selection of braid points. For $\Phi_1 \neq \Phi_2$ the metric is effectively 2-dimensional and would correspond to stringy dynamics.

One can also develop an objection to effective 1- or 2-dimensionality. The proposal for what preferred extremals of Kähler action as deformations of the known extremals of Kähler action could be leads to a beautiful ansatz relying on generalization of conformal invariance and minimal surface equations of string model [K5]. The field equations of TGD reduce to those of classical string model generalized to 4-D context.

If the proposed picture is correct, field equations reduce to purely algebraically conditions stating that the Maxwellian energy momentum tensor for the Kähler action has no common index pairs with the second fundamental form. For the deformations of $CP_2$ type vacuum extremals $T$ is a complex tensor of type (1,1) and second fundamental form $H^k$ a tensor of type (2,0) and (0,2) so that $Tr(TH^k) =$ is true. This requires that second light-like coordinate of $M^4$ is constant so that the $M^4$ projection is 3-dimensional. For Minkowskian signature of the induced metric Hamilton-Jacobi structure replaces conformal structure. Here the dependence of $CP_2$ coordinates on second light-like coordinate of $M^2(m)$ only plays a fundamental role. Note that now $T^{vv}$ is non-vanishing (and light-like). This picture generalizes to the deformations of cosmic strings and even to the case of vacuum extremals.

There is however an important consistency condition involved. The Maxwell energy momentum tensor for Kähler action must have vanishing covariant divergence. This is satisfied if it is linear combination of Einstein tensor and metric. This gives Einstein's equations with cosmological term in the general case. By the algebraic character of field equations also minimal surface equations are satisfied and Einstein's General Relativity would be exact part of TGD.

In the case of modified Dirac equation the result means that modified gamma matrices are contractions of linear combination of Einstein tensor and metric tensor with the induced gamma matrices so that the TGD counterpart of ordinary Dirac equation would be modified by the addition of a term proportional to Einstein tensor. The condition of effective 1- or 2-dimensionality seems to pose too strong conditions on this combination.

# Chapter 6

# The recent vision about preferred extremals and solutions of the modified Dirac equation

## 6.1 Introduction

During years several approaches to what preferred extremals of Kähler action and solutions of the modified Dirac equation could be have been proposed and the challenge is to see whether at least some of these approaches are consistent with each other. It is good to list various approaches first.

1. For preferred extremals generalization of conformal invariance to 4-D situation is very attractive approach and leads to concrete conditions formally similar to those encountered in string model [K5]. In particular, Einstein's equations with cosmological constant follow as consistency conditions and field equations reduce to a purely algebraic statements analogous to those appearing in equations for minimal surfaces if one assumes that space-time surface has Hermitian structure or its Minkowskian variant Hamilton-Jacobi structure (Appendix). The older approach based on basic heuristics for massless equations, on effective 3-dimensionality, and weak form of electric magnetic duality, and Beltrami flows is also promising. An alternative approach is inspired by number theoretical considerations and identifies space-time surfaces as associative or co-associative sub-manifolds of octonionic imbedding space [K54].

   The basic step of progress was the realization that the known extremals of Kähler action - certainly limiting cases of more general extremals - can be deformed to more general extremals having interpretation as preferred extremals.

   (a) The generalization boils down to the condition that field equations reduce to the condition that the traces $Tr(TH^k)$ for the product of energy momentum tensor and second fundamental form vanish. In string models energy momentum tensor corresponds to metric and one obtains minimal surface equations. The equations reduce to purely algebraic conditions stating that $T$ and $H^k$ have no common components. Complex structure of string world sheet makes this possible.

   Stringy conditions for metric stating $g_{zz} = g_{\overline{z}\overline{z}} = 0$ generalize. The condition that field equations reduce to $Tr(TH^k) = 0$ requires that the terms involving Kähler gauge current in field equations vanish. This is achieved if Einstein's equations hold true. The conditions guaranteeing the vanishing of the trace in turn boil down to the existence of Hermitian structure in the case of Euclidian signature and to the existence of its analog - Hamilton-Jacobi structure - for Minkowskian signature (Appendix). These conditions state that certain components of the induced metric vanish in complex coordinates or Hamilton-Jacobi coordinates.

   In string model the replacement of the imbedding space coordinate variables with quantized ones allows to interpret the conditions on metric as Virasoro conditions. In the recent case generalization of classical Virasoro conditions to four-dimensional ones would be in question.

An interesting question is whether quantization of these conditions could make sense also in TGD framework at least as a useful trick to deduce information about quantum states in WCW degrees of freedom.

The interpretation of the extended algebra as Yangian [A30] [B27] suggested previously [K63] to act as a generalization of conformal algebra in TGD Universe is attractive. There is also the conjecture that preferred extremals could be interpreted as quaternionic of co-quaternionic 4-surface of the octonionic imbedding space with octonionic representation of the gamma matrices defining the notion of tangent space quanternionicity.

2. There are also several approaches for solving the modified Dirac equation. The most promising approach is assumes that the solutions are restricted on 2-D stringy world sheets and/or partonic 2-surfaces. This strange looking view is a rather natural consequence of both strong form of holography and of number theoretic vision, and also follows from the notion of finite measurement resolution having discretization at partonic 2-surfaces as a geometric correlate. The conditions stating that electric charge is conserved for preferred extremals is an alternative very promising approach. One expects that stringy approach based on 4-D generalization of conformal invariance or its 2-D variant at 2-D preferred surfaces should also allow to understand the modified Dirac equation. In accordance with the earlier conjecture, all modes of the modified Dirac operator generate badly broken super-symmetries. Right-handed neutrino allows also holomorphic modes delocalized at entire space-time surface and the delocalization inside Euclidian region defining the line of generalized Feynman diagram is a good candidate for the right-handed neutrino generating the least broken super-symmetry. This super-symmetry seems however to differ from the ordinary one in that $\nu_R$ is expected to behave like a passive spectator in the scattering.

In the following the question whether these various approaches are mutually consistent is discussed. It indeed turns out that the approach based on the conservation of electric charge leads under rather general assumptions to the proposal that solutions of the modified Dirac equation are localized on 2-dimensional string world sheets and/or partonic 2-surfaces. Einstein's equations are satisfied for the preferred extremals and this implies that the earlier proposal for the realization of Equivalence Principle is not needed. This leads to a considerable progress in the understanding of super Virasoro representations for super-symplectic and super-Kac-Moody algebra. In particular, the proposal is that super-Kac-Moody currents assignable to string world sheets define duals of gauge potentials and their generalization for gravitons: in the approximation that gauge group is Abelian - motivated by the notion of finite measurement resolution - the exponents for the sum of KM charges would define non-integrable phase factors. One can also identify Yangian as the algebra generated by these charges. The approach allows also to understand the special role of the right handed neutrino in SUSY according to TGD.

## 6.2 About deformations of known extremals of Kähler action

I have done a considerable amount of speculative guesswork to identify what I have used to call preferred extremals of Kähler action. The problem is that the mathematical problem at hand is extremely non-linear and that there is no existing mathematical literature. One must proceed by trying to guess the general constraints on the preferred extremals which look physically and mathematically plausible. The hope is that this net of constraints could eventually chrystallize to Eureka! Certainly the recent speculative picture involves also wrong guesses. The need to find explicit ansatz for the deformations of known extremals based on some common principles has become pressing. The following considerations represent an attempt to combine the existing information to achieve this.

### 6.2.1 What might be the common features of the deformations of known extremals

The dream is to discover the deformations of all known extremals by guessing what is common to all of them. One might hope that the following list summarizes at least some common features.

### Effective three-dimensionality at the level of action

1. Holography realized as effective 3-dimensionality also at the level of action requires that it reduces to 3-dimensional effective boundary terms. This is achieved if the contraction $j^\alpha A_\alpha$ vanishes. This is true if $j^\alpha$ vanishes or is light-like, or if it is proportional to instanton current in which case current conservation requires that $CP_2$ projection of the space-time surface is 3-dimensional. The first two options for $j$ have a realization for known extremals. The status of the third option - proportionality to instanton current - has remained unclear.

2. As I started to work again with the problem, I realized that instanton current could be replaced with a more general current $j = *B \wedge J$ or concretely: $j^\alpha = \epsilon^{\alpha\beta\gamma\delta} B_\beta J_{\gamma\delta}$, where $B$ is vector field and $CP_2$ projection is 3-dimensional, which it must be in any case. The contractions of $j$ appearing in field equations vanish automatically with this ansatz.

3. Almost topological QFT property in turn requires the reduction of effective boundary terms to Chern-Simons terms: this is achieved by boundary conditions expressing weak form of electric magnetic duality. If one generalizes the weak form of electric magnetic duality to $J = \Phi * J$ one has $B = d\Phi$ and $j$ has a vanishing divergence for 3-D $CP_2$ projection. This is clearly a more general solution ansatz than the one based on proportionality of $j$ with instanton current and would reduce the field equations in concise notation to $Tr(TH^k) = 0$.

4. Any of the alternative properties of the Kähler current implies that the field equations reduce to $Tr(TH^k) = 0$, where $T$ and $H^k$ are shorthands for Maxwellian energy momentum tensor and second fundamental form and the product of tensors is obvious generalization of matrix product involving index contraction.

### Could Einstein's equations emerge dynamically?

For $j^\alpha$ satisfying one of the three conditions, the field equations have the same form as the equations for minimal surfaces except that the metric $g$ is replaced with Maxwell energy momentum tensor $T$.

1. This raises the question about dynamical generation of small cosmological constant $\Lambda$: $T = \Lambda g$ would reduce equations to those for minimal surfaces. For $T = \Lambda g$ modified gamma matrices would reduce to induced gamma matrices and the modified Dirac operator would be proportional to ordinary Dirac operator defined by the induced gamma matrices. One can also consider weak form for $T = \Lambda g$ obtained by restricting the consideration to sub-space of tangent space so that space-time surface is only "partially" minimal surface but this option is not so elegant although necessary for other than $CP_2$ type vacuum extremals.

2. What is remarkable is that $T = \Lambda g$ implies that the divergence of $T$ which in the general case equals to $j^\beta J^\alpha_\beta$ vanishes. This is guaranteed by one of the conditions for the Kähler current. Since also Einstein tensor has a vanishing divergence, one can ask whether the condition to $T = \kappa G + \Lambda g$ could the general condition. This would give Einstein's equations with cosmological term besides the generalization of the minimal surface equations. GRT would emerge dynamically from the non-linear Maxwell's theory although in slightly different sense as conjectured [K58]! Note that the expression for $G$ involves also second derivatives of the imbedding space coordinates so that actually a partial differential equation is in question. If field equations reduce to purely algebraic ones, as the basic conjecture states, it is possible to have $Tr(GH^k) = 0$ and $Tr(gH^k) = 0$ separately so that also minimal surface equations would hold true.

   What is amusing that the first guess for the action of TGD was curvature scalar. It gave analogs of Einstein's equations as a definition of conserved four-momentum currents. The recent proposal would give the analog of ordinary Einstein equations as a dynamical constraint relating Maxwellian energy momentum tensor to Einstein tensor and metric.

3. Minimal surface property is physically extremely nice since field equations can be interpreted as a non-linear generalization of massless wave equation: something very natural for non-linear variant of Maxwell action. The theory would be also very "stringy" although the fundamental action would not be space-time volume. This can however hold true only for Euclidian signature. Note that for $CP_2$ type vacuum extremals Einstein tensor is proportional to metric so that for

them the two options are equivalent. For their small deformations situation changes and it might happen that the presence of $G$ is necessary. The GRT limit of TGD discussed in [K58] [L12] indeed suggests that $CP_2$ type solutions satisfy Einstein's equations with large cosmological constant and that the small observed value of the cosmological constant is due to averaging and small volume fraction of regions of Euclidian signature (lines of generalized Feynman diagrams).

4. For massless extremals and their deformations $T = \Lambda g$ cannot hold true. The reason is that for massless extremals energy momentum tensor has component $T^{vv}$ which actually quite essential for field equations since one has $H^k_{vv} = 0$. Hence for massless extremals and their deformations $T = \Lambda g$ cannot hold true if the induced metric has Hamilton-Jacobi structure meaning that $g^{uu}$ and $g^{vv}$ vanish. A more general relationship of form $T = \kappa G + \Lambda G$ can however be consistent with non-vanishing $T^{vv}$ but require that deformation has at most 3-D $CP_2$ projection ($CP_2$ coordinates do not depend on $v$).

5. The non-determinism of vacuum extremals suggest for their non-vacuum deformations a conflict with the conservation laws. In, also massless extremals are characterized by a non-determinism with respect to the light-like coordinate but like-likeness saves the situation. This suggests that the transformation of a properly chosen time coordinate of vacuum extremal to a light-like coordinate in the induced metric combined with Einstein's equations in the induced metric of the deformation could allow to handle the non-determinism.

**Are complex structure of $CP_2$ and Hamilton-Jacobi structure of $M^4$ respected by the deformations?**

The complex structure of $CP_2$ and Hamilton-Jacobi structure of $M^4$ could be central for the understanding of the preferred extremal property algebraically.

1. There are reasons to believe that the Hermitian structure of the induced metric ((1,1) structure in complex coordinates) for the deformations of $CP_2$ type vacuum extremals could be crucial property of the preferred extremals. Also the presence of light-like direction is also an essential elements and 3-dimensionality of $M^4$ projection could be essential. Hence a good guess is that allowed deformations of $CP_2$ type vacuum extremals are such that (2,0) and (0,2) components the induced metric and/or of the energy momentum tensor vanish. This gives rise to the conditions implying Virasoro conditions in string models in quantization:

$$g_{\xi^i\xi^j} = 0 \ , \quad g_{\overline{\xi}^i\overline{\xi}^j} = 0 \ , \quad i,j = 1,2 \ . \tag{6.2.1}$$

   Holomorphisms of $CP_2$ preserve the complex structure and Virasoro conditions are expected to generalize to 4-dimensional conditions involving two complex coordinates. This means that the generators have two integer valued indices but otherwise obey an algebra very similar to the Virasoro algebra. Also the super-conformal variant of this algebra is expected to make sense.

   These Virasoro conditions apply in the coordinate space for $CP_2$ type vacuum extremals. One expects similar conditions hold true also in field space, that is for $M^4$ coordinates.

2. The integrable decomposition $M^4(m) = M^2(m)+E^2(m)$ of $M^4$ tangent space to longitudinal and transversal parts (non-physical and physical polarizations) - Hamilton-Jacobi structure- could be a very general property of preferred extremals and very natural since non-linear Maxwellian electrodynamics is in question. This decomposition led rather early to the introduction of the analog of complex structure in terms of what I called Hamilton-Jacobi coordinates $(u, v, w, \overline{w})$ for $M^4$. $(u, v)$ defines a pair of light-like coordinates for the local longitudinal space $M^2(m)$ and $(w, \overline{w})$ complex coordinates for $E^2(m)$. The Hamilton-Jacobi conditions on induced metric would be obtained by replacing imaginary unit in the definition of Hermitian metric for some complex coordinates with $e$, $e^2 = 1$ and defining hyper-complex conjugation as $u \to v$ for light-like-coordinate (Appendix).

   A good guess is that the deformations of massless extremals respect this structure. This condition gives rise to the analog of the constraints leading to Virasoro conditions stating the vanishing of the non-allowed components of the induced metric plus the analogs of hermiticity conditions. Again the generators of the algebra would involve two integers and the structure is that of Virasoro algebra and also generalization to super algebra is expected to make sense. The moduli space of Hamilton-Jacobi structures would be part of the moduli space of the preferred extremals and analogous to the space of all possible choices of complex coordinates. The analogs of infinitesimal holomorphic transformations would preserve the modular parameters and give rise to a 4-dimensional Minkowskian analog of Virasoro algebra. The conformal algebra acting on $CP_2$ coordinates acts in field degrees of freedom for Minkowskian signature.

#### Field equations as purely algebraic conditions

If the proposed picture is correct, field equations would reduce basically to purely algebraically conditions stating that the Maxwellian energy momentum tensor has no common index pairs with the second fundamental form. For the deformations of $CP_2$ type vacuum extremals $T$ is a complex tensor of type (1,1) and second fundamental form $H^k$ a tensor of type (2,0) and (0,2) so that $Tr(TH^k) =$ is true. This requires that second light-like coordinate of $M^4$ is constant so that the $M^4$ projection is 3-dimensional. For Minkowskian signature of the induced metric Hamilton-Jacobi structure replaces conformal structure. Here the dependence of $CP_2$ coordinates on second light-like coordinate of $M^2(m)$ only plays a fundamental role. Note that now $T^{vv}$ is non-vanishing (and light-like). This picture generalizes to the deformations of cosmic strings and even to the case of vacuum extremals.

### 6.2.2 What small deformations of $CP_2$ type vacuum extremals could be?

I was led to these arguments when I tried find preferred extremals of Kähler action, which would have 4-D $CP_2$ and $M^4$ projections - the Maxwell phase analogous to the solutions of Maxwell's equations that I conjectured long time ago. It however turned out that the dimensions of the projections can be $(D_{M^4} \leq 3, D_{CP_2} = 4)$ or $(D_{M^4} = 4, D_{CP_2} \leq 3)$. What happens is essentially breakdown of linear superposition so that locally one can have superposition of modes which have 4-D wave vectors in the same direction. This is actually very much like quantization of radiation field to photons now represented as separate space-time sheets and one can say that Maxwellian superposition corresponds to union of separate photonic space-time sheets in TGD. In the following I shall restrict the consideeration to the deformations of $CP_2$ type vacuum extremals.

#### Solution ansatz

I proceed by the following arguments to the ansatz.

1. Effective 3-dimensionality for action (holography) requires that action decomposes to vanishing $j^\alpha A_\alpha$ term + total divergence giving 3-D "boundary" terms. The first term certainly vanishes (giving effective 3-dimensionality) for

$$D_\beta J^{\alpha\beta} = j^\alpha = 0 \ .$$

   Empty space Maxwell equations, something extremely natural. Also for the proposed GRT limit these equations are true.

2. How to obtain empty space Maxwell equations $j^\alpha = 0$? The answer is simple: assume self duality or its slight modification:

$$J = *J$$

   holding for $CP_2$ type vacuum extremals or a more general condition

$$J = k * J \ ,$$

   In the simplest situation $k$ is some constant not far from unity. * is Hodge dual involving 4-D permutation symbol. $k = constant$ requires that the determinant of the induced metric is apart from constant equal to that of $CP_2$ metric. It does not require that the induced metric is proportional to the $CP_2$ metric, which is not possible since $M^4$ contribution to metric has Minkowskian signature and cannot be therefore proportional to $CP_2$ metric.

One can consider also a more general situation in which $k$ is scalar function as a generalization of the weak electric-magnetic duality. In this case the Kähler current is non-vanishing but divergenceless. This also guarantees the reduction to $Tr(TH^k) = 0$. In this case however the proportionality of the metric determinant to that for $CP_2$ metric is not needed. This solution ansatz becomes therefore more general.

3. Field equations reduce with these assumptions to equations differing from minimal surfaces equations only in that metric $g$ is replaced by Maxwellian energy momentum tensor $T$. Schematically:

$$Tr(TH^k) = 0 \ ,$$

where $T$ is the Maxwellian energy momentum tensor and $H^k$ is the second fundamental form - asymmetric 2-tensor defined by covariant derivative of gradients of imbedding space coordinates.

**How to satisfy the condition $Tr(TH^k) = 0$?**

It would be nice to have minimal surface equations since they are the non-linear generalization of massless wave equations. It would be also nice to have the vanishing of the terms involving Kähler current in field equations as a consequence of this condition. Indeed, $T = \kappa G + \Lambda g$ implies this. In the case of $CP_2$ vacuum extremals one cannot distinguish between these options since $CP_2$ itself is constant curvature space with $G \propto g$. Furthermore, if $G$ and $g$ have similar tensor structure the algebraic field equations for $G$ and $g$ are satisfied separately so that one obtains minimal surface property also now. In the following minimal surface option is considered.

1. The first opton is achieved if one has

$$T = \Lambda g \ .$$

Maxwell energy momentum tensor would be proportional to the metric! One would have dynamically generated cosmological constant! This begins to look really interesting since it appeared also at the proposed GRT limit of TGD [L12]. Note that here also non-constant value of $\Lambda$ can be considered and would correspond to a situation in which $k$ is scalar function: in this case the the determinant condition can be dropped and one obtains just the minimal surface equations.

2. Very schematically and forgetting indices and being sloppy with signs, the expression for $T$ reads as

$$T = JJ - g/4Tr(JJ) \ .$$

Note that the product of tensors is obtained by generalizing matrix product. This should be proportional to metric.

Self duality implies that $Tr(JJ)$ is just the instanton density and does not depend on metric and is constant.

For $CP_2$ type vacuum extremals one obtains

$$T = -g + g = 0 \ .$$

Cosmological constant would vanish in this case.

3. Could it happen that for deformations a small value of cosmological constant is generated?

The condition would reduce to

$$JJ = (\Lambda - 1)g \ .$$

$\Lambda$ must relate to the value of parameter $k$ appearing in the generalized self-duality condition. For the most general ansatz $\Lambda$ would not be constant anymore.

This would generalize the defining condition for Kähler form

$$JJ = -g \;\; (i^2 = -1 \;\; geometrically)$$

stating that the square of Kähler form is the negative of metric. The only modification would be that index raising is carried out by using the induced metric containing also $M^4$ contribution rather than $CP_2$ metric.

4. Explicitly:

$$J_{\alpha\mu}J^{\mu}_{\;\;\beta} = (\Lambda - 1)g_{\alpha\beta} \;\; .$$

Cosmological constant would measure the breaking of Kähler structure. By writing $g = s+m$ and defining index raising of tensors using $CP_2$ metric and their product accordingly, this condition can be also written as

$$Jm = (\Lambda - 1)mJ \;\; .$$

If the parameter $k$ is constant, the determinant of the induced metric must be proportional to the $CP_2$ metric. If $k$ is scalar function, this condition can be dropped. Cosmological constant would not be constant anymore but the dependence on $k$ would drop out from the field equations and one would hope of obtaining minimal surface equations also now. It however seems that the dimension of $M^4$ projection cannot be four. For 4-D $M^4$ projection the contribution of the $M^2$ part of the $M^4$ metric gives a non-holomorphic contribution to $CP_2$ metric and this spoils the field equations.

For $T = \kappa G + \Lambda g$ option the value of the cosmological constant is large - just as it is for the proposed GRT limit of TGD [K58] [L12]. The interpretation in this case is that the average value of cosmological constant is small since the portion of space-time volume containing generalized Feynman diagrams is very small.

### More detailed ansatz for the deformations of $CP_2$ type vacuum extremals

One can develop the ansatz to a more detailed form. The most obvious guess is that the induced metric is apart from constant conformal factor the metric of $CP_2$. This would guarantee self-duality apart from constant factor and $j^\alpha = 0$. Metric would be in complex $CP_2$ coordinates tensor of type (1,1) whereas $CP_2$ Riemann connection would have only purely holomorphic or anti-holomorphic indices. Therefore $CP_2$ contributions in $Tr(TH^k)$ would vanish identically. $M^4$ degrees of freedom however bring in difficulty. The $M^4$ contribution to the induced metric should be proportional to $CP_2$ metric and this is impossible due to the different signatures. The $M^4$ contribution to the induced metric breaks its Kähler property but would preserve Hermitian structure.

A more realistic guess based on the attempt to construct deformations of $CP_2$ type vacuum extremals is following.

1. Physical intuition suggests that $M^4$ coordinates can be chosen so that one has integrable decomposition to longitudinal degrees of freedom parametrized by two light-like coordinates $u$ and $v$ and to transversal polarization degrees of freedom parametrized by complex coordinate $w$ and its conjugate. $M^4$ metric would reduce in these coordinates to a direct sum of longitudinal and transverse parts. I have called these coordinates Hamilton Jacobi coordinates.

2. $w$ would be holomorphic function of $CP_2$ coordinates and therefore satisfy massless wave equation. This would give hopes about rather general solution ansatz. $u$ and $v$ cannot be holomorphic functions of $CP_2$ coordinates. Unless wither $u$ or $v$ is constant, the induced metric would receive contributions of type (2,0) and (0,2) coming from $u$ and $v$ which would break Kähler structure and complex structure. These contributions would give no-vanishing contribution to all minimal surface equations. Therefore either $u$ or $v$ is constant: the coordinate line for non-constant coordinate -say $u$- would be analogous to the $M^4$ projection of $CP_2$ type vacuum extremal.

3. With these assumptions the induced metric would remain $(1,1)$ tensor and one might hope that $Tr(TH^k)$ contractions vanishes for all variables except $u$ because the there are no common index pairs (this if non-vanishing Christoffel symbols for $H$ involve only holomorphic or anti-holomorphic indices in $CP_2$ coordinates). For $u$ one would obtain massless wave equation expressing the minimal surface property.

4. If the value of $k$ is constant the determinant of the induced metric must be proportional to the determinant of $CP_2$ metric. The induced metric would contain only the contribution from the transversal degrees of freedom besides $CP_2$ contribution. Minkowski contribution has however rank 2 as $CP_2$ tensor and cannot be proportional to $CP_2$ metric. It is however enough that its determinant is proportional to the determinant of $CP_2$ metric with constant proportionality coefficient. This condition gives an additional non-linear condition to the solution. One would have wave equation for $u$ (also $w$ and its conjugate satisfy massless wave equation) and determinant condition as an additional condition.

   The determinant condition reduces by the linearity of determinant with respect to its rows to sum of conditions involved 0,1,2 rows replaced by the transversal $M^4$ contribution to metric given if $M^4$ metric decomposes to direct sum of longitudinal and transversal parts. Derivatives with respect to derivative with respect to particular $CP_2$ complex coordinate appear linearly in this expression they can depend on $u$ via the dependence of transversal metric components on $u$. The challenge is to show that this equation has (or does not have) non-trivial solutions.

5. If the value of $k$ is scalar function the situation changes and one has only the minimal surface equations and Virasoro conditions.

What makes the ansatz attractive is that special solutions of Maxwell empty space equations are in question, equations reduces to non-linear generalizations of Euclidian massless wave equations, and possibly space-time dependent cosmological constant pops up dynamically. These properties are true also for the GRT limit of TGD [L12].

### 6.2.3 Hamilton-Jacobi conditions in Minkowskian signature

The maximally optimistic guess is that the basic properties of the deformations of $CP_2$ type vacuum extremals generalize to the deformations of other known extremals such as massless extremals, vacuum extremals with 2-D $CP_2$ projection which is Lagrangian manifold, and cosmic strings characterized by Minkowskian signature of the induced metric. These properties would be following.

1. The recomposition of $M^4$ tangent space to longitudinal and transversal parts giving Hamilton-Jacobi structure. The longitudinal part has hypercomplex structure but the second light-like coordinate is constant: this plays a crucial role in guaranteeing the vanishing of contractions in $Tr(TH^k)$. It is the algebraic properties of $g$ and $T$ which are crucial. $T$ can however have light-like component $T^{vv}$. For the deformations of $CP_2$ type vacuum extremals $(1,1)$ structure is enough and is guaranteed if second light-like coordinate of $M^4$ is constant whereas $w$ is holomorphic function of $CP_2$ coordinates.

2. What could happen in the case of massless extremals? Now one has 2-D $CP_2$ projection in the initial situation and $CP_2$ coordinates depend on light-like coordinate $u$ and single real transversal coordinate. The generalization would be obvious: dependence on single light-like coordinate $u$ and holomorphic dependence on $w$ for complex $CP_2$ coordinates. The constraint is $T = \Lambda g$ cannot hold true since $T^{vv}$ is non-vanishing (and light-like). This property restricted to transversal degrees of freedom could reduce the field equations to minimal surface equations in transversal degrees of freedom. The transversal part of energy momentum tensor would be proportional to metric and hence covariantly constant. Gauge current would remain light-like but would not be given by $j = *d\phi \wedge J$. $T = \kappa G + \Lambda g$ seems to define the attractive option.

It therefore seems that the essential ingredient could be the condition

$$T = \kappa G + \lambda g \ ,$$

which has structure (1,1) in both $M^2(m)$ and $E^2(m)$ degrees of freedom apart from the presence of $T^{vv}$ component with deformations having no dependence on $v$. If the second fundamental form has (2,0)+(0,2) structure, the minimal surface equations are satisfied provided Kähler current satisfies on of the proposed three conditions and if $G$ and $g$ have similar tensor structure.

One can actually pose the conditions of metric as complete analogs of stringy constraints leading to Virasoro conditions in quantization to give

$$g_{uu} = 0 \ , \quad g_{vv} = 0 \ , \quad g_{ww} = 0 \ , \quad g_{\overline{w}\overline{w}} = 0 \ . \tag{6.2.2}$$

This brings in mind the generalization of Virasoro algebra to four-dimensional algebra for which an identification in terms of non-local Yangian symmetry has been proposed [K63]. The number of conditions is four and the same as the number of independent field equations. One can consider similar conditions also for the energy momentum tensor $T$ but allowing non-vanishing component $T^{vv}$ if deformations has no $v$-dependence. This would solve the field equations if the gauge current vanishes or is light-like. On this case the number of equations is 8. First order differential equations are in question and they can be also interpreted as conditions fixing the coordinates used since there is infinite number of manners to choose the Hamilton-Jacobi coordinates.

One can can try to apply the physical intuition about general solutions of field equations in the linear case by writing the solution as a superposition of left and right propagating solutions:

$$\xi^k = f_+^k(u,w) + f_+^k(v,w) \ . \tag{6.2.3}$$

This could guarantee that second fundamental form is of form (2,0)+(0,2) in both $M^2$ and $E^2$ part of the tangent space and these terms if $Tr(TH^k)$ vanish identically. The remaining terms involve contractions of $T^{uw}$, $T^{u\overline{w}}$ and $T^{vw}$, $T^{v\overline{w}}$ with second fundamental form. Also these terms should sum up to zero or vanish separately. Second fundamental form has components coming from $f_+^k$ and $f_-^k$

Second fundamental form $H^k$ has as basic building bricks terms $\hat{H}^k$ given by

$$\hat{H}^k_{\alpha\beta} \quad = \quad \partial_\alpha\partial_\beta h^k + \left({k \atop l\ m}\right) \partial_\alpha h^l \partial_\beta h^m \ . \tag{6.2.4}$$

For the proposed ansatz the first terms give vanishing contribution to $H^k_{uv}$. The terms containing Christoffel symbols however give a non-vanishing contribution and one can allow only $f_+^k$ or $f_-^k$ as in the case of massless extremals. This reduces the dimension of $CP_2$ projection to $D = 3$.

What about the condition for Kähler current? Kähler form has components of type $J_{w\overline{w}}$ whose contravariant counterpart gives rise to space-like current component. $J_{uw}$ and $J_{u\overline{w}}$ give rise to light-like currents components. The condition would state that the $J^{w\overline{w}}$ is covariantly constant. Solutions would be characterized by a constant Kähler magnetic field. Also electric field is represent. The interpretation both radiation and magnetic flux tube makes sense.

### 6.2.4 Deformations of cosmic strings

In the physical applications it has been assumed that the thickening of cosmic strings to Kähler magnetic flux tubes takes place. One indeed expects that the proposed construction generalizes also to the case of cosmic strings having the decomposition $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$, where $X^2$ is minimal surface and $Y^2$ a complex homologically non-trivial sub-manifold of $CP_2$. Now the starting point structure is Hamilton-Jacobi structure for $M^2_m \times Y^2$ defining the coordinate space.

1. The deformation should increase the dimension of either $CP_2$ or $M^4$ projection or both. How this thickening could take place? What comes in mind that the string orbits $X^2$ can be interpreted as a distribution of longitudinal spaces $M^2(x)$ so that for the deformation $w$ coordinate becomes a holomorphic function of the natural $Y^2$ complex coordinate so that $M^4$ projection becomes 4-D but $CP_2$ projection remains 2-D. The new contribution to the $X^2$ part of the induced metric is vanishing and the contribution to the $Y^2$ part is of type $(1,1)$ and the the ansatz $T = \kappa G + \Lambda g$ might be needed as a generalization of the minimal surface equations The ratio of $\kappa$ and $G$ would be determined from the form of the Maxwellian energy momentum tensor and be fixed at the

limit of undeformed cosmic strong to $T = (ag(Y^2) - bg(Y^2))$. The value of cosmological constant is now large, and overall consistency suggests that $T = \kappa G + \Lambda g$ is the correct option also for the $CP_2$ type vacuum extremals.

2. One could also imagine that remaining $CP_2$ coordinates could depend on the complex coordinate of $Y^2$ so that also $CP_2$ projection would become 4-dimensional. The induced metric would receive holomorphic contributions in $Y^2$ part. As a matter fact, this option is already implied by the assumption that $Y^2$ is a complex surface of $CP_2$.

### 6.2.5 Deformations of vacuum extremals?

What about the deformations of vacuum extremals representable as maps from $M^4$ to $CP_2$?

1. The basic challenge is the non-determinism of the vacuum extremals. One should perform the deformation so that conservation laws are satisfied. For massless extremals there is also non-determinism but it is associated with the light-like coordinate so that there are no problems with the conservation laws. This would suggest that a properly chosen time coordinate consistent with Hamilton-Jacobi decomposition becomes light-like coordinate in the induced metric. This poses a conditions on the induced metric.

2. Physical intuition suggests that one cannot require $T = \Lambda g$ since this would mean that the rank of $T$ is maximal whereas the original situation corresponds to the vanishing of $T$. For small deformations rank two for $T$ looks more natural and one could think that $T$ is proportional to a projection of metric to a 2-D subspace. The vision about the long length scale limit of TGD is that Einstein's equations are satisfied and this would suggest $T = kG$ or $T = \kappa G + \Lambda g$. The rank of $T$ could be smaller than four for this ansatz and this conditions binds together the values of $\kappa$ and $G$.

3. These extremals have $CP_2$ projection which in the generic case is 2-D Lagrangian sub-manifold $Y^2$. Again one could assume Hamilton-Jacobi coordinates for $X^4$. For $CP_2$ one could assume Darboux coordinates $(P_i, Q_i)$, $i = 1, 2$, in which one has $A = P_i dQ^i$, and that $Y^2 \subset CP_2$ corresponds to $Q_i = constant$. In principle $P_i$ would depend on arbitrary manner on $M^4$ coordinates. It might be more convenient to use as coordinates $(u, v)$ for $M^2$ and $(P_1, P_2)$ for $Y^2$. This covers also the situation when $M^4$ projection is not 4-D. By its 2-dimensionality $Y^2$ allows always a complex structure defined by its induced metric: this complex structure is not consistent with the complex structure of $CP_2$ ($Y^2$ is not complex sub-manifold).

   Using Hamilton-Jacobi coordinates the pre-image of a given point of $Y^2$ is a 2-dimensional sub-manifold $X^2$ of $X^4$ and defines also 2-D sub-manifold of $M^4$. The following picture suggests itself. The projection of $X^2$ to $M^4$ can be seen for a suitable choice of Hamilton-Jacobi coordinates as an analog of Lagrangian sub-manifold in $M^4$ that is as surface for which $v$ and $Im(w)$ vary and $u$ and $Re(w)$ are constant. $X^2$ would be obtained by allowing $u$ and $Re(w)$ to vary: as a matter fact, $(P_1, P_2)$ and $(u, Re(w))$ would be related to each other. The induced metric should be consistent with this picture. This would requires $g_{uRe(w)} = 0$.

   For the deformations $Q_1$ and $Q_2$ would become non-constant and they should depend on the second light-like coordinate $v$ only so that only $g_{uu}$ and $g_{uw}$ and $gu\overline{w}$ $g_{w,w}$ and $g_{\overline{w},\overline{w}}$ receive contributions which vanish. This would give rise to the analogs of Virasoro conditions guaranteeing that $T$ is a tensor of form $(1,1)$ in both $M^2$ and $E^2$ indices and that there are no cross components in the induced metric. A more general formulation states that energy momentum tensor satisfies these conditions. The conditions on $T$ might be equivalent with the conditions for $g$ and $G$ separately.

4. Einstein's equations provide an attractive manner to achieve the vanishing of effective 3-dimensionality of the action. Einstein equations would be second order differential equations and the idea that a deformation of vacuum extremal is in question suggests that the dynamics associated with them is in directions transversal to $Y^2$ so that only the deformation is dictated partially by Einstein's equations.

5. Lagrangian manifolds do not involve complex structure in any obvious manner. One could however ask whether the deformations could involve complex structure in a natural manner in $CP_2$ degrees of freedom so that the vanishing of $g_{ww}$ would be guaranteed by holomorphy of $CP_2$ complex coordinate as function of $w$.

   One should get the complex structure in some natural manner: in other words, the complex structure should relate to the geometry of $CP_2$ somehow. The complex coordinate defined by say $z = P_1 + iQ^1$ for the deformation suggests itself. This would suggest that at the limit when one puts $Q_1 = 0$ one obtains $P_1 = P_1(Re(w))$ for the vacuum extremals and the deformation could be seen as an analytic continuation of real function to region of complex plane. This is in spirit with the algebraic approach. The vanishing of Kähler current requires that the Kähler magnetic field is covariantly constant: $D_z J^{z\overline{z}} = 0$ and $D_{\overline{z}} J^{z\overline{z}} = 0$ .

6. One could consider the possibility that the resulting 3-D sub-manifold of $CP_2$ can be regarded as contact manifold with induced Kähler form non-vanishing in 2-D section with natural complex coordinates. The third coordinate variable- call it $s$- of the contact manifold and second coordinate of its transversal section would depend on time space-time coordinates for vacuum extremals. The coordinate associated with the transversal section would be continued to a complex coordinate which is holomorphic function of $w$ and $u$.

7. The resulting thickened magnetic flux tubes could be seen as another representation of Kähler magnetic flux tubes: at this time as deformations of vacuum flux tubes rather than cosmic strings. For this ansatz it is however difficult to imagine deformations carrying Kähler electric field.

### 6.2.6 About the interpretation of the generalized conformal algebras

The long-standing challenge has been finding of the direct connection between the super-conformal symmetries assumed in the construction of the geometry of the "world of classical worlds" (WCW) and possible conformal symmetries of field equations. 4-dimensionality and Minkowskian signature have been the basic problems. The recent construction provides new insights to this problem.

1. In the case of string models the quantization of the Fourier coefficients of coordinate variables of the target space gives rise to Kac-Moody type algebra and Virasoro algebra generators are quadratic in these. Also now Kac-Moody type algebra is expected. If one were to perform a quantization of the coefficients in Laurents series for complex $CP_2$ coordinates, one would obtain interpretation in terms of $su(3) = u(2) + t$ decomposition, where $t$ corresponds to $CP_3$: the oscillator operators would correspond to generators in $t$ and their commutator would give generators in $u(2)$. SU(3)/SU(2) coset representation for Kac-Moody algebra would be in question. Kac-Moody algebra would be associated with the generators in both $M^4$ and $CP_2$ degrees of freedom. This kind of Kac-Moody algebra appears in quantum TGD.

2. The constraints on induced metric imply a very close resemblance with string models and a generalization of Virasoro algebra emerges. An interesting question is how the two algebras acting on coordinate and field degrees of freedom relate to the super-conformal algebras defined by the symplectic group of $\delta M^4_+ \times CP_2$ acting on space-like 3-surfaces at boundaries of $CD$ and to the Kac-Moody algebras acting on light-like 3-surfaces. It has been conjectured that these algebras allow a continuation to the interior of space-time surface made possible by its slicing by 2-surfaces parametrized by 2-surfaces. The proposed construction indeed provides this kind of slicings in both $M^4$ and $CP_2$ factor.

3. In the recent case, the algebras defined by the Fourier coefficients of field variables would be Kac-Moody algebras. Virasoro algebra acting on preferred coordinates would be expressed in terms of the Kac-Moody algebra in the standard Sugawara construction applied in string models. The algebra acting on field space would be analogous to the conformal algebra assignable to the symplectic algebra so that also symplectic algebra is present. Stringy pragmatist could imagine quantization of symplectic algebra by replacing $CP_2$ coordinates in the expressions of Hamiltonians with oscillator operators. This description would be counterpart for the construction of spinor harmonics in WCW and might provide some useful insights.

4. For given type of space-time surface either $CP_2$ or $M^4$ corresponds to Kac-Moody algebra but not both. From the point of view of quantum TGD it looks as that something were missing. An analogous problem was encountered at GRT limit of TGD [L12]. When Euclidian space-time regions are allowed Einstein-Maxwell action is able to mimic standard model with a surprising accuracy but there is a problem: one obtains either color charges or $M^4$ charges but not both. Perhaps it is not enough to consider either $CP_2$ type vacuum extremal or its exterior but both to describe particle: this would give the direct product of the Minkowskian and Euclidian algebras acting on tensor product. This does not however seem to be consistent with the idea that the two descriptions are duality related (the analog of T-duality).

## 6.3 Under what conditions electric charge is conserved for the modified Dirac equation?

One might think that talking about the conservation of electric charge at 21st century is a waste of time. In TGD framework this is certainly not the case and the following arguments suggests that the conservation of electric charge is the Golden Road to the understanding of the spinorial dynamics.

1. In quantum field theories there are two manners to define em charge: as electric flux over 2-D surface sufficiently far from the source region or in the case of spinor field quantum mechanically as combination of fermion number and vectorial isospin. The latter definition is quantum mechanically more appropriate.

2. There is however a problem. In standard approach to gauge theory Dirac equation in the presence of charged classical gauge fields does not conserve the electric charge: electron is transformed to neutrino and vice versa. Quantization solves the problem since the non-conservation can be interpreted in terms of emission of gauge bosons. In TGD framework this does not work since one does not have path integral quantization anymore. Preferred extremals carry classical gauge fields and the question whether em charge is conserved arises. Heuristic picture suggests that em charge must be conserved. This condition might be actually one of the conditions defining what it is to be a preferred extremal. It is not however trivial whether this kind of additional condition can be posed.

### 6.3.1 Conditions guaranteing the conservation of em charge

What does the conservation of em charge imply in the case of the modified Dirac equation? The obvious guess that the em charged part of the modified Dirac operator must annihilate the solutions, turns out to be correct as the following argument demonstrates.

1. Em charge as coupling matrix can be defined as a linear combination $Q = aI + bI_3$, $I_3 = J_{kl}\Sigma^{kl}$, where I is unit matrix and $I_3$ vectorial isospin matrix, $J_{kl}$ is the Kähler form of $CP_2$, $\Sigma^{kl}$ denotes sigma matrices, and $a$ and $b$ are numerical constants different for quarks and leptons. $Q$ is covariantly constant in $M^4 \times CP_2$ and its covariant derivatives at space-time surface are also well-defined and vanish.

2. The modes of the modified Dirac equation should be eigen modes of $Q$. This is the case if the modified Dirac operator $D$ commutes with $Q$. The covariant constancy of $Q$ can be used to derive the condition

$$\begin{aligned} [D,Q]\,\Psi &= D_1\Psi = 0 \ , \\ D &= \hat{\Gamma}^\mu D_\mu \ , \ \ D_1 = [D,Q] = \hat{\Gamma}_1^\mu D_\mu \ , \ \ \hat{\Gamma}_1^\mu = \left[\hat{\Gamma}^\mu, Q\right] \ . \end{aligned} \tag{6.3.1}$$

Covariant constancy of $J$ is absolutely essential: without it the resulting conditions would not be so simple.

It is easy to find that also $[D_1,Q]\Psi = 0$ and its higher iterates $[D_n,Q]\Psi = 0$, $D_n = [D_{n-1},Q]$ must be true. The solutions of the modified Dirac equation would have an additional symmetry.

3. The commutator $D_1 = [D,Q]$ reduces to a sum of terms involving the commutators of the vectorial isospin $I_3 = J_{kl}\Sigma^{kl}$ with the $CP_2$ part of the gamma matrices:

$$D_1 = [Q,D] = [I_3,\Gamma_r]\partial_\mu s^r T^{\alpha\mu} D_\alpha \ . \tag{6.3.2}$$

   In standard complex coordinates in which $U(2)$ acts linearly the complexified gamma matrices can be chosen to be eigenstates of vectorial isospin. Only the charged flat space complexified gamma matrices $\Gamma^A$ denoted by $\Gamma^+$ and $\Gamma^-$ possessing charges +1 and -1 contribute to the right hand side. Therefore the additional Dirac equation $D_1\Psi = 0$ states

$$\begin{aligned} D_1\Psi &= [Q,D]\Psi = I_3(A)e_{Ar}\Gamma^A\partial_\mu s^r T^{\alpha\mu} D_\alpha\Psi \\ &= (e_{+r}\Gamma^+ - e_{-r}\Gamma^-)\partial_\mu s^r T^{\alpha\mu} D_\alpha\Psi = 0 \ . \end{aligned} \tag{6.3.3}$$

   The next condition is

$$D_2\Psi = [Q,D]\Psi = (e_{+r}\Gamma^+ + e_{-r}\Gamma^-)\partial_\mu s^r T^{\alpha\mu} D_\alpha\Psi = 0 \ . \tag{6.3.4}$$

   Only the relative sign of the two terms has changed. The remaining conditions give nothing new.

4. These equations imply two separate equations for the two charged gamma matrices

$$\begin{aligned} D_+\Psi &= T^\alpha_+\Gamma^+ D_\alpha\Psi = 0 \ , \\ D_-\Psi &= T^\alpha_-\Gamma^- D_\alpha\Psi = 0 \ , \\ T^\alpha_\pm &= e_{\pm r}\partial_\mu s^r T^{\alpha\mu} \ . \end{aligned} \tag{6.3.5}$$

   These conditions state what one might have expected: the charged part of the modified Dirac operator annihilates separately the solutions. The reason is that the classical W fields are proportional to $e_{r\pm}$.

   The above equations can be generalized to define a decomposition of the energy momentum tensor to charged and neutral components in terms of vierbein projections. The equations state that the analogs of the modified Dirac equation defined by charged components of the energy momentum tensor are satisfied separately.

5. In complex coordinates one expects that the two equations are complex conjugates of each other for Euclidian signature. For the Minkowskian signature an analogous condition should hold true. The dynamics enters the game in an essential manner: whether the equations can be satisfied depends on the coefficients $a$ and $b$ in the expression $T = aG + bg$ implied by Einstein's equations in turn guaranteeing that the solution ansatz generalizing minimal surface solutions holds true [K5].

6. As a result one obtains three separate Dirac equations corresponding to the the neutral part $D_0\Psi = 0$ and charged parts $D_\pm\Psi = 0$ of the modified Dirac equation. By acting on the equations with these Dirac operators one obtains also that the commutators $[D_+, D_-]$, $[D_0, D_\pm]$ and also higher commutators obtained from these annihilate the induced spinor field mode. Therefore possibly infinite-dimensional algebra would annihilate the induced spinor fields unless the charged parts of the energy momentum tensor vanish identically.

### 6.3.2 Dirac equation in $CP_2$ as a test bench

What could the conservation of electric charge mean from the point of view of the solutions of the modified Dirac equation? The field equations for the preferred extremals of Kähler action reduce to purely algebraic conditions in the same manner as the field equations for the minimal surfaces in string model. Could something similar happen also for the modified Dirac equation and could the condition on charged part of the Dirac operator help to achieve this?

1. For $CP_2$ type vacuum extremals the modified Dirac operator vanishes identically for the Kähler action. For volume action it reduces to the ordinary Dirac operator in $CP_2$ and one can ask whether ordinary Dirac operator could in this case allow solutions with a well-defined em charge. Since also spinor harmonics of the imbedding space are expected to be important and associated with the representations of conformal symmetries assignable to the boundary of light cone involving symplectic group of $\delta M^4_\pm \times CP_2$, it would be nice if this construction would work for $CP_2$.

2. One can construct the solutions of the ordinary Dirac equation from covariantly constant right-handed neutrino spinor playing the role of fermionic vacuum annihilated by the second half of complexified gamma matrices. Dirac equation reduces to Laplace equation for a scalar function and solution can be constructed from this "vacuum" by multiplying with the spherical harmonics of $CP_2$ and applying Dirac operator [K29]. Similar construction works quite generally thanks to the existence of covariantly constant right handed neutrino spinor. Spherical harmonics of $CP_2$ are only replaced with those of space-time surface possessing either hermitian structure of Hamilton-Jacobi structure (corresponding to Euclidian and Minkowskian signatures of the induced metric (Appendix)).

3. A good guess is that holomorphy codes the statement that only spinor harmonics of form $(n, 0)$ and $(0, n)$ should be allowed. The first problem is that these modes are not actually holomorphic since color triplet partial waves are proportional to $1/\sqrt{1 + |\xi^1|^2 + |\xi^2|^2}$: The are good hopes that covariant derivative containing also a term proportional to Kähler gauge potential and coupling to leptons and quarks differently takes care of this.

   Dirac equation in $CP_2$ allows only modes with $(m, m+3)$, $(m+3, m)$ for leptons and anti-leptons and modes $(m + 1, m)$ and $(m, m + 1)$ for quarks. More general solutions could be possible but would not be global solutions. If this picture is correct, the dynamics in fermionic degrees of freedom would be extremely restricted and only only very few color partial waves would survive.

4. Most holomorphic and anti-holomorphic modes for leptons and quarks would represent gauge degrees of freedom. The remaining three modes for quarks could be interpreted in terms of color of ground state. At first this looks good since only color neutral leptons and color triplet quarks would be allowed. This result just what has been observed and the experimental absence has indeed a challenge for quantum TGD. The experimental absence of higher modes would be due to Kac-Moody gauge invariance.

   Lepto-pion hypothesis in its original form and postulating color octet leptons would be however wrong. This might not be a catastrophe: a variant of this hypothesis identifies lepto-pion as quark antiquark pair associated with scaled down variant of hadron physics [K57].

5. A further work described in the sequel however shows that the correct identification of partial waves of imbedding space spinors is in terms of cm degrees of freedom of the partonic 2-surface and can be assigned to the super-symplectic conformal invariance dictating the ground states of Super-Kac-Moody representations. There is therefore no need to modify the earlier well-tested picture.

### 6.3.3 How to satisfy the conditions guaranteeing the conservation of em charge?

There are two manners to satisfy the conditions guaranteeing the conservation of em charge leading to three separate Dirac equations. The first option is inspired by string model and solutions are annihilated either by second charged gamma matrix or holomorphic covariant derivative or by the conjugates of these. For the second option the charged modified gamma matrices vanish identically.

### Holomorphy of the solutions

In string model holomorphy/antiholomorphy for the modes of the induced spinor field is essential from the point of view of Super-Virasoro conditions. For preferred extremals the holomorphy seems to be in a key role and it would not be surprising if this were the case also in the fermionic sector. Could the additional Dirac equations associated with charged parts of the modified Dirac operator be solved by a generalization of holomorphy or anti-holomorphy? For the second charged Dirac operator - say $D_+$ - complexified gamma matrices would annihilate spinor mode and for the second one - say $D_-$ - holomorphic covariant derivative would annihilate the spinor. Note that the gamma matrices $\Gamma_+$ and $\Gamma_-$ are hermitian conjugates of each other.

The condition that either charged gamma matrix annihilates the spinor mode for the space-time sheet in question requires super-symmetry. For $CP_2$ one has this kind of supersymmetry but covariant constancy allows only right handed neutrino spinor: in this case however both holomorphic gamma matrices annihilate the spinor. In super-string models in flat target space, one has maximal supersymmetry allowing maximal number of covariantly constant spinor modes. For modified gamma matrices this kind of situation might be realized. If one allows the restriction of the induced spinor fields to string world sheets or partonic 2-surfaces, the situation simplifies further, and it might be possible to assume holomorphy in the complex coordinate parametrizing the 2-surface.

Either $\Gamma_+$ or $\Gamma_-$ should annihilate the spinor mode. For right-handed neutrino second half of complexified gamma matrices annihilate it. This condition is analogous to the condition that fermionic annihilation operators annihilate the state. In the recent case the condition is weaker since only single complexified gamma matrix annihilates the state. For $CP_2$ Dirac operator one obtains four basic solutions corresponding to $\nu_R$, $\Gamma_+\nu_R$, $\Gamma_0\nu_R$, $\Gamma_0\Gamma_+\nu_R$. $\Gamma_0$ is the second holomorphic complexitifed gamma matrix. Therefore it seems that one might be able to obtain at least two charged states for both quarks and lepton as required by the standard model plus possible higher color partial waves.

A stronger condition is that both $\Gamma_+$ and $\Gamma_-$ vanish and Dirac operator reduces to neutral Dirac operator acting on complex coordinate and its conjugate. If the vanishing of $\Gamma_+$ or $\Gamma_-$ takes place everywhere the energy momentum tensor must be effectively 2-dimensional. This option has been proposed earlier as a solution ansatz. If the vanishing occurs only at 2-D surface, effective 2-dimensionality holds true only at this surface. This option looks more plausible one.

### Option for which charged parts of energy momentum tensor vanish

The vanishing of the charged parts of the modified Dirac operator $D$ - or equivalently, those of the energy momentum tensor - would reduce $D$ to its neutral part and the conditions would trivialize. There would be no need for the full holomorphy, which could be an un-physical condition for $CP_2$ and not favored if one assumes that all color partial waves in $CP_2$ correspond to physical states in the construction of representations of the symplectic conformal algebra. On the other hand, the reduction of allowed modes to just the observed one (singlet for leptons and triplet for quarks) is an attractive property. In string models one would speak about super-symmetry breaking. Note that holomorphy in the remaining neutral complex or hyper-complex coordinate could provide elegant solution of the modified Dirac equation exactly in the same manner as it does in string models.

It is however not at all clear whether the charged part of the energy momentum tensor can vanish everywhere. Note however that since imvbedding space projections of the energy momentum tensor are in question, the conditions do not mean reduction of the rank of energy momentum tensor to at most two. The possibility that the vanishing occurs only at 2-D surfaces analogous to string world sheets and that induced spinor fields must be restricted to these, is consistent with the existent vision

One can study the option allowing non-vanishing charged Dirac operators and requiring holomorphy, covariant constancy, and extended supersymmetry in more detail. To get some perspective one can the situation in the case of $CP_2$ type vacuum extremals. In this case the energy momentum tensor and therefore also modified gamma matrices vanish identically. Therefore the situation trivializes and one might hope that for the deformations of $CP_2$ type vacuum extremals the charged parts of the modified Dirac operator vanish or that holomorphy makes sense.

### 6.3.4 Could the solutions of the modified Dirac equation be restricted to 2-D surfaces?

The condition that the charged Dirac operators (and also neutral one) annihilate physical states is rather strong. If the charged parts $T^{\alpha}_{\pm}$ of the energy momentum tensor vanish, these conditions apply only to the space-time surface. These conditions are however quite strong and the question is whether one could require them for 2-D sub-manifolds of space-time surface- the analogs of string world sheets- only, and assume that the modes of Dirac equation are restricted on these.

There are several other manners to end up with this view.

1. The vision inspired by the finite measurement resolution is that the solutions of the modified Dirac equation are singular and restricted to 2-dimensional surfaces identifiable as string world sheets in Minkowskian signature and as partonic 2-surfaces in Euclidian signature. For 3-D light-like surfaces solutions would be restricted at word lines defining strands of braid defining discretization as space-time correlate for the finite measurement resolution. The interesting self-referential aspect would be that physical system itself would define the finite measurement resolution.

2. Another interpretation is in terms of the number theoretical vision [K54]. Space-time surfaces define associative or co-associative 4-surfaces with tangent space allowing quaternionic of co-quaternionic structure or its Minkowskian variant. This is a formulation for the idea that classical dynamics is determined by associativity condition. One can however go further and require also commutativity or co-commutativity and this leads to string world sheets or partonic 2-surfaces. The vanishing of the charged components of the energy momentum tensor could be indeed seen as a condition stating that the surface is complex or co-complex sub-manifold (or hyper-complex or co-hyper-complex one).

3. Also strong form of holography leads to the idea that 2-D partonic surfaces or string world sheets and the 4-D tangent space data at them should be enough for the formulation of quantum theory and 4-D space-time surfaces are necessary only for the realization of quantum classical correspondence. One could say that space-time surface is analogous to phase space and that in quantum theory only 2-D slice of it analogous to Lagrangian sub-manifold can be used.

The general vision about preferred extremals involves a non-trivial aspect not yet mentioned [K5] and this allows to developed an argument in favor of reductionof spinorial dynamics to that at 2-D surfaces.

1. Various conserved currents are suggested to define integrable flows meaning that one can identify a global coordinate varying along the flow lines. Could the charged parts of the energy momentum tensor defined as currents define Beltrami flow? If so, these currents have expression of form $J_{\pm} = \Phi_{\pm}\nabla\Psi_{\pm}$, where $\Phi_{\pm}$ and $\Psi_{\pm}$ are complex scalar functions such that the latter ones define the global coordinate. If this is the case, then the surface at which $J_{\pm}$ vanishes corresponds to the surface $\Phi_{+} = \overline{\Phi}_{-} = 0$ and by complex valuedness of $\Phi_{\pm} = 0$ is 2-dimensional rather than 0-dimensional as for a generic vector field. The charged parts of energy momentum tensor vanish identically as do the corresponding modified gamma matrices.

2. Vanishing of $\Phi_{\pm}$ would reduce the 4-D conformal algebra to 2-dimensional conformal algebra associated with the string world sheet or partonic 2-surface, and this is just what is expected on basis of physical intuition. One could say that locally space-time surface reduces to effectively 2-D surface. Charge conservation would select 2-dimensional string world sheets and/or partonic 2-surfaces and reproduce the earlier picture inspired by the notion of finite measurement resolution, by number theoretical considerations, and by strong form of holography.

A couple of further comments about are in order.

1. A natural consistency condition is that the modified gamma matrices in the modified Dirac operator are parallel to the 2-D surface. Otherwise one obtains covariant derivatives in transversal direction giving delta functions. This requires that modified gamma matrices generate a 2-D subspace tangential to $X^2$ at $X^2$. This condition need not hold true elsewhere. This would mean

that with respect to the effective metric defined by the anticommutators of the modified gamma matrices space-time surface becomes effectively 2-dimensional locally. Effective 2-dimensionality of the effective metric was conjectured already earlier but now it is restricted to string world sheets and partonic 2-surfaces thus appearing as singularities of the preferred extremals. String world sheets and partonic 2-surfaces must obey some dynamics and minimal surface equation in the effective metric is a good guess since it automatically would reduce the situation to 2-D one.

2. Weak form of electric magnetic duality states that at partonic 2-surface $X^2$ the Kähler electric field strength $J^{n\beta}$ in 2-dimensional tangent plane of $X^4$ transversal to $X^2$ is proportional to the 4-D dual of the Kähler magnetic field strength $J_{\alpha\beta}$ at $X^2$ : $J^{n\beta} = k\epsilon^{n\beta\gamma\delta}J_{\gamma\delta}$ , $k = constant$. The transversal plane is not unique without some additional condition and the natural condition is that it defines tangent plane to the string world sheet.

### 6.3.5 The algebra spanned by the modified Dirac operators

The conservation of em charge for the modified Dirac equation implies that the electromagnetic charge determined defined as $Q = aI + bI_3$ is conserved in the classical electro-weak gauge fields identified as induced gauge fields. This condition is highly non-trivial and as has been found could hold true only at 2-D surfaces implying a stringy localization of fermions.

For the modified Dirac equation additional consistency conditions analogous to Super-Virasoro conditions follow from the conditions that electromagnetic charge is constant for the modes of the modified Dirac equation. The conditions state that the possibly infinite-dimensional algebra or super-algebra generated by the neutral part and two charged parts with charges $\pm 1$ of the modified Dirac operator annihilates the preferred solutions of the modified Dirac equation.

For super-algebra option one would start with anti-commutators of the Dirac operators and consider commutators when either generators has even value of em charge. For algebra option one would consider only commutators which are formally Lie commutators of gamma matrix valued vector fields $\Gamma_i^\alpha D_\alpha$ which ordinary derivatives replaced with covariant derivatives and components of the vector fields replaced with the modified gamma matrices obtained by contracting the neutral or charged part of the energy momentum tensor with flat space gamma matrices.

The commutator is the more feasible option as following arguments show.

1. Ordinary modified Dirac equation gives rise to a conserved fermionic current and its conservation could be seen as a consequence of the modified Dirac equation. The statement that the modified Dirac operators annihilate the induced spinors could thus be equivalent with the statement that SU(2) triplet of fermionic currents is conserved. In old fashioned hadron physics this corresponds to conserved vector current hypothesis.

2. The algebra defined by the commutators has the structure of vectorial SU(2) algebra and the natural guess is that this algebra relates closely to $\mathcal{N} = 2$ super-conformal algebra for which super-generators $G$ form SU(2) triplets and which allows conserved $U(1)$ currents besides energy momentum tensor.

   In the recent case the U(1) charge would be em charge. As a matter fact, $\mathcal{N} = 2$ algebra is accompanied also by SU(2) algebra of conserved currents and the attractive interpretation is that the Dirac operators generate this algebra.

3. The algebra of Dirac operators annihilating the induced spinor field would define algebra of divergences of fermionic currents of form $J_i^\alpha = \overline{\Psi}\Gamma_i^\alpha\Psi$. These currents are conserved if the modified Dirac equations are satisfied. The algebra generated by the commutators of these fermionic currents assuming anti-commutation relations for the induced spinor fields should be equivalent with the algebra of Dirac operators. This should fix the anti-commutation relations for the induced spinor fields.

4. The outcome is a bosonic algebra of vector currents. By replacing $\overline{\Psi}$ or $\Psi$ with a mode of the induced spinor field one would obtain super-algebra generators of extended super-algebra. The divergences of fermionic and bosonic generators would generate algebra which vanishes identically for the solutions of the modified Dirac equation. The currents themselves would be non-vanishing.

5. If the charged currents vanish at 2-surface then the commutator of charged Dirac operator vanishes identically. The commutators of neutral and charged Dirac operators need not vanish identically and it might be necessary to pose this as an additional conditions. The vanishing conditions reduce to the vanishing of the ordinary commutator $[T_0, T_\pm]$ of vector fields $T_0$ and $T_\pm$.

What about the super algebra part of the Super-Virasoro algebra. Is it also present?

1. One must notice that it is the "gamma matrix fields" defined by neutral and charged parts of the modified gamma matrices $\Gamma_i^A$ forming an SU(2) triplet and these anti-commute classically to parts of the modified metric for which certain parts should vanish. These "certain parts" should vanish also for the induced metric resulting as anti-commutators of the induced gamma matrices. The conditions are not expected to be independent and should correspond to Virasoro conditions for the induced metric.

2. The SU(2) Super Virasoro algebra discussed above would naturally relate to $\mathcal{N}=2$ variant of the ordinary Super-Virasoro algebra and electromagnetic charge would take the role of conserved $U(1)$ current accompanying $\mathcal{N}=2$ algebra. This algebra indeed involves SU(2) as an additional symmetry algebra. Modified Dirac equations would correspond to conservation of the SU(2) currents and the vanishing conditions on the induced metric and/or its analog defined by the anti-commutators of the modified gamma matrices would correspond to Virasoro conditions. It is however not clear whether the modified gamma matrices - or rather, their second quantized variants - should annihilate the physical states. This condition would correspond for the induced spinor fields a condition stating that second half of complexified modified gamma matrices annihilates the right handed neutrino spinor serving as the analog of fermionic Fock vacuum.

### 6.3.6 Connection with the number theoretical vision about field equations

The recent progress in the understanding of preferred extremals of Kḧler action suggests also an interesting connection to the number theoretic vision about field equations [K54]. In particular, it might be possible to understand how one can have Hermitian/Harmilton-Jacobi structure simultaneously with quaternionic structure and how quaternionic structure is possible for the Minkowskian signature of the induced metric.

One can imagine two manners of introducing octonionic and quaternionic structures. The first one is based on the introduction of octonionic representation of gamma matrices and second on the notion of octonion real-analycity.

1. If quaternionic structure is defined in terms of the octonionic representation of the imbedding space gamma matrices, there seems to be no obvious problems since one considers automatically complexification of quaternions represented in terms of gamma matrices. For the approach based on the notion of quaternion real analyticity, one is forced to use Wick rotation to define the quaternionic structure in Minkowskian regions or to introduce what I have called hyper-quaternionic structure by imbedding the space-time surface to a sub-space $M^8$ of complexified octonions. This is admittedly artificial.

2. The octonionic representation effectively replaces $SO(7,1)$ as tangent space group with $G_2$ and means selection of preferred $M^2 \subset M^4$ having interpretation complex plane of octonionic space. A more general condition is that the tangent space of space-time surface at each point contains preferred sub-space $M^2(x) \subset M^4$ forming an integrable distribution. The same condition is involved with the definition of Hamilton-Jacobi structure. What puts bells ringing is that the modified Dirac equation for the octonionic representation of gamma matrices allows the conservation of electromagnetic charge in the proposed sense a observed for years ago. One can ask whether the conditions on the charged part of energy momentum tensor could relate to the reduction of $SO(7,1)$ to $G_2$.

3. Octonionic gamma matrices appear also in the proposal stating that space-time surfaces are quaternionic in the sense that tangent space of the space-time surface is quaternionic in the sense that induced octonionic gamma matrices generate a quaternionic sub-space at a given point of space-time time. Besides this the already mentioned additional condition stating that

the tangent space contains preferred sub-space $M^2 \subset M^4$ or integrable distribution of this kind of sub-spaces is required. It must be emphasized that induced rather than modified gamma matrices are natural in these conditions.

**Definition of quaternionicity based on gamma matrices**

The definition of quaternionicity in terms of gamma matrices looks more promising. This however raises two questions.

1. Can the quaternionicity of the space-time surface together with a preferred distribution of tangent planes $M^2(x) \subset M^4$ or $E^2(x) \subset CP_2$ be equivalent with the reduction of the field equations to the analogs of minimal surface equations stating that certain components of the induced metric in complex/Hamilton-Jacobi coordinates vanish in turn guaranteeing that field equations reduce to algebraic identifies following from the fact that energy momentum tensor and second fundamental form have no common components? This should be the case if one requires that the two solution ansätze are equivalent.

2. Can the conditions for the modified Dirac equation select complex of co-complex 2-sub-manifold of space-time surface identified as quaternionic or co-quaternionic 4-surface? Could the conditions stating the vanishing of charged energy momentum currents state that the spinor fields are localized to complex or co-complex (hyper-complex or co-hypercomplex) 2-surfaces?

One should assign to the space-time sheets both quaternionic and Hermitian or Hamilton-Jacobi structure. There are two structures involved. Euclidian metric is an essential aspect of what it is to be quaternionic or octonions. It however seems that one can assign to the induced metric only Hermitian or Hamilton-Jacobi structure. This leads to a serious of innocent questions.

1. Could these two structures be associated with canonical momentum currents and metric respectively? Anti-commutators of the modified gamma matrices define an effective metric expressible in terms of canonical momentum currents as

$$G^{\alpha\beta} = \Pi_k^{\alpha} \Pi^{\beta k} \quad .$$

   Here $\Pi_k^{\alpha} = \partial L / \partial_{\alpha} h^k$ is the canonical momentum current. This effective metric should have a deep physical and mathematical meaning but this meaning has remained a mystery.

2. Could $G$ be assigned with the quaternionic structure and induced metric to the Hermitian/Hamilton-Jacobi structures? Or perhaps vice versa? Could the neutral and charged components of the energy momentum tensor somehow correspond to quaternionic units?

The basic potential problem with the assignment of quaternionic structure to the induced gamma matrices is the signature of the metric in Minkowskian regions.

1. If quaternionic structrures is defined in terms of the octonionic representation of the imbedding space gamma matrices, there seems to be no obvious problems since one considers automatically complexification of quaternions.

2. For the approach based on the notion of quaternion real analyticity, one is forced to use Wick rotation to define the quaternionic structure or to introduce hyper-quaternionic structure by imbedding the space-time surface to a sub-space M¡sup¿8¡/sup¿ of complexified octonions. This is admittedly artificial.

Could one pose the additional requirement that the signature of the effective metric $G$ defined by the modified gamma marices (and to be distinguished from Einstein tensor) is Euclidian in the sense that all four eigenvalues of this tensor would have same sign.

1. For the induced metric the projections of gamma matrices are given by

$$\Gamma_{\alpha} = \Gamma^a e_{a\alpha} \quad , \quad e_{a\alpha} = e_{ak} \partial_{\alpha} h^k \quad .$$

For the modified gamma matrices their analogs would be given by

$$\Gamma^\alpha = \Gamma^a E^\alpha_a \ , \ E^\alpha_a = e^k_a \Pi^\alpha_k \ .$$

One cannot induce $G$ from any metric defined in the imbedding space but the notion of tangent space quaternionicity is well-defined.

2. What quaternionic structure for $G$ could mean? One can imagine several options.

   (a) For the ordinary complex structure metric has vanishing diagonal components and the inner product for infinitesimal vectors is just $g_{z\bar{z}}(dz_1 d\overline{z}_2 + dz_2 d\overline{z}_1)$. Could this formula generalize to $g_{Q\bar{Q}}(dQ_1 d\overline{Q}_2 + dQ_2 d\overline{Q}_1)$? The generalization would be a direct generalization of conformal invariance to 4-D context stating that 4-metric is quaternion-conformally equivalent to flat metric. This would give additional strong condition on energy momentum tensor:

   $$G = \Pi^\alpha_k \Pi^{\beta k} = T^2 \delta_{\alpha\beta} \ .$$

   The proportionality to Euclidian metric means in Minkowskian realm that the $G$ is of form $G = T^2(2u_\alpha u_\beta - g_{\alpha\beta}$. Here $u$ is time-like vector field satisfying $u^\alpha u_\alpha = 1$ and having interpretation as a local four-velocity (in Robertson-Walker cosmology similar situation is encountered). The eigen value problem in the form $G^\alpha_\beta x^\beta = \lambda x^\alpha$ makes sense and eigenvectors would be $u^\alpha$ with eigenvalue $\lambda = T^2$ and three vectors orthogonal to it with eigenvalue $-T^2$. This requires integrable flow defined by $u$ and defining a preferred time coordinate. In number theoretic vision this kind of time coordinate is introduced and corresponds to the direction assignable to the octonionic real unit. Note that the vanishing of charged projections of the energy momentum tensor does not imply a reduction of the rank of $T$ so that this options might work.

   (b) Quaternionicity could mean also the structure of hyper-Kähler manifold. Metric and Kähler form for Kähler manifold are generalized to metric representing quaternion real unit and three covariantly constant Kähler forms $I_i$ obeying the multiplication rules for quaternions. The necessary condition is that the holonomy group equals to SU(2) identifiable as automorphism group of quaternions. One can also define quaternionic structure: there would exist three antisymmetric tensors, whose squares give the negative of the metric. $CP_2$ allows quaternionic structure in this sense and only one of these forms is covariantly constant.

   Could space-time surface allow Hyper-Kähler or quaternionic structure somehow induced from that of $CP_2$? This does not work for $G$. $G$ is quadratic in energy momentum tensor and therefore involves four power of $J$ rather than being square of projection of $J$ or two other quaternionic imaginary units of $CP_2$. One can of course ask whether the induced quaternionic units could obey the multiplication of quaternionic units and have same square given by the projection of $CP_2$ metric. In this case $CP_2$ metric would define the effective metric and would be indeed Euclidian. For the ansatz for preferred extremals with Minkowskian signature $CP_2$ projection is at most 3-dimensional but also in this case the imaginary units might allow a realization as projections.

### Definition of quaternionicity based on octonion real-analyticity

Second definition of quaternionicity is on more shaky basis and motivated by the solutions of 2-D Laplace equation: quaternionic space-time surfaces would be obtained as zero loci of octonion real–analytic functions. Unfortunately octonion real–analyticity does not make sense in Minkowskian signature.

One could understand octonion real-analyticity in Minkowskian signature if one could understand the deeper meaning of Wick rotation. Octonion real analyticity formulated as a condition for the vanishing of the imaginary part of octonion real-analytic function makes sense for in octonionic coordinates for $E^4 \times CP_2$ with Euclidian signature of metric. $M^4 \times CP_2$ is however only a subspace of complexified octonions and not closed with respect to multiplication so that octonion real-analytic functions do not make sense in $M^4 \times CP_2$ . Wick rotation should transform the solution candidate defined by an octonion real-analytic function to that defined in $M^4 \times CP_2$. A natural additional condition is that Wick rotation should reduce to that taking $M^2 \subset M^4$ to $E^2 \subset E^4$.

The following trivial observation made in the construction of Hamilton-Jacobi structure in $M^4$ with Minkowskian signature of the induced metric (see the Appendix) as a Wick rotation of Hermitian structure in $E^4$ might help here.

1. The components of the metric of $E^2$ in complex coordinates $(z, \overline{z})$ for $E^2$ are given by $g_{w\overline{w}} = -1$ whereas the metric of $M^2$ in light-like coordinates $(u = x+t, v = x-t)$ is given by $g_{uv} = -1$. The metric is same and $M^2$ and $E^2$ correspond only to different interpretations for the coordinates! One could say that $M^4 \times CP_2$ and $E^4 \times CP_2$ have same metric tensor, Kähler structure, and spinor structure. Since only these appear in field equations, one could hope that the solutions of field equations in $M^4 \times CP_2$ and $E^4 \times CP_2$ are obtained by Wick rotation. This for preferred extremals at least and if the field equations reduce to purely algebraic ones.

2. If one accepts the proposed construction of preferred extremals of Kähler action discussed in [K67], the field equations indeed reduce to purely algebraic conditions satisfied if space-time surface possesses Hermitian structure in the case of Euclidian signature of the induced metric and Hamilton-Jacobi structure in the case of Minkowskian signature. Just as in the case of minimal surfaces, energy momentum tensor and second fundamental form have no common non-vanishing components. The algebraization requires as a consistency condition Einstein's equations with a cosmological term. Gravitational constant and cosmological constant follow as predictions.

3. If Wick rotation in the replacement of $E^2$ coordinates $(z, \overline{z})$ with $M^2$ coordinates $(u, v)$ makes sense, one can hope that field equations for the preferred extremals hold true also for a Wick rotated surfaces obtained by mapping $M^2 \subset M^4$ to $E^2 \subset E^4$. Also Einstein's equations should be satisfied by the Wick rotated metric with Euclidian signature.

4. Wick rotation makes sense also for the surfaces defined by the vanishing of the imaginary part (complementary to quaternionic part) of octonion real-analytic function. Therefore one can hope that this ansatz could work. Wick rotation is non-trivial geometrically. For instance, light-like lines $v = 0$ of hyper-complex plane $M^2$ are taken to $\overline{z} = 0$ defining a point of complex plane $E^2$. Note that non-invertible hyper-complex numbers correspond to the two light-like lines $u = 0$ and $v = 0$ whereas non-invertible complex numbers correspond to the origin of $E^2$.

5. If the conjecture holds true, one can apply to both factors in $E^4 = E^2 \times E^2$ and to get preferred extremals in $M^{2,2} \times CP_2$. Minkowski space $M^{2,2}$ is essential in twistor approach and the possibility to carry out Wick rotation for preferred extremals could justify Wick rotation in quantum theory.

### 6.3.7 Modification of the measurement interaction term

By quantum classical correspondence the momenta and other quantum numbers should have correlates in the geometry of the space-time sheet. This suggests an inclusion to the modified Dirac action of a general coordinate invariant measurement interaction term invariant under appropriate subgroup of isometries characterizing the choice of the measurement axis. In the following only the measurement interaction term assignable to four-momentum is discussed. One could assign this term only to 3-D space-like ends of space-time surface and the light-like wormhole throats. Somewhat surprisingly the effective gauge character of this term allows also the assignment to space-time interior.

The first guess for the measurement interaction term for 4-momentum would be as $\lambda\overline{\Psi}\Gamma^\alpha p_\alpha\Psi$ restricted to 3-D preferred surface in question. This term however vanishes at the light-like orbits of wormhole throats since the modified gamma matries defined by the Chern-Simons term contain only $CP_2$ gamma matrices. This forced to replace the term with $\lambda\overline{\Psi}\gamma^k p_k\Psi$ in the original approach [K19]. This term does not possess a formal gauge character and treats $M^4$ and $CP_2$ asymmetrically. Second problem is that measurement interaction term is proportional to a constant $\lambda$ with dimensions of mass and unless one can relate it to gravitational constant, is un-natural.

As already noticed, the measurement interaction term formally corresponds to a gauge transform of Kähler gauge potential by the gradient $p_\alpha = p_k\partial_\alpha m^k$ defining the momentum projection. The change of the gauge eliminating this term introduces plane wave factor to the induced spinor field. The gauge transformation eliminating the measurement interaction term does not become trivial asymptotically and might therefore carry physical information. Therefore one can consider also the possibility that measurement interaction term is introduced at the entire 4-D space-time sheet. In this case the change of gauge by a phase transformation introducing a plane-wave factor would lead to the equation without measurement interaction term and one would obtain holomorphic solutions.

Consider now the 4-D option in more detail.

1. One can argue that the measurement interaction term in the interior can be transformed away by a gauge transformation $A_\alpha \rightarrow A_\alpha - p_\alpha$ so that the holomorphic solutions are not lost. The global nature of the gauge transformation gives hopes that it indeed codes information via the plane wave phase multiplying the holomorphic solutions. The projection $p_\alpha$ appearing in the contraction with the modified gamma matrices is automatically parallel to the tangent space of the string world sheet or partonic 2-surface.

2. The question whether there is a connection between gravitational and ordinary Planck constants led to the conjecture that the gravitational momentum squared defined by the modified gamma matrices would be equal to inertial momentum squared [K18] just as Equivalence Principle requires. In other words, the gravitational longitudinal 2-momentum squared $p_{gr}^2 = g_{eff}^{\alpha\beta} p_\alpha p_\beta$ would be equal to the inertial 2-momentum squared $p_I^2 = m_2^{kl} p_k p_l$ at respective tangent spaces $M^2$ *resp.* $E^2$ of string world sheet *resp.* partonic 2-surface. At the ends of braid strands defining the intersections of string world sheets and partonic 2-surfaces, one would have

$$p_{gr}^2 = g_{eff,s}^{\alpha\beta} p_\alpha p_\beta + g_{eff,p}^{\alpha\beta} p_\alpha p_\beta = p^2 \ .$$

   Here the subscripts 's' and 'p' refer to string world sheet and partonic 2-surface respectively.

3. It would be nice if this condition would somehow follow from the proposed field equation for the induced spinors at the edges of string world sheet, where one should treat the gauge conditions carefully without doing the gauge transformation. At the intersection point it would seem necessary to assume that the ordinary derivatives - maybe even covariant derivatives - vanish. If covariant derivatives vanish, the modified Dirac equation in 4-D sense would reduce to the condition that the sum of the measurement interaction terms annihilates the spinor modes. This would give

$$\Gamma^\alpha p_\alpha \Psi = 0$$

   at the ends of braid strands and this would give massless condition in 4-D sense stating $p_{gr,||}^2 + p_{gr,\perp}^2 = 0$.

4. The modified Dirac equation contains a boundary term $\Gamma^n \Psi$ at the boundaries of the string world sheet. The vanishing of $\Gamma^n$ proportional to the canonical momentum current in the normal direction at wormhole throats could be forced by the condition that classical charges do not leak between Minkowskian and Euclidian regions of the space-time sheet. This condition cannot be posed at space-like 3-surfaces since they represent initial data.

To sum up, this option is favored because no dimensional coupling is needed and because one obtains a connection between ordinary Planck constant and gravitational Planck constant as discussed in [K18]. Also a close connection with braid picture and generalized Feynman diagrams with lines identified as massless wormhole throats emerges.

## 6.4 Preferred extremals and solutions of the modified Dirac equation and super-conformal symmetries

The new vision about preferred exrtremals and modified Dirac equations is bound to check the existing vision about super-conformal symmetries. One important discovery is that Einstein's equations follow from the vanishing of terms proportional to Kähler current in field equations for preferred extremals and Equivalence Principle at the classical level is realized automatically in all scales in contrast to the earlier belief. This obviously must have implications to the general vision about Super-Virasoro representations and one must be ready to modify the existing picture based on the assumption that quantum version of Equivalence Principle is realized in terms coset representations.

The very special role of right handed neutrino is also bound to have profound implications. A further important outcome is the identification of gauge potentials as duals of Kac-Moody currents at the boundaries of string world sheets: quantum gauge potentials are defined only where they are needed that is the curves defining the non-integrable phase factors. This gives also rise to the realization of the conjecture Yangian in terms of the Kac-Moody charges and commutators in accordance with the earlier conjecture.

### 6.4.1 Super-conformal symmetries

It is good to summarize first the basic ideas about Super-Virasoro representations. TGD allows two kinds of super-conformal symmetries.

1. The first super-conformal symmetry is associated with $\delta M^4_\pm \times CP_2$ and corresponds to symplectic symmetries of $\delta M^4_\pm \times CP_2$. The reason for extension of conformal symmetries is metric 2-dimensionality of the light-like boundary $\delta M^4_\pm$ defining upper/lower boundary of causal diamond ($CD$). This super-conformal symmetry is something new and corresponds to replacing finite-dimensional Lie-group $G$ for Kac-Moody symmetry with infinite-dimensional symplectic group. The light-like radial coordinate of $\delta M^4_\pm$ takes the role of the real part of complex coordinate $z$ for ordinary conformal symmetry. Together with complex coordinate of $S^2$ it defines 3-D restriction of Hamilton-Jacobi variant of 4-D super-conformal symmetries. One can continue the conformal symmetries from light-cone boundary to $CD$ by forming a slicing by parallel copies of $\delta M^4_\pm$. There are two possible slicings corresponding to the choices $\delta M^4_+$ and $\delta M^4_-$ assignable to the upper and lower boundaries of $CD$. These two choices correspond to two arrows of geometric time for the basis of zero energy states in ZEO.

2. Super-symplectic degrees of freedom determine the electroweak and color quantum numbers of elementary particles. Bosonic emergence implies that ground states assignable to partonic 2-surfaces correspond to partial waves in $\delta M^4_\pm$ and one obtains color partial waves in particular. These partial waves correspond to the solutions for the Dirac equation in imbedding space and the correlation between color and electroweak quantum numbers is not quite correct. Super-Kac-Moody generators give the compensating color for massless states obtained from tachyonic ground states guaranteeing that standard correlation is obtained. Super-symplectic degrees are therefore directly visible in particle spectrum. One can say that at the pointlike limit the WCW spinors reduce to tensor products of imbedding space spinors assignable to the center of mass degrees of freedom for the partonic 2-surfaces defining wormhole throats.

   I have proposed a physical interpretation of super-symplectic vibrational degrees of freedom in terms of degrees of freedom assignable to non-perturbative QCD. These degrees of freedom would be responsible for most of the baryon masses but their theoretical understanding is lacking in QCD framework.

3. The second super-conformal symmetry is assigned light-like 3-surfaces and to the isometries and holonomies of the imbedding space and is analogous to the super-Kac-Moody symmetry of string models. Kac-Moody symmetries could be assigned to the light-like deformations of light-like 3-surfaces. Isometries give tensor factor $E^2 \times SU(3)$ and holonomies factor $SU(2)_L \times U(1)$. Altogether one has 5 tensor factors to super-conformal algebra. That the number is just five is essential for the success p-adic mass calculations [K34, K29].

   The construction of solutions of the modified Dirac equation suggests strongly that the fermionic representation of the Super-Kac-Moody algebra can be assigned as conserved charges associated with the space-like braid strands at both the 3-D space-like ends of space-time surfaces and with the light-like (or space-like with a small deformation) associated with the light-like 3-surfaces. The extension to Yangian algebra involving higher multilinears of super-Kac Moody generators is also highly suggestive. These charges would be non-local and assignable to several wormhole contacts simultaneously. The ends of braids would correspond points of partonic 2-surfaces defining a discretization of the partonic 2-surface having interpretation in terms of finite measurement resolution.

These symmetries would correspond to electroweak and strong gauge fields and to gravitation. The duals of the currents giving rise to Kac-Moody charges would define the counterparts of gauge potentials and the conserved Kac-Moody charges would define the counterparts of non-integrable phase factors in gauge theories. The higher Yangian charges would define generalization of non-integrable phase factors. This would suggest a rather direct connection with the twistorial program for calculating the scattering amplitudes implies also by zero energy ontology.

Quantization recipes have worked in the case of super-string models and one can ask whether the application of quantization to the coefficients of powers of complex coordinates or Hamilton-Jacobi coordinates could lead to the understanding of the 4-D variants of the conformal symmetries and give detailed information about the representations of the Kac-Moody algebra too.

### 6.4.2 What is the role of the right-handed neutrino?

A highly interesting aspect of Super-Kac-Moody symmetry is the special role of right handed neutrino.

1. Only right handed neutrino allows besides the modes restricted to 2-D surfaces also the 4D modes delocalized to the entire space-time surface. The first ones are holomorphic functions of single coordinate and the latter ones holomorphic functions of two complex/Hamilton-Jacobi coordinates. Only $\nu_R$ has the full $D = 4$ counterpart of the conformal symmetry and the localization to 2-surfaces has interpretation as super-conformal symmetry breaking halving the number of super-conformal generators.

2. This forces to ask for the meaning of super-partners. Are super-partners obtained by adding $\nu_R$ neutrino localized at partonic 2-surface or delocalized to entire space-time surface or its Euclidian or Minkowskian region accompanying particle identified as wormhole throat? Only the Euclidian option allows to assign right handed neutrino to a unique partonic 2-surface. For the Minkowskian regions the assignment is to many particle state defined by the partonic 2-surfaces associated with the 3-surface. Hence for spartners the 4-D right-handed neutrino must be associated with the 4-D Euclidian line of the generalized Feynman diagram.

3. The orthogonality of the localized and de-localized right handed neutrino modes requires that 2-D modes correspond to higher color partial waves at the level of imbedding space. If color octet is in question, the 2-D right handed neutrino as the candidate for the generator of standard SUSY would combine with the left handed neutrino to form a massive neutrino. If 2-D massive neutrino acts as a generator of super-symmetries, it is is in the same role as badly broken supersymmeries generated by other 2-D modes of the induced spinor field (SUSY with rather large value of $\mathcal{N}$) and one can argue that the counterpart of standard SUSY cannot correspond to this kind of super-symmetries. The right-handed neutrinos delocalized inside the lines of generalized Feynman diagrams, could generate $\mathcal{N} = 2$ variant of the standard SUSY.

#### How particle and right handed neutrino are bound together?

Ordinary SUSY means that apart from kinematical spin factors sparticles and particles behave identically with respect to standard model interactions. These spin factors would allow to distinguish between particles and sparticles. But is this the case now?

1. One can argue that 2-D particle and 4-D right-handed neutrino behave like independent entities, and because $\nu_R$ has no standard model couplings this entire structure behaves like a particle rather than sparticle with respect to standard model interactions: the kinematical spin dependent factors would be absent.

2. The question is also about the internal structure of the sparticle. How the four-momentum is divided between the $\nu_R$ and and 2-D fermion. If $\nu_R$ carries a negligible portion of four-momentum, the four-momentum carried by the particle part of sparticle is same as that carried by particle for given four-momentum so that the distinctions are only kinematical for the ordinary view about sparticle and trivial for the view suggested by the 4-D character of $\nu_R$.

Could sparticle character become manifest in the ordinary scattering of sparticle?

1. If $\nu_R$ behaves as an independent unit not bound to the particle, it would continue in the original direction as particle scatters: sparticle would decay to particle and right-handed neutrino. If $\nu_R$ carries a non-negligible energy the scattering could be detected via a missing energy. If not, then the decay could be detected by the interactions revealing the presence of $\nu_R$. $\nu_R$ can have only gravitational interactions. What these gravitational interactions are is not however quite clear since the proposed identification of gravitational gauge potentials is as duals of Kac-Moody currents analogous to gauge potentials located at the boundaries of string world sheets. Does this mean that 4-D right-handed neutrino has no quantal gravitational interactions? Does internal consistency require $\nu_R$ to have a vanishing gravitational and inertial masses and does this mean that this particle carries only spin?

2. The cautious conclusion would be following: if delocalized $\nu_R$ and parton are un-correlated particle and sparticle cannot be distinguished experimentally and one might perhaps understand the failure to detect standard SUSY at LHC. Note however that the 2-D fermionic oscillator algebra defines badly broken large $\mathcal{N}$ SUSY containing also massive (longitudinal momentum square is non-vanishing) neutrino modes as generators.

**Taking a closer look on sparticles**

It is good to take a closer look at the delocalized right handed neutrino modes.

1. At imbedding space level that is in cm mass degrees of freedom they correspond to covariantly constant $CP_2$ spinors carrying light-like momentum which for causal diamond could be discretized. For non-vanishing momentum one can speak about helicity having opposite sign for $\nu_R$ and $\overline{\nu}_R$. For vanishing four-momentum the situation is delicate since only spin remains and Majorana like behavior is suggestive. Unless one has momentum continuum, this mode might be important and generate additional SUSY resembling standard $\mathcal{N}=1$ SUSY.

2. At space-time level the solutions of modified Dirac equation are holomorphic or anti-holomorphic.

   (a) For non-constant holomorphic modes these characteristics correlate naturally with fermion number and helicity of $\nu_R$ . One can assign creation/annihilation operator to these two kinds of modes and the sign of fermion number correlates with the sign of helicity.

   (b) The covariantly constant mode is naturally assignable to the covariantly constant neutrino spinor of imbedding space. To the two helicities one can assign also oscillator operators $\{a_\pm, a_\pm^\dagger\}$. The effective Majorana property is expressed in terms of non-orthogonality of $\nu_R$ and and $\overline{\nu}_R$ translated to the the non-vanishing of the anti-commutator $\{a_+^\dagger, a_-\} = \{a_-^\dagger, a_+\} = 1$. The reduction of the rank of the $4\times 4$ matrix defined by anti-commutators to two expresses the fact that the number of degrees of freedom has halved. $a_+^\dagger = a_-$ realizes the conditions and implies that one has only $\mathcal{N}=1$ SUSY multiplet since the state containing both $\nu_R$ and $\overline{\nu}_R$ is same as that containing no right handed neutrinos.

   (c) One can wonder whether this SUSY is masked totally by the fact that sparticles with all possible conformal weights $n$ for induced spinor field are possible and the branching ratio to $n=0$ channel is small. If momentum continuum is present, the zero momentum mode might be equivalent to nothing.

What can happen in spin degrees of freedom in super-symmetric interaction vertices if one accepts this interpretation? As already noticed, this depends solely on what one assumes about the correlation of the four-momenta of particle and $\nu_R$.

1. For SUSY generated by covariantly constant $\nu_R$ and $\overline{\nu}_R$ there is no neutrino four-momentum involved so that only spin matters. One cannot speak about the change of direction for $\nu_R$. In the scattering of sparticle the direction of particle changes and introduces different spin quantization axes. $\nu_R$ retains its spin and in new system it is superposition of two spin projections. The presence of both helicities requires that the transformation $\nu_R \to \overline{\nu}_R$ happens with an amplitude determined purely kinematically by spin rotation matrices. This is consistent with fermion number conservation modulo 2. $\mathcal{N}=1$ SUSY based on Majorana spinors is highly suggestive.

2. For SUSY generated by non-constant holomorphic and anti-holomorphic modes carrying fermion number the behavior in the scattering is different. Suppose that the sparticle does not split to particle moving in the new direction and $\nu_R$ moving in the original direction so that also $\nu_R$ or $\overline{\nu}_R$ carrying some massless fraction of four-momentum changes its direction of motion. One can form the spin projections with respect to the new spin axis but must drop the projection which does not conserve fermion number. Therefore the kinematics at the vertices is different. Hence $\mathcal{N}=2$ SUSY with fermion number conservation is suggestive when the momentum directions of particle and $\nu_R$ are completely correlated.

3. Since right-handed neutrino has no standard model couplings, p-adic thermodynamics for 4-D right-handed neutrino must correspond to a very low p-adic temperature $T=1/n$. This implies that the excitations with non-vanishing conformal weights are effectively absent and one would have $\mathcal{N}=1$ SUSY effectively.

   The simplest assumption is that particle and sparticle correspond to the same p-adic mass scale and have degenerate masses: it is difficult to imagine any good reason for why the p-adic mass scales should differ. This should have been observed -say in decay widths of weak bosons - unless the spartners correspond to large hbar phase and therefore to dark matter. Note that for the badly broken 2-D N=2 SUSY in fermionic sector this kind of almost degeneracy cannot be excluded and I have considered an explanation for the mysterious X and Y mesons in terms of this degeneracy [K32].

**Why space-time SUSY is not possible in TGD framework?**

LHC suggests that one does not have $\mathcal{N}=1$ SUSY in standard sense. Why one cannot have standard space-time SUSY in TGD framework. Let us begin by listing all arguments popping in mind.

1. Could covariantly constant $\nu_R$ represents a gauge degree of freedom? This is plausible since the corresponding fermion current is non-vanishing.

2. The original argument for absence of space-time SUSY years ago was indirect: $M^4 \times CP_2$ does not allow Majorana spinors so that $\mathcal{N}=1$ SUSY is excluded.

3. One can however consider $\mathcal{N}=2$ SUSY by including both helicities possible for covariantly constant $\nu_R$. For $\nu_R$ the four-momentum vanishes so that one cannot distinguish the modes assigned to the creation operator and its conjugate via complex conjugation of the spinor. Rather, one oscillator operator and its conjugate correspond to the two different helicities of right-handed neutrino with respect to the direction determined by the momentum of the particle. The spinors can be chosen to be real in this basis. This indeed gives rise to an irreducible representation of spin 1/2 SUSY algebra with right-handed neutrino creation operator acting as a ladder operator. This is however $\mathcal{N}=1$ algebra and right-handed neutrino in this particular basis behaves effectively like Majorana spinor. One can argue that the system is mathematically inconsistent. By choosing the spin projection axis differently the spinor basis becomes complex. In the new basis one would have $\mathcal{N}=2$ , which however reduces to $\mathcal{N}=1$ in the real basis.

4. Or could it be that fermion and sfermion do exist but cannot be related by SUSY? In standard SUSY fermions and sfermions forming irreducible representations of super Poincare algebra are combined to components of superfield very much like finite-dimensional representations of Lorentz group are combined to those of Poincare. In TGD framework $\nu_R$ generates in space-time interior generalization of 2-D super-conformal symmetry but covarianlty constant $\nu_R$ cannot give rise to space-time SUSY.

   This would be very natural since right-handed neutrinos do not have any electroweak interactions and are are delocalized into the interior of the space-time surface unlike other particles localized at 2-surfaces. It is difficult to imagine how fermion and $\nu_R$ could behave as a single coherent unit reflecting itself in the characteristic spin and momentum dependence of vertices implied by SUSY. Rather, it would seem that fermion and sfermion should behave identically with respect to electroweak interactions.

The third argument looks rather convincing and can be developed to a precise argument.

1. If sfermion is to represent elementary bosons, the products of fermionic oscillator operators with the oscillator operators assignable to the covariantly constant right handed neutrinos must define might-be bosonic oscillator operators as $b_n = a_n a$ and $b_n^\dagger = a_n^\dagger a^\dagger$ One can calculate the commutator for the product of operators. If fermionic oscillator operators commute, so do the corresponding bosonic operators. The commutator $[b_n, b_n^\dagger]$ is however proportional to occupation number for $\nu_R$ in $\mathcal{N}=1$ SUSY representation and vanishes for the second state of the representation. Therefore $\mathcal{N}=1$ SUSY is a pure gauge symmetry.

2. One can however have both irreducible representations of SUSY: for them either fermion or sfermion has a non-vanishing norm. One would have both fermions and sfermions but they would not belong to the same SUSY multiplet, and one cannot expect SUSY symmetries of 3-particle vertices.

3. For instance, $\gamma FF$ vertex is closely related to $\gamma\tilde{F}\tilde{\overline{F}}$ in standard SUSY. Now one expects this vertex to decompose to a product of $\gamma F\overline{F}$ vertex and amplitude for the creation of $\nu_R\tilde{\nu}_R$ from vacuum so that the characteristic momentum and spin dependent factors distinguishing between the couplings of photon to scalar and and fermion are absent. Both states behave like fermions. The amplitude for the creation of $\nu_R\tilde{\nu}_R$ from vacuum is naturally equal to unity as an occupation number operator by crossing symmetry. The presence of right-handed neutrinos would be invisible if this picture is correct. Whether this invisible label can have some consequences is not quite clear: one could argue that the decay rates of weak bosons to fermion pairs are doubled unless one introduces $1/\sqrt{2}$ factors to couplings.

   Where the sfermions might make themselves visible are loops. What loops are? Consider boson line first. Boson line is replaced with a sum of two contributions corresponding to ordinary contribution with fermion and antifermion at opposite throats and second contribution with fermion and antifermion accompanied by right-handed neutrino $\nu_R$ and its antiparticle which now has opposite helicity to $\nu_R$. The loop for $\nu_R$ decomposes to four pieces since also the propagation from wormhole throat to the opposite wormhole throat must be taken into account. Each of the four propagators equals to $a_{1/2}^\dagger a_{-1/2}^\dagger$ or its hermitian conjugate. The product of these is slashed between vacuum states and anti-commutations give imaginary unit per propagator giving $i^4 = 1$. The two contributions are therefore identical and the scaling $g \to g/\sqrt{2}$ for coupling constants guarantees that sfermions do not affect the scattering amplitudes at all. The argument is identical for the internal fermion lines.

### 6.4.3 WCW geometry and super-conformal symmetries

The vision about the geometry of WCW has been roughly the following and the recent steps of progress induce to it only small modifications if any.

1. Kähler geometry is forced by the condition that hermitian conjugation allows geometrization. Kähler function is given by the Kähler action coming from space-time regions with Euclidian signature of the induced metric identifiable as lines of generalized Feynman diagrams. Minkowskian regions give imaginary contribution identifiable as the analog of Morse function and implying interference effects and stationary phase approximation. The vision about quantum TGD as almost topological QFT inspires the proposal that Kähler action reduces to 3-D terms reducing to Chern-Simons terms by the weak form of electric-magnetic duality. The recent proposal for preferred extremals is consistent with this property realizing also holography implied by general coordinate invariance. Strong form of general coordinate invariance implying effective 2-dimensionality in turn suggests that Kähler action is expressible in terms of areas of partonic 2-surfaces and string world sheets.

2. The complexified gamma matrices of WCW come as hermitian conjugate pairs and anti-commute to the Kähler metric of WCW. Also bosonic generators of symplectic transformations of $\delta M^4_\pm \times CP_2$ a assumed to act as isometries of WCW geometry can be complexified and appear as similar pairs. The action of isometry generators co-incides with that of symplectic generators at partonic 2-surfaces and string world sheets but elsewhere inside the space-time surface it is expected to be deformed from the symplectic action. The super-conformal transformations of

$\delta M^4_\pm \times CP_2$ acting on the light-like radial coordinate of $\delta M^4_\pm$ act as gauge symmetries of the geometry meaning that the corresponding WCW vector fields have zero norm.

3. WCW geometry has also zero modes which by definition do not contribute to WCW metric expect possibly by the dependence of the elements of WCW metric on zero modes through a conformal factor. In particular, induced $CP_2$ Kähler form and its analog for sphere $r_M = constant$ of light cone boundary are symplectic invariants, and one can define an infinite number of zero modes as invariants defined by Kähler fluxes over partonic 2-surfaces and string world sheets. This requires however the slicing of $CD$ parallel copies of $\delta M^4_+$ or $\delta M^4_-$. The physical interpretation of these non-quantum fluctuating degrees of freedom is as classical variables necessary for the interpretation of quantum measurement theory. Classical variable would metaphorically correspond the position of the pointer of the measurement instrument.

4. The construction receives a strong philosophical inspiration from the geometry of loop spaces. Loop spaces allow a unique Kähler geometry with maximal isometry group identifiable as Kac-Moody group. The reason is that otherwise Riemann connection does not exist. The only problem is that curvature scalar diverges since the Riemann tensor is by constant curvature property proportional to the metric. In 3-D case one would have union of constant curvature spaces labelled by zero modes and the situation is expected to be even more restrictive. The conjecture indeed is that WCW geometry exists only for $H = M^4 \times CP_2$: infinite-D Kähler geometric existence and therefore physics would be unique. One can also hope that Ricci scalar is finite and therefore zero by the constant curvature property so that Einstein's equations are satisfied.

5. WCW Hamiltonians determined the isometry currents and WCW metric is given in terms of the anti-commutators of the Killing vector fields associated with symplectic isometry currents. The WCW Hamiltonians generating symplectic isometries correspond to the Hamiltonians spanning the symplectic group of $\delta M^4_\pm \times CP_2$. One can say that the space of quantum fluctuating degrees of freedom is this symplectic group of $\delta M^4_\pm \times CP_2$ or its subgroup or coset space: this must have very deep implications for the structure of the quantum TGD.

6. Zero energy ontology brings in additional delicacies. Basic objects are now unions of partonic 2-surfaces at the ends of $CD$. Also string world sheets would naturally contribute. One can generalize the expressions for the isometry generators in a straightforward manner by requiring that given isometry restricts to a symplectic transformation at partonic 2-surfaces and string world sheets.

7. One could criticize the effective metric 2-dimensionality forced by general consistency arguments as something non-physical. The Hamiltonians are expressed using only the data at partonic 2-surfaces: this includes also 4-D tangent space data via the weak form of electric-magnetic duality so that one has only effective 2-dimensionality. Obviously WCW geometry must have large gauge symmetries besides zero modes. The super-conformal symmetries indeed represent gauge symmetries of this kind. Effective 2-dimensionality realizing strong form of holography in turn is induced by the strong form of general coordinate invariance. Light-like 3-surfaces at which the signature of the induced metric changes must be equivalent with the 3-D space-like ends of space-time surfaces at the light-boundaries of space-time surfaces as far as WCW geometry is considered. This requires that the data from their 2-D intersections defining partonic 2-surfaces should dictate the WCW geometry. Note however that Super-Kac-Moody charges giving information about the interiors of 3-surfaces appear in the construction of the physical states.

What is the role of the right handed neutrino in this construction?

1. In the construction of components of WCW metric as anti-commutators of super-generators only the covariantly constant right-handed neutrino appears in the super-generators analogous to super-Kac-Moody generators. All holomorphic modes of right handed neutrino characterized by two integers could in principle contribute to the WCW gamma matrices identified as fermionic super-symplectic generators anti-commuting to the metric. At the space-like ends of space-time surface the holomorphic generators would restrict to symplectic generators since the radial

light-like coordinate $r_M$ identified and complex coordinate of $CP_2$ allowing identification as restrictions of two complex coordinates or Hamilton-Jacobi coordinates to light-like boundary.

2. The non-covariantly constant modes could also correspond to purely super-conformal gauge degrees of freedom. Originally the restriction to right-handed neutrino looked somewhat unsatisfactory but the recent view about Super-Kac-Moody symmetries makes its special role rather natural. One could say that WCW geometry possesses the maximal $D = 4$ supersymmetry.

3. One can of course ask whether the Super-Kac-Moody generators assignable to the isometries of $H$ and expressible as conserved charges associated with the boundaries of string world sheets could contribute to the WCW geometry via the anti-commutators. This option cannot be excluded but in this case the interpretation in terms of Hamiltonians is not obvious.

### 6.4.4 Equivalence Principle

An important physical input has been the condition that a generalization of Equivalence Principle is obtained.

1. The proposal has been that inertial and gravitational masses can be assigned with the super-symplectic and super-Kac-Moody representations via the condition that the scaling generator $L_0$ defined as a difference of the corresponding generators for the two representations annihilates physical states. This requires that super-Kac-Moody algebra can be regarded in some sense as a sub-algebra of super-symplectic algebra. For isometries this would be natural but in the case of holonomies the situation is problematic. The idea has been that the ordinary realization of Equivalence Principle follows as Einstein's equations for fluctuations around vacuum extremals expressing the average energy momentum tensor for the fluctuations.

2. The emergence of Einstein's equations for preferred extremals as additional conditions [K5, K58] allowing the algebraization of the equations to analogs of minimal surface equations changes the situation completely. Is there anymore need to realize Equivalence Principle at quantum level? If one drops this condition one can imagine very simple option obtained as tensor product of the super-symplectic and super-Kac-Moody representations. Of course, coset representations for the symplectic group and its suitable subgroup - say subgroup defining measurement resolution - can be present but would not nothing to do with Equivalence Principle.

3. One can of course argue that one has very naturally to different mass squared operators and therefore inertial and gravitational masses. Inertial mass squared would be naturally assignable to the representations of the super-symplectic algebra imbedding space d'Alembertian and gravitational mass squared with the spinor d'Alembertian at string world sheets at space-time surfaces. Quantum level realization for Equivalence Principle could mean that these two mass squared operators are identical or something analogous to this. One can however criticize this idea as un-necessary and also because the signature of the effective metric defined by the modified Dirac gamma matrices is speculated to be Euclidian.

### 6.4.5 Constraints from p-adic mass calculations and ZEO

A further important physical input comes from p-adic thermodynamics forming a core element of p-adic mass calculations.

1. The first thing that one can get worried about relates to the extension of conformal symmetries. If the conformal symmetries generalize to $D = 4$, how can one take seriously the results of p-adic mass calculations based on 2-D conformal invariance? There is no reason to worry. The reduction of the conformal invariance to 2-D one for the preferred extremals takes care of this problem. This however requires that the fermionic contributions assignable to string world sheets and/or partonic 2-surfaces - Super- Kac-Moody contributions - should dictate the elementary particle masses. For hadrons also symplectic contributions should be present. This is a valuable hint in attempts to identify the mathematical structure in more detail.

2. ZEO suggests that all particles, even virtual ones correspond to massless wormhole throats carrying fermions. As a consequenc, twistor approach would work and the kinematical constraints to vertices would allow th cancellation of divergences. This would suggest that the p-adic thermal expectation value is for the longitudinal $M^2$ momentum squared (the definition of $CD$ selects $M^1 \subset M^2 \subset M^4$ as also does number theoretic vision). Also propagator would be determined by $M^2$ momentum. Lorentz invariance would be obtained by integration of the moduli for $CD$ including also Lorentz boosts of $CD$.

3. In the original approach one allows states with arbitrary large values of $L_0$ as physical states. Usually one would require that $L_0$ annihilates the states. In the calculations however mass squared was assumed to be proportional $L_0$ apart from vacuum contribution. This is a questionable assumption. ZEO suggests that total mass squared vanishes and that one can decompose mass squared to a sum of longitudinal and transversal parts. If one can do the same decomposition to longitudinal and transverse parts also for the Super Virasoro algebra then one can calculate longitudinal mass squared as a p-adic thermal expectation in the transversal super-Virasoro algebra and only states with $L_0 = 0$ would contribute and one would have conformal invariance in the standard sense.

4. In the original approach the assumption motivated by Lorentz invariance has been that mass squared is replaced with conformal weight in thermodynamics, and that one first calculates the thermal average of the conformal weight and then equates it with mass squared. This assumption is somewhat ad hoc. ZEO however suggests an alternative interpretation in which one has zero energy states for which longitudinal mass squared of positive energy state derive from p-adic thermodynamics. Thermodynamics - or rather, its square root - would become part of quantum theory in ZEO. $M$-matrix is indeed product of hermitian square root of density matrix multiplied by unitary S-matrix and defines the entanglement coefficients between positive and negative energy parts of zero energy state.

5. The crucial constraint is that the number of super-conformal tensor factors is $N = 5$: this suggests that thermodynamics applied in Super-Kac-Moody degrees of freedom assignable to string world sheets is enough, when one is interested in the masses of fermions and gauge bosons. Super-symplectic degrees of freedom can also contribute and determine the dominant contribution to baryon masses. Should also this contribution obey p-adic thermodynamics in the case when it is present? Or does the very fact that this contribution need not be present mean that it is not thermal? The symplectic contribution should correspond to hadronic p-adic length prime rather the one assignable to (say ) u quark. Hadronic p-adic mass squared and partonic p-adic mass squared cannot be summed since primes are different. If one accepts the basic rules [K35], longitudinal energy and momentum are additive as indeed assumed in perturbative QCD.

6. Calculations work if the vacuum expectation value of the mass squared must be assumed to be tachyonic. There are two options depending on whether one whether p-adic thermodynamics gives total mass squared or longitudinal mass squared.

   (a) One could argue that the total mass squared has naturally tachyonic ground state expectation since for massless extremals longitudinal momentum is light-like and transversal momentum squared is necessary present and non-vanishing by the localization to topological light ray of finite thickness of order p-adic length scale. Transversal degrees of freedom would be modeled with a particle in a box.

   (b) If longitudinal mass squared is what is calculated, the condition would require that transversal momentum squared is negative so that instead of plane wave like behavior exponential damping would be required. This would conform with the localization in transversal degrees of freedom.

7. What about Equivalence Principle in this framework? A possible quantum counterpart of Equivalence Principle could be that the longitudinal parts of the imbedding space mass squared operator for a given massless state equals to that for d'Alembert operator assignable to the modified Dirac action. The attempts to formulate this in more precise manner however seem to produce only additional troubles.

### 6.4.6 The emergence of Yangian symmetry and gauge potentials as duals of Kac-Moody currents

Yangian symmetry plays a key role in $\mathcal{N} = 4$ super-symmetric gauge theories. What is special in Yangian symmetry is that the algebra contains also multi-local generators. In TGD framework multi-locality would naturally correspond to that with respect to partonic 2-surfaces and string world sheets and the proposal has been that the Super-Kac-Moody algebras assignable to string worlds sheets could generalize to Yangian.

Witten has written a beautiful exposition of Yangian algebras [B27]. Yangian is generated by two kinds of generators $J^A$ and $Q^A$ by a repeated formation of commutators. The number of commutations tells the integer characterizing the multi-locality and provides the Yangian algebra with grading by natural numbers. Witten describes a 2-dimensional QFT like situation in which one has 2-D situation and Kac-Moody currents assignable to real axis define the Kac-Moody charges as integrals in the usual manner. It is also assumed that the gauge potentials defined by the 1-form associated with the Kac-Moody current define a flat connection:

$$\partial_\mu j^A_\nu - \partial_\nu j^A_\nu + [j^A_\mu, j^A_\nu] = 0 \ . \tag{6.4.1}$$

This condition guarantees that the generators of Yangian are conserved charges. One can however consider alternative manners to obtain the conservation.

1. The generators of first kind - call them $J^A$ - are just the conserved Kac-Moody charges. The formula is given by

$$J_A = \int_{-\infty}^{\infty} dx j^{A0}(x,t) \ . \tag{6.4.2}$$

2. The generators of second kind contain bi-local part. They are convolutions of generators of first kind associated with different points of string described as real axis. In the basic formula one has integration over the point of real axis.

$$Q^A = f^A_{BC} \int_{-\infty}^{\infty} dx \int_x^{\infty} dy j^{B0}(x,t) j^{C0}(y,t) - 2 \int_{-\infty}^{\infty} j^A_x dx \ . \tag{6.4.3}$$

These charges are indeed conserved if the curvature form is vanishing as a little calculation shows.

How to generalize this to the recent context?

1. The Kac-Moody charges would be associated with the braid strands connecting two partonic 2-surfaces - Strands would be located either at the space-like 3-surfaces at the ends of the space-time surface or at light-like 3-surfaces connecting the ends. Modified Dirac equation would define Super-Kac-Moody charges as standard Noether charges. Super charges would be obtained by replacing the second quantized spinor field or its conjugate in the fermionic bilinear by particular mode of the spinor field. By replacing both spinor field and its conjugate by its mode one would obtain a conserved c-number charge corresponding to an anti-commutator of two fermionic super-charges. The convolution involving double integral is however not number theoretically attactive whereas single 1-D integrals might make sense.

2. An encouraging observation is that the Hodge dual of the Kac-Moody current defines the analog of gauge potential and exponents of the conserved Kac-Moody charges could be identified as analogs for the non-integrable phase factors for the components of this gauge potential. This identification is precise only in the approximation that generators commute since only in this case the ordered integral $P(exp(i\int Adx))$ reduces to $P(exp(i\int Adx))$.Partonic 2-surfaces connected

by braid strand would be analogous to nearby points of space-time in its discretization implying that Abelian approximation works. This conforms with the vision about finite measurement resolution as discretization in terms partonic 2-surfaces and braids.

This would make possible a direct identification of Kac-Moody symmetries in terms of gauge symmetries. For isometries one would obtain color gauge potentials and the analogs of gauge potentials for graviton field (in TGD framework the contraction with $M^4$ vierbein would transform tensor field to 4 vector fields). For Kac-Moody generators corresponding to holonomies one would obtain electroweak gauge potentials. Note that super-charges would give rise to a collection of spartners of gauge potentials automatically. One would obtain a badly broken SUSY with very large value of $\mathcal{N}$ defined by the number of spinor modes as indeed speculated earlier [K20].

3. The condition that the gauge field defined by 1-forms associated with the Kac-Moody currents are trivial looks unphysical since it would give rise to the analog of topological QFT with gauge potentials defined by the Kac-Moody charges. For the duals of Kac-Moody currents defining gauge potentials only covariant divergence vanishes implying that curvature form is

$$F_{\alpha\beta} = \epsilon_{\alpha\beta}[j_\mu, j^\mu] \ , \tag{6.4.4}$$

so that the situation does not reduce to topological QFT unless the induced metric is diagonal. This is not the case in general for string world sheets.

4. It seems however that there is no need to assume that $j_\mu$ defines a flat connection. Witten mentions that although the discretization in the definition of $J^A$ does not seem to be possible, it makes sense for $Q^A$ in the case of $G = SU(N)$ for any representation of $G$. For general $G$ and its general representation there exists no satisfactory definition of $Q$. For certain representations, such as the fundamental representation of $SU(N)$, the definition of $Q^A$ is especially simple. One just takes the bi-local part of the previous formula:

$$Q^A = f^A_{BC}\sum_{i<j} J^B_i J^C_j \ . \tag{6.4.5}$$

What is remarkable that in this formula the summation need not refer to a discretized point of braid but to braid strands ordered by the label $i$ by requiring that they form a connected polygon. Therefore the definition of $J^A$ could be just as above.

5. This brings strongly in mind the interpretation in terms of twistor diagrams. Yangian would be identified as the algebra generated by the logarithms of non-integrable phase factors in Abelian approximation assigned with pairs of partonic 2-surfaces defined in terms of Kac-Moody currents assigned with the modified Dirac action. Partonic 2-surfaces connected by braid strand would be analogous to nearby points of space-time in its discretization. This would fit nicely with the vision about finite measurement resolution as discretization in terms partonic 2-surfaces and braids.

The resulting algebra satisfies the basic commutation relations

$$\left[J^A, J^B\right] = f^{AB}_C J^C \ , \ \left[J^A, Q^B\right] = f^{AB}_C Q^C \ . \tag{6.4.6}$$

plus the rather complex Serre relations described in [B27].

### 6.4.7 Quantum criticality and electro-weak gauge symmetries

Quantum criticality is one of the basic guiding principles of Quantum TGD. What it means mathematically is however far from clear.

1. What is obvious is that quantum criticality implies quantization of Kähler coupling strength as a mathematical analog of critical temperature so that the theory becomes mathematically unique if only single critical temperature is possible. Physically this means the presence of long range fluctuations characteristic for criticality and perhaps assignable to the effective hierarchy of Planck constants having explanation in terms of effective covering spaces of the imbedding space. This hierarchy follows from the vacuum degeneracy of Kähler action, which in turn implies 4-D spin-glass degeneracy. It is easy to interpret the degeneracy in terms of criticality.

2. At more technical level one would expect criticality to corresponds deformations of a given preferred extremal defining a vanishing second variation of Kähler action. This is analogous to the vanishing of also second derivatives of potential function at extremum in certain directions so that the matrix defined by second derivatives does not have maximum rank. Entire hierarchy of criticalities is expected and a good finite-dimensional model is provided by the catastrophe theory of Thom [?]. Cusp catastrophe [A3] is the simplest catastrophe one can think of, and here the folds of cusp where discontinuous jump occurs correspond to criticality with respect to one control variable and the tip to criticality with respect to both control variables.

3. I have discussed what criticality could mean for modified Dirac action [K19] and claimed that it leads to the existence of additional conserved currents defined by the variations which do not affect the value of Kähler action. These arguments are far from being mathematically rigorous and the recent view about the solutions of the modified Dirac equation predicting that the spinor modes are restricted to 2-D string world sheets requires a modification of these arguments.

In the following these arguments are updated. The unexpected result is that critical deformations induce conformal scalings of the modified metric and electro-weak gauge transformations of the induced spinor connection at $X^2$. Therefore holomorphy brings in the Kac-Moody symmetries associated with isometries of $H$ (gravitation and color gauge group) and quantum criticality those associated with the holonomies of $H$ (electro-weak-gauge group) as additional symmetries.

#### The variation of modes of the induced spinor field in a variation of space-time surface respecting the preferred extremal property

Consider first the variation of the induced spinor field in a variation of space-time surface respecting the preferred extremal property. The deformation must be such that the deformed modified Dirac operator $D$ annihilates the modified mode. By writing explicitly the variation of the modified Dirac action (the action vanishes by modified Dirac equation) one obtains deformations and requiring its vanishing one obtains

$$\delta\Psi \quad = \quad D^{-1}(\delta D)\Psi \ . \tag{6.4.7}$$

$D^{-1}$ is the inverse of the modified Dirac operator defining the analog of Dirac propagator and $\delta D$ defines vertex completely analogous to $\gamma^k\delta A_k$ in gauge theory context. The functional integral over preferred extremals can be carried out perturbatively by expressing $\delta D$ in terms of $\delta h^k$ and one obtains stringy perturbation theory around $X^2$ associated with the preferred extremal defining maximum of Kähler function in Euclidian region and extremum of Kähler action in Minkowskian region (stationary phase approximation).

What one obtains is stringy perturbation theory for calculating n-points functions for fermions at the ends of braid strands located at partonic 2-surfaces and representing intersections of string world sheets and partonic 2-surfaces at the light-like boundaries of $CD$s. $\delta D$- or more precisely, its partial derivatives with respect to functional integration variables - appear atthe vertices located anywhere in the interior of $X^2$ with outcoming fermions at braid ends. Bosonic propagators are replaced with correlation functions for $\delta h^k$. Fermionic propagator is defined by $D^{-1}$.

After 35 years or hard work this provides for the first time a reasonably explicit formula for the N-point functions of fermions. This is enough since by bosonic emergence [K39] these N-point functions define the basic building blocks of the scattering amplitudes. Note that bosonic emergence states that bosons corresponds to wormhole contacts with fermion and antifermion at the opposite wormhole throats.

### What critical modes could mean for the induced spinor fields?

What critical modes could mean for the induced spinor fields at string world sheets and partonic 2-surfaces. The problematic part seems to be the variation of the modified Dirac operator since it involves gradient. One cannot require that covariant derivative remains invariant since this would require that the components of the induced spinor connection remain invariant and this is quite too restrictive condition. Right handed neutrino solutions delocalized into entire $X^2$ are however an exception since they have no electro-weak gauge couplings and in this case the condition is obvious: modified gamma matrices suffer a local scaling for critical deformations:

$$\delta\Gamma^{\mu} = \Lambda(x)\Gamma^{\mu} \ . \tag{6.4.8}$$

This guarantees that the modified Dirac operator $D$ is mapped to $\Lambda D$ and still annihilates the modes of $\nu_R$ labelled by conformal weight, which thus remain unchanged.

What is the situation for the 2-D modes located at string world sheets? The condition is obvious. $\Psi$ suffers an electro-weak gauge transformation as does also the induced spinor connection so that $D_{\mu}$ is not affected at all. Criticality condition states that the deformation of the space-time surfaces induces a conformal scaling of $\Gamma^{\mu}$ at $X^2$. It might be possible to continue this conformal scaling of the entire space-time sheet but this might be not necessary and this would mean that all critical deformations induced conformal transformations of the effective metric of the space-time surface defined by $\{\Gamma^{\mu},\Gamma^{\nu}\} = 2G^{\mu\nu}$. Thus it seems that effective metric is indeed central concept (recall that if the conjectured quaternionic structure is associated with the effective metric, it might be possible to avoid problem related to the Minkowskian signature in an elegant manner).

In fact, one can consider even more general action of critical deformation: the modes of the induced spinor field would be mixed together in the infinitesimal deformation besides infinitesimal electroweak gauge transformation, which is same for all modes. This would extend electroweak gauge symmetry. Modified Dirac equation holds true also for these deformations. One might wonder whether the conjectured dynamically generated gauge symmetries assignable to finite measurement resolution could be generated in this manner.

The infinitesimal generator of a critical deformation $J_M$ can be expressed as tensor product of matrix $A_M$ acting in the space of zero modes and of a generator of infinitemal electro-weak gauge transformation $T_M(x)$ acting in the same manner on all modes: $J_M = A_M \otimes T_M(x)$. $A_M$ is a spatially constant matrix and $T_M(x)$ decomposes to a direct sum of left- and right-handed $SU(2)\times U(1)$ Lie-algebra generators. Left-handed Lie-algebra generator can be regarded as a quaternion and right handed as a complex number. One can speak of a direct sum of left-handed local quaternion $q_{M,L}$ and right-handed local complex number $c_{M,R}$. The commutator $[J_M, J_N]$ is given by $[J_M, J_N] = [A_M, A_N] \otimes \{T_M(x), T_N(x)\} + \{A_M, A_N\} \otimes [T_M(x), T_N(x)]$. One has $\{T_M(x), T_N(x)\} = \{q_{M,L}(x), q_{N,L}(x)\} \oplus \{c_{M,R}(x), c_{N,R}(x)\}$ and $[T_M(x), T_N(x)] = [q_{M,L}(x), q_{N,L}(x)]$. The commutators make sense also for more general gauge group but quaternion/complex number property might have some deeper role.

Thus the critical deformations would induce conformal scalings of the effective metric and dynamical electro-weak gauge transformations. Electro-weak gauge symmetry would be a dynamical symmetry restricted to string world sheets and partonic 2-surfaces rather than acting at the entire space-time surface. For 4-D delocalized right-handed neutrino modes the conformal scalings of the effective metric are analogous to the conformal transformations of $M^4$ for $\mathcal{N}=4$ SYMs. Also ordinary conformal symmetries of $M^4$ could be present for string world sheets and could act as symmetries of generalized Feynman graphs since even virtual wormhole throats are massless. An interesting question is whether the conformal invariance associated with the effective metric is the analog of dual conformal invariance in $\mathcal{N}=4$ theories.

Critical deformations of space-time surface are accompanied by conserved fermionic currents. By using standard Noetherian formulas one can write

$$J_i^{\mu} = \overline{\Psi}\Gamma^{\mu}\delta_i\Psi + \delta_i\overline{\Psi}\Gamma^{\mu}\Psi \ . \tag{6.4.9}$$

Here $\delta\Psi_i$ denotes derivative of the variation with respect to a group parameter labeled by $i$. Since $\delta\Psi_i$ reduces to an infinitesimal gauge transformation of $\Psi$ induced by deformation, these currents are the analogs of gauge currents. The integrals of these currents along the braid strands at the ends of string world sheets define the analogs of gauge charges. The interpretation as Kac-Moody charges is also very attractive and I have proposed that the 2-D Hodge duals of gauge potentials could be identified as Kac-Moody currents. If so, the 2-D Hodge duals of $J$ would define the quantum analogs of dynamical electro-weak gauge fields and Kac-Moody charge could be also seen as non-integral phase factor associated with the braid strand in Abelian approximation (the interpretation in terms of finite measurement resolution is discussed earlier).

One can also define super currents by replacing $\overline{\Psi}$ or $\Psi$ by a particular mode of the induced spinor field as well as c-number valued currents by performing the replacement for both $\overline{\Psi}$ or $\Psi$. As expected, one obtains a super-conformal algebra with all modes of induced spinor fields acting as generators of super-symmetries restricted to 2-D surfaces. The number of the charges which do not annihilate physical states as also the effective number of fermionic modes could be finite and this would suggest that the integer $\mathcal{N}$ for the supersymmetry in question is finite. This would conform with the earlier proposal inspired by the notion of finite measurement resolution implying the replacement of the partonic 2-surfaces with collections of braid ends.

Note that Kac-Moody charges might be associated with "long" braid strands connecting different wormhole throats as well as short braid strands connecting opposite throats of wormhole contacts. Both kinds of charges would appear in the theory.

#### What is the interpretation of the critical deformations?

Critical deformations bring in an additional gauge symmetry. Certainly not all possible gauge transformations are induced by the deformations of preferred extremals and a good guess is that they correspond to holomorphic gauge group elements as in theories with Kac-Moody symmetry. What is the physical character of this dynamical gauge symmetry?

1. Do the gauge charges vanish? Do they annihilate the physical states? Do only their positive energy parts annihilate the states so that one has a situation characteristic for the representation of Kac-Moody algebras. Or could some of these charges be analogous to the gauge charges associated with the constant gauge transformations in gauge theories and be therefore non-vanishing in the absence of confinement. Now one has electro-weak gauge charges and these should be non-vanishing. Can one assign them to deformations with a vanishing conformal weight and the remaining deformations to those with non-vanishing conformal weight and acting like Kac-Moody generators on the physical states?

2. The simplest option is that the critical Kac-Moody charges/gauge charges with non-vanishing positive conformal weight annihilate the physical states. Critical degrees of freedom would not disappear but make their presence known via the states labelled by different gauge charges assignable to critical deformations with vanishing conformal weight. Note that constant gauge transformations can be said to break the gauge symmetry also in the ordinary gauge theories unless one has confinement.

3. The hierarchy of quantum criticalities suggests however entire hierarchy of electro-weak Kac-Moody algebras. Does this mean a hierarchy of electro-weak symmetries breakings in which the number of Kac-Moody generators not annihilating the physical states gradually increases as also modes with a higher value of positive conformal weight fail to annihilate the physical state?

   The only manner to have a hierarchy of algebras is by assuming that only the generators satisfying $n \ mod \ N = 0$ define the sub-Kac-Moody algebra annihilating the physical states so that the generators with $n \ mod \ N \neq 0$ would define the analogs of gauge charges. I have suggested for long time ago the relevance of kind of fractal hierarchy of Kac-Moody and Super-Virasoro algebras for TGD but failed to imagine any concrete realization.

   A stronger condition would be that the algebra reduces to a finite dimensional algebra in the sense that the actions of generators $Q_n$ and $Q_{n+kN}$ are identical. This would correspond to periodic boundary conditions in the space of conformal weights. The notion of finite measurement resolution suggests that the number of independent fermionic oscillator operators is proportional to the number of braid ends so that an effective reduction to a finite algebra is expected.

Whatever the correct interpretation is, this would obviously refine the usual view about electro-weak symmetry breaking.

These arguments suggests the following overall view. The holomorphy of spinor modes gives rise to Kac-Moody algebra defined by isometries and includes besides Minkowskian generators associated with gravitation also SU(3) generators associated with color symmetries. Vanishing second variations in turn define electro-weak Kac-Moody type algebra.

Note that criticality suggests that one must perform functional integral over WCW by decomposing it to an integral over zero modes for which deformations of $X^4$ induce only an electro-weak gauge transformation of the induced spinor field and to an integral over moduli corresponding to the remaining degrees of freedom.

### 6.4.8 The importance of being light-like

The singular geometric objects associated with the space-time surface have become increasingly important in TGD framework. In particular, the recent progress has made clear that these objects might be crucial for the understanding of quantum TGD. The singular objects are associated not only with the induced metric but also with the effective metric defined by the anti-commutators of the modified gamma matrices appearing in the modified Dirac equation and determined by the Kähler action.

#### The singular objects associated with the induced metric

Consider first the singular objects associated with the induced metric.

1. At light-like 3-surfaces defined by wormhole throats the signature of the induced metric changes from Euclidian to Minkowskian so that 4-metric is degenerate. These surfaces are carriers of elementary particle quantum numbers and the 4-D induced metric degenerates locally to 3-D one at these surfaces.

2. Braid strands at light-like 3-surfaces are most naturally light-like curves: this correspond to the boundary condition for open strings. One can assign fermion number to the braid strands. Braid strands allow an identification as curves along which the Euclidian signature of the string world sheet in Euclidian region transforms to Minkowskian one. Number theoretic interpretation would be as a transformation of complex regions to hyper-complex regions meaning that imaginary unit $i$ satisfying $i^2 = -1$ becomes hyper-complex unit $e$ satisfying $e^2 = 1$. The complex coordinates $(z, \overline{z})$ become hyper-complex coordinates $(u = t + ex, v = t - ex)$ giving the standard light-like coordinates when one puts $e = 1$.

#### The singular objects associated with the effective metric

There are also singular objects assignable to the effective metric. According to the simple arguments already developed, string world sheets and possibly also partonic 2-surfaces are singular objects with respect to the effective metric defined by the anti-commutators of the modified gamma matrices rather than induced gamma matrices. Therefore the effective metric seems to be much more than a mere formal structure.

1. For instance, quaternionicity of the space-time surface could allow an elegant formulation in terms of the effective metric avoiding the problems due to the Minkowski signature. This is achieved if the effective metric has Euclidian signature $\epsilon \times (1, 1, 1, 1)$, $\epsilon = \pm 1$ or a complex counterpart of the Minkowskian signature $\epsilon(1, 1, -1, -1)$.

2. String word sheets and perhaps also partonic 2-surfaces could be understood as singularities of the effective metric. What happens that the effective metric with Euclidian signature $\epsilon \times (1, 1, 1, 1)$ transforms to the signature $\epsilon(1, 1, -1, -1)$ (say) at string world sheet so that one would have the degenerate signature $\epsilon \times (1, 1, 0, 0)$ at the string world sheet.

   What is amazing is that this works also number theoretically. It came as a total surprise to me that the notion of hyper-quaternions as a closed algebraic structure indeed exists. The hyper-quaternionic units would be given by $(1, I, iJ, iK)$, where $i$ is a commuting imaginary unit satisfying $i^2 = -1$. Hyper-quaternionic numbers defined as combinations of these units with real coefficients do form a closed algebraic structure which however fails to be a number field just like hyper-complex numbers do. Note that the hyper-quaternions obtained with real coefficients from the basis $(1, iI, iJ, iK)$ fail to form an algebra since the product is not hyper-quaternion in this sense but belongs to the algebra of complexified quaternions. The same problem is encountered in the case of hyper-octonions defined in this manner. This has been a stone in my shoe since I feel strong disrelish towards Wick rotation as a trick for moving between different signatures.

3. Could also partonic 2-surfaces correspond to this kind of singular 2-surfaces? In principle, 2-D surfaces of 4-D space intersect at discrete points just as string world sheets and partonic 2-surfaces do so that this might make sense. By complex structure the situation is algebraically equivalent to the analog of plane with non-flat metric allowing all possible signatures $(\epsilon_1, \epsilon_2)$ in various regions. At light-like curve either $\epsilon_1$ or $\epsilon_2$ changes sign and light-like curves for these two kinds of changes can intersect as one can easily verify by drawing what happens. At the intersection point the metric is completely degenerate and simply vanishes.

4. Replacing real 2-dimensionality with complex 2-dimensionality, one obtains by the universality of algebraic dimension the same result for partonic 2-surfaces and string world sheets. The braid ends at partonic 2-surfaces representing the intersection points of 2-surfaces of this kind would have completely degenerate effective metric so that the modified gamma matrices would vanish implying that energy momentum tensor vanishes as does also the induced Kähler field.

5. The effective metric suffers a local conformal scaling in the critical deformations identified in the proposed manner. Since ordinary conformal group acts on Minkowski space and leaves the boundary of light-cone invariant, one has two conformal groups. It is not however clear whether the $M^4$ conformal transformations can act as symmetries in TGD, where the presence of the induced metric in Kähler action breaks $M^4$ conformal symmetry. As found, also in TGD framework the Kac-Moody currents assigned to the braid strands generate Yangian: this is expected to be true also for the Kac-Moody counterparts of the conformal algebra associated with quantum criticality. On the other hand, in twistor program one encounters also two conformal groups and the space in which the second conformal group acts remains somewhat mysterious object. The Lie algebras for the two conformal groups generate the conformal Yangian and the integrands of the scattering amplitudes are Yangian invariants. Twistor approach should apply in TGD if zero energy ontology is right. Does this mean a deep connection?

   What is also intriguing that twistor approach in principle works in strict mathematical sense only at signatures $\epsilon \times (1, 1, -1 - 1)$ and the scattering amplitudes in Minkowski signature are obtained by analytic continuation. Could the effective metric give rise to the desired signature? Note that the notion of massless particle does not make sense in the signature $\epsilon \times (1, 1, 1, 1)$.

These arguments provide genuine a support for the notion of quaternionicity and suggest a connection with the twistor approach.

### 6.4.9 Realization of large $\mathcal{N}$ SUSY in TGD

The generators large $\mathcal{N}$ SUSY algebras are obtained by taking fermionic currents for second quantized fermions and replacing either fermion field or its conjugate with its particular mode. The resulting super currents are conserved and define super charges. By replacing both fermion and its conjugate with modes one obtains c number valued currents. Therefore $\mathcal{N} = \infty$ SUSY - presumably equivalent with super-conformal invariance - or its finite $\mathcal{N}$ cutoff is realized in TGD framework and the challenge is to understand the realization in more detail.

#### Super-space viz. Grassmann algebra valued fields

Standard SUSY induces super-space extending space-time by adding anti-commuting coordinates as a formal tool. Many mathematicians are not enthusiastic about this approach because of the purely formal nature of anti-commuting coordinates. Also I regard them as a non-sense geometrically and there is actually no need to introduce them as the following little argument shows.

Grassmann parameters (anti-commuting theta parameters) are generators of Grassmann algebra and the natural object replacing super-space is this Grassmann algebra with coefficients of Grassmann algebra basis appearing as ordinary real or complex coordinates. This is just an ordinary space with additional algebraic structure: the mysterious anti-commuting coordinates are not needed. To me this notion is one of the conceptual monsters created by the over-pragmatic thinking of theoreticians.

This allows allows to replace field space with super field space, which is completely well-defined object mathematically, and leave space-time untouched. Linear field space is simply replaced with its Grassmann algebra. For non-linear field space this replacement does not work. This allows to formulate the notion of linear super-field just in the same manner as it is done usually.

The generators of super-symmetries in super-space formulation reduce to super translations , which anti-commute to translations. The super generators $Q_\alpha$ and $\overline{Q}_{\dot{\beta}}$ of super Poincare algebra are Weyl spinors commuting with momenta and anti-commuting to momenta:

$$\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \quad . \tag{6.4.10}$$

One particular representation of super generators acting on super fields is given by

$$\begin{aligned} D_\alpha &= i\frac{\partial}{\partial_{\theta_\alpha}} \ , \\ D_{\dot{\alpha}} &= i\frac{\partial}{\partial_{\overline{\theta}_{\dot{\alpha}}}} + \theta^\beta \sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \end{aligned} \tag{6.4.11}$$

Here the index raising for 2-spinors is carried out using antisymmetric 2-tensor $\epsilon^{\alpha\beta}$. Super-space interpretation is not necessary since one can interpret this action as an action on Grassmann algebra valued field mixing components with different fermion numbers.

Chiral superfields are defined as fields annihilated by $D_{\dot{\alpha}}$. Chiral fields are of form $\Psi(x^\mu + i\overline{\theta}\sigma^\mu\theta, \theta)$. The dependence on $\overline{\theta}_{\dot{\alpha}}$ comes only from its presence in the translated Minkowski coordinate annihilated by $D_{\dot{\alpha}}$. Super-space enthusiast would say that by a translation of $M^4$ coordinates chiral fields reduce to fields, which depend on $\theta$ only.

### The space of fermionic Fock states at partonic 2-surface as TGD counterpart of chiral super field

As already noticed, another manner to realize SUSY in terms of representations the super algebra of conserved super-charges. In TGD framework these super charges are naturally associated with the modified Dirac equation, and anti-commuting coordinates and super-fields do not appear anywhere. One can however ask whether one could identify a mathematical structure replacing the notion of chiral super field.

In [K20] it was proposed that generalized chiral super-fields could effectively replace induced spinor fields and that second quantized fermionic oscillator operators define the analog of SUSY algebra. One would have $\mathcal{N} = \infty$ if all the conformal excitations of the induced spinor field restricted on 2-surface are present. For right-handed neutrino the modes are labeled by two integers and delocalized to the interior of Euclidian or Minkowskian regions of space-time sheet.

The obvious guess is that chiral super-field generalizes to the field having as its components many-fermions states at partonic 2-surfaces with theta parameters and their conjugates in one-one correspondence with fermionic creation operators and their hermitian conjugates.

1. Fermionic creation operators - in classical theory corresponding anti-commuting Grassmann parameters - replace theta parameters. Theta parameters and their conjugates are not in one-one correspondence with spinor components but with the fermionic creation operators and their hermitian conjugates. One can say that the super-field in question is defined in the "world of classical worlds" (WCW) rather than in space-time. Fermionic Fock state at the partonic 2-surface is the value of the chiral super field at particular point of WCW.

2. The matrix defined by the $\sigma^\mu\partial_\mu$ is replaced with a matrix defined by the modified Dirac operator $D$ between spinor modes acting in the solution space of the modified Dirac equation. Since modified Dirac operator annihilates the modes of the induced spinor field, super covariant derivatives

reduce to ordinary derivatives with respect the theta parameters labeling the modes. Hence the chiral super field is a field that depends on $\theta_m$ or conjugates $\overline{\theta}_m$ only. In second quantization the modes of the chiral super-field are many-fermion states assigned to partonic 2-surfaces and string world sheets. Note that this is the only possibility since the notion of super-coordinate does not make sense now.

3. It would seem that the notion of super-field does not bring anything new. This is not the case. First of all, the spinor fields are restricted to 2-surfaces. Second point is that one cannot assign to the fermions of the many-fermion states separate non-parallel or even parallel four-momenta. The many-fermion state behaves like elementary particle. This has non-trivial implications for propagators and a simple argument [K20] leads to the proposal that propagator for N-fermion partonic state is proportional to $1/p^N$. This would mean that only the states with fermion number equal to 1 or 2 behave like ordinary elementary particles.

### How the fermionic anti-commutation relations are determined?

Understanding the fermionic anti-commutation relations is not trivial since all fermion fields except right-handed neutrino are assumed to be localized at 2-surfaces. Since fermionic conserved currents must give rise to well-defined charges as 3-D integrals the spinor modes must be proportional to a square root of delta function in normal directions. Furthermore, the modified Dirac operator must act only in the directions tangential to the 2-surface in order that the modified Dirac equation can be satisfied.

The square root of delta function can be formally defined by starting from the expansion of delta function in discrete basis for a particle in 1-D box. The product of two functions in x-space is convolution of Fourier transforms and the coefficients of Fourier transform of delta function are apart from a constant multiplier equal to 1: $\delta(x) = K\sum_n exp(inx/2\pi L)$. Therefore the Fourier transform of square root of delta function is obtained by normalizing the Fourier transform of delta function by $1/\sqrt{N}$, where $N \to \infty$ is the number of plane waves. In other words: $\sqrt{\delta(x)} = \sqrt{\frac{K}{N}}\sum_n \sum exp(inx/2\pi L)$.

Canonical quantization defines the standard approach to the second quantization of the Dirac equation.

1. One restricts the consideration to time=constant slices of space-time surface. Now the 3-surfaces at the ends of $CD$ are natural slices. The intersection of string world sheet with these surfaces is 1-D whereas partonic 2-surfaces have 2-D Euclidian intersection with them.

2. The canonical momentum density is defined by

$$\begin{aligned} \Pi_\alpha &= \frac{\partial L}{\partial_t \overline{\Psi}_\alpha(x)} = \Gamma^t \Psi \ , \\ \Gamma^t &= \frac{\partial L_K}{\partial(\partial_t h^k)} \ . \end{aligned} \tag{6.4.12}$$

$L_K$ denotes Kähler action density: consistency requires $D_\mu \Gamma^\mu = 0$, and this is guaranteed only by using the modified gamma matrices defined by Kähler action. Note that $\Gamma^t$ contains also the $\sqrt{g_4}$ factor. Induced gamma matrices would require action defined by four-volume. $t$ is time coordinate varying in direction tangential to 2-surface.

3. The standard equal time canonical anti-commutation relations state

$$\{\Pi_\alpha, \overline{\Psi}_\beta\} = \delta^3(x,y)\delta_{\alpha\beta} \ . \tag{6.4.13}$$

Can these conditions be applied both at string world sheets and partonic 2-surfaces.

1. Sttring world sheets do not pose problems. The restriction of the modes to string world sheets means that the square root of delta function in the normal direction of string world sheet takes care of the normal dimensions and the dynamical part of anti-commutation relations is 1-dimensional just as in the case of strings.

2. Partonic 2-surfaces are problematic. The $\sqrt{g_4}$ factor in $\Gamma^t$ implies that $\Gamma^t$ approaches zero at partonic 2-surfaces since they belong to light-like wormhole throats at which the signature of the induced metric changes. Energy momentum tensor appearing in $\Gamma^t$ involves to index raisins by induced metric so that it can grow without limit as one approaches partonic two-surface. Therefore it is quite possible that the limit is finite and the boundary conditions defined by the weak form of electric magnetic duality might imply that the limit is finite. The open question is whether one can apply canonical quantization at partonic 2-surfaces. One can also ask whether one can define induced spinor fields at wormhole throats only at the ends of string world sheets so that partonic 2-surface would be effectively discretized. This cautious conclusion emerged in the earlier study of the modified Dirac equation [K19].

3. Suppose that one can assume spinor modes at partonic 2-surfaces. 2-D conformal invariance suggests that the situation reduces to effectively one-dimensional also at the partonic two-surfaces. If so, one should pose the anti-commutation relations at some 1-D curves of the partonic 2-surface only. This is the only sensical option. The point is that the action of the modified Dirac operator is tangential so that also the canonical momentum current must be tangential and one can fix anti-commutations only at some set of curves of the partonic 2-surface.

One can of course worry what happens at the limit of vacuum extremals. The problem is that $\Gamma^t$ vanishes for space-time surfaces reducing to vacuum extremals at the 2-surfaces carrying fermions so that the anti-commutations are inconsistent. Should one require - as done earlier- that the anti-commutation relations make sense at this limit and cannot therefore have the standard form but involve the scalar magnetic flux formed from the induced Kähler form by permuting it with the 2-D permutations symbl? The restriction to preferred extremals, which are always non-vacuum extremals, might allow to avoid this kind of problems automatically.

In the case of right-handed neutrino the situation is genuinely 3-dimensional and in this case non-vacuum extremal property must hold true in the regions where the modes of $\nu_R$ are non-vanishing. The same mechanism would save from problems also at the partonic 2-surfaces. The dynamics of induced spinor fields must avoid classical vacuum. Could this relate to color confinement? Could hadrons be surrounded by an insulating layer of Kähler vacuum?

## 6.5 Twistor revolution and TGD

Lubos Motl wrote a nice summary about the talk of Nima Arkani Hamed about twistor revolution in Strings 2012 and gave also a link to the talk [B13]. It seems that Nima and collaborators are ending to a picture about scattering amplitudes which strongly resembles that provided bt generalized Feynman diagrammatics in TGD framework

TGD framework is much more general than $\mathcal{N} = 4$ SYM and is to it same as general relativity for special relativity whereas the latter is completely explicit. Of course, I cannot hope that TGD view could be taken seriously - at least publicly. One might hope that these approaches could be combined some day: both have a lot to give for each other. Below I compare these approaches.

### 6.5.1 The origin of twistor diagrammatics

In TGD framework zero energy ontology forces to replace the idea about continuous unitary evolution in Minkowski space with something more general assignable to causal diamonds (CDs), and S-matrix is replaced with a square root of density matrix equal to a hermitian l square root of density matrix multiplied by unitary S-matrix. Also in twistor approach unitarity has ceased to be a star actor. In p-Adic context continuous unitary time evolution fails to make sense also mathematically.

Twistor diagrammatics involves only massless on mass shell particles on both external and internal lines. Zero energy ontology (ZEO) requires same in TGD: wormhole lines carry parallely moving massless fermions and antifermions. The mass shell conditions at vertices are enormously powerful and imply UV finiteness. Also IR finiteness follows if external particles are massive.

What one means with mass is however a delicate matter. What does one mean with mass? I have pondered 35 years this question and the recent view is inspired by p-adic mass calculations and ZEO, and states that observed mass is in a well-defined sense expectation value of longitudinal mass squared for all possible choices of $M^2 \subset M^4$ characterizing the choices of quantization axis for energy and spin at the level of "world of classical worlds" (WCW) assignable with given causal diamond $CD$.

The choice of quantization axis thus becomes part of the geometry of WCW. All wormhole throats are massless but develop non-vanishing longitudinal mass squared. Gauge bosons correspond to wormhole contacts and thus consist of pairs of massless wormhole throats. Gauge bosons could develop 4-D mass squared but also remain massless in 4-D sense if the throats have parallel massless momenta. Longitudinal mass squared is however non-vanishing andp-adic thermodynamics predicts it.

### 6.5.2 The emergence of 2-D sub-dynamics at space-time level

Nima et al introduce ordering of the vertices in 4-D case. Ordering and related braiding are however essentially 2-D notions. Somehow 2-D theory must be a part of the 4-D theory also at space-time level, and I understood that understanding this is the challenge of the twistor approach at this moment.

The twistor amplitude can be represented as sum over the permutations of $n$ external gluons and all diagrams corresponding to the same permutation are equivalent. Permutations are more like braidings since they carry information about how the permutation proceeded as a homotopy. Yang-Baxter equation emerges and states associativity of the braid group. The allowed braidings are minimal braidings in the sense that the repetitions of permutations of two adjacent vertices are not considered to be separate. Minimal braidings reduce to ordinary permutations. Nima also talks about affine braidings which I interpret as analogs of Kac-Moody algebras meaning that one uses projective representations which for Kac-Moody algebra mean non-trivial central extension. Perhaps the condition is that the square of a permutation permuting only two vertices which each other gives only a non-trivial phase factor. Lubos suggests an alternative interpretation which would select only special permutations and cannot be therefore correct.

There are rules of identifying the permutation associated with a given diagram involving only basic 3-gluon vertex with white circle and its conjugate. Lubos explains this "Mickey Mouse in maze" rule in his posting in detail: to determine the image $p(n)$ of vertex $n$ in the permutation put a mouse in the maze defined by the diagram and let it run around obeying single rule: if the vertex is black turn to the right and if the vertex is white turn to the left. The mouse cannot remain in a loop: if it would do so, the rule would force it to run back to $n$ after single full loop and one would have a fixed point: $p(n) = n$. The reduction in the number of diagrams is enormous: the infinity of different diagrams reduces to $n!$ diagrams!

What happens in TGD framework?

1. In TGD framework string world sheets and partonic 2-surfaces (or either or these if they are dual notions as conjectured) at space-time surface would define the sought for 2-D theory, and one obtains indeed perturbative expansion with fermionic propagator defined by the inverse of the modified Dirac operator and bosonic propagator defined by the correlation function for small deformations of the string world sheet. The vertices of twistor diagrams emerge as braid ends defining the intersections of string world sheets and partonic 2-surfaces.

   String model like description becomes part of TGD and the role of string world sheets in $X^4$ is highly analogous to that of string world sheets connecting branes in $AdS^5 \times S^5$ of $\mathcal{N} = 4$ SYM. In TGD framework 10-D $AdS^5 \times S^5$ is replaced with 4-D space-time surface in $M^4 \times CP_2$. The meaning of the analog of $AdS^5$ duality in TGD framework should be understood. In particular, it could it be that the descriptions involving string world sheets on one hand and partonic 2-surfaces - or 3-D orbits of wormhole throats defining the generalized Feynman diagram- on the other hand are dual to each other. I have conjectured something like this earlier but it takes some time for this kind of issues to find their natural answer.

2. As described in the article, string world sheets and partonic 2-surfaces emerge directly from the construction of the solutions of the modified Dirac equation by requiring conservation of em charge. This result has been conjectured already earlier but using other less direct arguments.

2-D "string world sheets" as sub-manifolds of the space-time surface make the ordering possible, and guarantee the finiteness of the perturbation theory involving n-point functions of a conformal QFT for fermions at wormhole throats and n-point functions for the deformations of the space-time surface. Conformal invariance should dictate these n-point functions to a high degree. In TGD framework the fundamental 3-vertex corresponds to joining of light-like orbits of three wormhole contacts along their 2-D ends (partonic 2-surfaces).

### 6.5.3 The emergence of Yangian symmetry

Yangian symmetry associated with the conformal transformations of $M^4$ is a key symmetry of Grassmannian approach. Is it possible to derive it in TGD framework?

1. TGD indeed leads to a concrete representation of Yangian algebra as generalization of color and electroweak gauge Kac-Moody algebra using general formula discussed in Witten's article about Yangian algebras (see the article).

2. Article discusses also a conjecture about 2-D Hodge duality of quantized YM gauge potentials assignable to string world sheets with Kac-Moody currents. Quantum gauge potentials are defined only where they are needed - at string world sheets rather than entire 4-D space-time.

3. Conformal scalings of the effective metric defined by the anticommutators of the modified gamma matrices emerges as realization of quantum criticality. They are induced by critical deformations (second variations not changing Kähler action) of the space-time surface. This algebra can be generalized to Yangian using the formulas in Witten's article (see the article).

4. Critical deformations induce also electroweak gauge transformations and even more general symmetries for which infinitesimal generators are products of U(n) generators permuting $n$ modes of the modified Dirac operator and infinitesimal generators of local electro-weak gauge transformations. These symmetries would relate in a natural manner to finite measurement resolution realized in terms of inclusions of hyperfinite factors with included algebra taking the role of gauge group transforming to each other states not distinguishable from each other.

5. How to end up with Grassmannian picture in TGD framework? This has inspired some speculations in the past. From Nima's lecture one however learns that Grassmannian picture emerges as a convenient parametrization. One starts from the basic 3-gluon vertex or its conjugate expressed in terms of twistors. Momentum conservation implies that with the three twistors $\lambda_i$ or their conjugates are proportional to each other (depending on which is the case one assigns white or black dot with the vertex). This constraint can be expressed as a delta function constraint by introducing additional integration variables and these integration variables lead to the emergence of the Grassmannian $G_{n,k}$ where $n$ is the number of gluons, and $k$ the number of positive helicity gluons.

   Since only momentum conservation is involved, and since twistorial description works because only massless on mass shell virtual particles are involved, one is bound to end up with the Grassmannian description also in TGD.

### 6.5.4 The analog of $AdS^5$ duality in TGD framework

The generalization of AdS$^5$ duality of $\mathcal{N} = 4$ SYMs to TGD framework is highly suggestive and states that string world sheets and partonic 2-surfaces play a dual role in the construction of M-matrices. Some terminology first.

1. Let us agree that string world sheets and partonic 2-surfaces refer to 2-surfaces in the slicing of space-time region defined by Hermitian structure or Hamilton-Jacobi structure.

2. Let us also agree that *singular* string world sheets and partonic 2-surfaces are surfaces at which the *effective* metric defined by the anticommutators of the modified gamma matrices degenerates to effectively 2-D one.

3. Braid strands at wormhole throats in turn would be loci at which the *induced* metric of the string world sheet transforms from Euclidian to Minkowskian as the signature of induced metric changes from Euclidian to Minkowskian.

$AdS^5$ duality suggest that string world sheets are in the same role as string world sheets of 10-D space connecting branes in $AdS^5$ duality for $\mathcal{N} = 4$ SYM. What is important is that there should exist a duality meaning two manners to calculate the amplitudes. What the duality could mean now?

1. Also in TGD framework the first manner would be string model like description using string world sheets. The second one would be a generalization of conformal QFT at light-like 3-surfaces (allowing generalized conformal symmetry) defining the lines of generalized Feynman diagram. The correlation functions to be calculated would have points at the intersections of partonic 2-surfaces and string world sheets and would represent braid ends.

2. General Coordinate Invariance (GCI) implies that physics should be codable by 3-surfaces. Light-like 3-surfaces define 3-surfaces of this kind and same applies to space-like 3-surfaces. There are also preferred 3-surfaces of this kind. The orbits of 2-D wormhole throats at which 4-metric degenerates to 3-dimensional one define preferred light-like 3-surfaces. Also the space-like 3-surfaces at the ends of space-time surface at light-like boundaries of causal diamonds ($CD$s) define preferred space-like 3-surfaces. Both light-like and space-like 3-surfaces should code for the same physics and therefore their intersections defining partonic 2-surfaces plus the 4-D tangent space data at them should be enough to code for physics. This is strong form of GCI implying effective 2-dimensionality. As a special case one obtains singular string world sheets at which the effective metric reduces to 2-dimensional and singular partonic 2-surfaces defining the wormhole throats. For these 2-surfaces situation could be especially simple mathematically.

3. The guess inspired by strong GCI is that string world sheet -partonic 2-surface duality holds true. The functional integrals over the deformations of 2 kinds of 2-surfaces should give the same result so tthat functional integration over either kinds of 2-surfaces should be enough. Note that the members of a given pair in the slicing intersect at discrete set of points and these points define braid ends carrying fermion number. Discretization and braid picture follow automatically.

4. Scattering amplitudes in the twistorial approach could be thus calculated by using *any* pair in the slicing - or only either member of the pair if the analog of $AdS^5$ duality holds true as argued. The possibility to choose any pair in the slicing means general coordinate invariance as a symmetry of the Kähler metric of WCW and of the entire theory suggested already early: Kähler functions for difference choices in the slicing would differ by a real part of holomorphic function and give rise to same Kähler metric of "world of classical worlds" (WCW). For a general pair one obtains functional integral over deformations of space-time surface inducing deformations of 2-surfaces with only other kind 2-surface contributing to amplitude. This means the analog of stringy QFT: Minkowskian or Euclidian string theory depending on choice.

5. For singular string world sheets and partonic 2-surfaces an enormous simplification results. The propagators for fermions and correlation functions for deformations reduce to 1-D instead of being 2-D: the propagation takes place only along the light-like lines at which the string world sheets with Euclidian signature (inside $CP_2$ like regions) change to those with Minkowskian signature of induced metric. The local reduction of space-time dimension would be very real for particles moving along sub-manifolds at which higher dimensional space-time has reduced metric dimension: they cannot get out from lower-D sub-manifold. This is like ending down to 1-D black hole interior and one would obtain the analog of ordinary Feynman diagrammatics. This kind of Feynman diagrammatics involving only braid strands is what I have indeed ended up earlier so that it seems that I can trust good intuition combined with a sloppy mathematics sometimes works;-).

   These singular lines represent orbits of point like particles carrying fermion number at the orbits of wormhole throats. Furthermore, in this representation the expansions coming from string world sheets and partonic 2-surfaces are identical automatically. This follows from the fact that only the light-like lines connecting points common to singular string world sheets and singular partonic 2-surfaces appear as propagator lines!

6. The TGD analog of AdS$^5$ duality of $\mathcal{N}=4$ SUSYs would be trivially true as an identity in this special case, and the good guess is that it is true also generally. One could indeed use integral over either string world sheets or partonic 2-sheets to deduce the amplitudes.

What is important to notice that singularities of Feynman diagrams crucial for the Grassmannian approach of Nima and others would correspond at space-time level 2-D singularities of the effective metric defined by the modified gamma matrices defined as contractions of canonical momentum currents for Kähler action with ordinary gamma matrices of the imbedding space and therefore directly reflecting classical dynamics.

### 6.5.5 Problems of the twistor approach from TGD point of view

Twistor approach has also its problems and here TGD suggests how to proceed. Signature problem is the first problem.

1. Twistor diagrammatics works in a strict mathematical sense only for $M^{2,2}$ with metric signature (1,1,-1,-1) rather than $M^4$ with metric signature (1,-1,-1,-1). Metric signature is wrong in the physical case. This is a real problem which must be solved eventually.

2. Effective metric defined by anticommutators of the modified gamma matrices (to be distinguished from the induced gamma matrices) could solve that problem since it would have the correct signature in TGD framework (see the article). String world sheets and partonic 2-surfaces would correspond to the 2-D singularities of this effective metric at which the even-even signature (1,1,1,1) changes to even-even signature (1,1,-1,-1). Space-time at string world sheet would become locally 2-D with respect to effective metric just as space-time becomes locally 3-D with respect to the induced metric at the light-like orbits of wormhole throats. String world sheets become also locally 1-D at light-like curves at which Euclidian signature of world sheet in induced metric transforms to Minkowskian.

3. Twistor amplitudes are indeed singularities and string world sheets implied in TGD framework by conservation of em charge would represent these singularities at space-time level. At the end of the talk Nima conjectured about lower-dimensional manifolds of space-time as representation of space-time singularities. Note that string world sheets and partonic 2-surfaces have been part of TGD for years. TGD is of course to $\mathcal{N}=4$ SYM what general relativity is for the special relativity. Space-time surface is dynamical and possesses induced and effective metrics rather than being flat.

Second limitation is that twistor diagrammatics works only for planar diagrams. This is a problem which must be also fixed sooner or later.

1. This perhaps dangerous and blasphemous statement that I will regret it some day but I will make it;-). Nima and others have not yet discovered that $M^2 \subset M^4$ must be there but will discover it when they begin to generalize the results to non-planar diagrams and realize that Feynman diagrams are analogous to knot diagrams in 2-D plane (with crossings allowed) and that this 2-D plane must correspond to $M^2 \subset M^4$. The different choices of causal diamond $CD$ correspond to different choices of $M^2$ representing choice of quantization axes 4-momentum and spin. The integral over these choices guarantees Lorentz invariance. Gauge conditions are modified: longitudinal $M^2$ projection of massless four-momentum is orthogonal to polarization so that three polarizations are possible: states are massive in longitudinal sense.

2. In TGD framework one replaces the lines of Feynman diagrams with the light-like 3-surfaces defining orbits of wormhole throats. These lines carry many fermion states defining braid strands at light-like 3-surfaces. There is internal braiding associated with these braid strands. String world sheets connect fermions at different wormhole throats with space-like braid strands. The $M^2$ projections of generalized Feynman diagrams with 4-D "lines" replaced with genuine lines define the ordinary Feynman diagram as the analog of braid diagram. The conjecture is that one can reduce non-planar diagrams to planar diagrams using a procedure analogous to the construction of knot invariants by un-knotting the knot in Alexandrian manner by allowing it to be cut temporarily.

3. The permutations of string vertices emerge naturally as one constructs diagrams by adding to the interior of polygon sub-polygons connected to the external vertices. This corresponds to the addition of internal partonic two-surfaces. There are very many equivalent diagrams of this kind. Only permutations matter and the permutation associated with a given diagram of this kind can be deduced by the Mickey-Mouse rule described explicitly by Lubos. A connection with planar operads is highly suggestive and also conjecture already earlier in TGD framework.

### 6.5.6 Could $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM be a part of TGD after all?

Whether right-handed neutrinos generate a supersymmetry in TGD has been a long standing open question. $\mathcal{N} = 1$ SUSY is certainly excluded by fermion number conservation but already $\mathcal{N} = 2$ defining a "complexification" of $\mathcal{N} = 1$ SUSY is possible and could generate right-handed neutrino and its antiparticle. These states should however possess a non-vanishing light-like momentum since the fully covariantly constant right-handed neutrino generates zero norm states. So called massless extremals (MEs) allow massless solutions of the modified Dirac equation for right-handed neutrino in the interior of space-time surface, and this seems to be case quite generally in Minkowskian signature for preferred extremals. This suggests that particle represented as magnetic flux tube structure with two wormhole contacts sliced between two MEs could serve as a starting point in attempts to understand the role of right handed neutrinos and how $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM emerges at the level of space-time geometry. The following arguments inspired by the article of Nima Arkani-Hamed et al [B20] about twistorial scattering amplitudes suggest a more detailed physical interpretation of the possible SUSY associated with the right-handed neutrinos.

The fact that right handed neutrinos have only gravitational interaction suggests a radical reinterpretation of SUSY: no SUSY breaking is needed since it is very difficult to distinguish between mass degenerate spartners of ordinary particles. In order to distinguish between different spartners one must be able to compare the gravitomagnetic energies of spartners in slowly varying external gravimagnetic field: this effect is extremely small.

#### Scattering amplitudes and the positive Grassmannian

The work of Nima Arkani-Hamed and others represents something which makes me very optimistic and I would be happy if I could understand the horrible technicalities of their work. The article Scattering Amplitudes and the Positive Grassmannian by Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, and Trnka [B20] summarizes the recent situation in a form, which should be accessible to ordinary physicist. Lubos has already discussed the article. The following considerations do not relate much to the main message of the article (positive Grassmannians) but more to the question how this approach could be applied in TGD framework.

*1. All scattering amplitudes have on shell amplitudes for massless particles as building bricks*

The key idea is that all planar amplitudes can be constructed from on shell amplitudes: all virtual particles are actually real. In zero energy ontology I ended up with the representation of TGD analogs of Feynman diagrams using only mass shell massless states with both positive and negative energies. The enormous number of kinematic constraints eliminates UV and IR divergences and also the description of massive particles as bound states of massless ones becomes possible.

In TGD framework quantum classical correspondence requires a space-time correlate for the on mass shell property and it indeed exists. The mathematically ill-defined path integral over all 4-surfaces is replaced with a superposition of preferred extremals of Kähler action analogous to Bohr orbits, and one has only a functional integral over the 3-D ends at the light-like boundaries of causal diamond (Euclidian/Minkowskian space-time regions give real/imaginary Chern-Simons exponent to the vacuum functional). This would be obviously the deeper principle behind on mass shell representation of scattering amplitudes that Nima and others are certainly trying to identify. This principle in turn reduces to general coordinate invariance at the level of the world of classical worlds.

Quantum classical correspondence and quantum ergodicity would imply even stronger condition: the quantal correlation functions should be identical with classical correlation functions for any preferred extremal in the superposition: all preferred extremals in the superposition would be statistically equivalent [K67]. 4-D spin glass degeneracy of Kähler action however suggests that this is is probably too strong a condition applying only to building bricks of the superposition.

Minimal surface property is the geometric counterpart for masslessness and the preferred extremals are also minimal surfaces: this property reduces to the generalization of complex structure at space-time surfaces, which I call Hamilton-Jacobi structure for the Minkowskian signature of the induced metric. Einstein Maxwell equations with cosmological term are also satisfied.

*2. Massless extremals and twistor approach*

The decomposition $M^4 = M^2 \times E^2$ is fundamental in the formulation of quantum TGD, in the number theoretical vision about TGD, in the construction of preferred extremals, and for the vision about generalized Feynman diagrams. It is also fundamental in the decomposition of the degrees of string to longitudinal and transversal ones. An additional item to the list is that also the states appearing in thermodynamical ensemble in p-adic thermodynamics correspond to four-momenta in $M^2$ fixed by the direction of the Lorentz boost. In twistor approach to TGD the possibility to decompose also internal lines to massless states at parallel space-time sheets is crucial.

Can one find a concrete identification for $M^2 \times E^2$ decomposition at the level of preferred extremals? Could these preferred extremals be interpreted as the internal lines of generalized Feynman diagrams carrying massless momenta? Could one identify the mass of particle predicted by p-adic thermodynamics with the sum of massless classical momenta assignable to two preferred extremals of this kind connected by wormhole contacts defining the elementary particle?

Candidates for this kind of preferred extremals indeed exist. Local $M^2 \times E^2$ decomposition and light-like longitudinal massless momentum assignable to $M^2$ characterizes "massless extremals" (MEs, "topological light rays"). The simplest MEs correspond to single space-time sheet carrying a conserved light-like $M^2$ momentum. For several MEs connected by wormhole contacts the longitudinal massless momenta are not conserved anymore but their sum defines a time-like conserved four-momentum: one has a bound states of massless MEs. The stable wormhole contacts binding MEs together possess Kähler magnetic charge and serve as building bricks of elementary particles. Particles are necessary closed magnetic flux tubes having two wormhole contacts at their ends and connecting the two MEs.

The sum of the classical massless momenta assignable to the pair of MEs is conserved even when they exchange momentum. Quantum classical correspondence requires that the conserved classical rest energy of the particle equals to the prediction of p-adic mass calculations. The massless momenta assignable to MEs would naturally correspond to the massless momenta propagating along the internal lines of generalized Feynman diagrams assumed in zero energy ontology. Masslessness of virtual particles makes also possible twistor approach. This supports the view that MEs are fundamental for the twistor approach in TGD framework.

*3. Scattering amplitudes as representations for braids whose threads can fuse at 3-vertices*

Just a little comment about the content of the article. The main message of the article is that non-equivalent contributions to a given scattering amplitude in $\mathcal{N} = 4$ SYM represent elements of the group of permutations of external lines - or to be more precise - decorated permutations which replace permutation group $S_n$ with $n!$ elements with its decorated version containing $2^n n!$ elements. Besides 3-vertex the basic dynamical process is permutation having the exchange of neighboring lines as a generating permutation completely analogous to fundamental braiding. BFCW bridge has interpretation as a representations for the basic braiding operation.

This supports the TGD inspired proposal (TGD as almost topological QFT) that generalized Feynman diagrams are in some sense also knot or braid diagrams allowing besides braiding operation also two 3-vertices [K25]. The first 3-vertex generalizes the standard stringy 3-vertex but with totally different interpretation having nothing to do with particle decay: rather particle travels along two paths simultaneously after $1 \rightarrow 2$ decay. Second 3-vertex generalizes the 3-vertex of ordinary Feynman diagram (three 4-D lines of generalized Feynman diagram identified as Euclidian space-time regions meet at this vertex). The main idea is that in TGD framework knotting and braiding emerges at two levels.

1. At the level of space-time surface string world sheets at which the induced spinor fields (except right-handed neutrino [K67]) are localized due to the conservation of electric charge can form 2-knots and can intersect at discrete points in the generic case. The boundaries of strings world sheets at light-like wormhole throat orbits and at space-like 3-surfaces defining the ends of the space-time at light-like boundaries of causal diamonds can form ordinary 1-knots, and get linked and braided. Elementary particles themselves correspond to closed loops at the ends of

space-time surface and can also get knotted (possible effects are discussed in [K25]).

2. One can assign to the lines of generalized Feynman diagrams lines in $M^2$ characterizing given causal diamond. Therefore the 2-D representation of Feynman diagrams has concrete physical interpretation in TGD. These lines can intersect and what suggests itself is a description of non-planar diagrams (having this kind of intersections) in terms of an algebraic knot theory. A natural guess is that it is this knot theoretic operation which allows to describe also non-planar diagrams by reducing them to planar ones as one does when one constructs knot invariant by reducing the knot to a trivial one. Scattering amplitudes would be basically knot invariants.

"Almost topological" has also a meaning usually not assigned with it. Thurston's geometrization conjecture stating that geometric invariants of canonical representation of manifold as Riemann geometry, defined topological invariants, could generalize somehow. For instance, the geometric invariants of preferred extremals could be seen as topological or more refined invariants (symplectic, conformal in the sense of 4-D generalization of conformal structure). If quantum ergodicity holds true, the statistical geometric invariants defined by the classical correlation functions of various induced classical gauge fields for preferred extremals could be regarded as this kind of invariants for sub-manifolds. What would distinguish TGD from standard topological QFT would be that the invariants in question would involve length scale and thus have a physical content in the usual sense of the word!

### Could $\mathcal{N}$ =2 or $\mathcal{N} = 4$ SUSY have something to do with TGD?

$\mathcal{N} = 4$ SYM has been the theoretical laboratory of Nima and others. $\mathcal{N} = 4$ SYM is definitely a completely exceptional theory, and one cannot avoid the question whether it could in some sense be part of fundamental physics. In TGD framework right handed neutrinos have remained a mystery: whether one should assign space-time SUSY to them or not. Could they give rise to something resembpling $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SUSY with fermion number conservation?

*1. Earlier results*

My latest view is that *fully* covariantly constant right-handed neutrinos decouple from the dynamics completely. I will repeat first the earlier arguments which consider only fully covariantly constant right-handed neutrinos.

1. $\mathcal{N} = 1$ SUSY is certainly excluded since it would require Majorana property not possible in TGD framework since it would require superposition of left and right handed neutrinos and lead to a breaking of lepton number conservation. Could one imagine SUSY in which both MEs between which particle wormhole contacts reside have $\mathcal{N} = 2$ SUSY which combine to form an $\mathcal{N} = 4$ SUSY?

2. Right-handed neutrinos which are covariantly constant right-handed neutrinos in both $M^4$ degrees of freedom cannot define a non-trivial theory as shown already earlier. They have no electroweak nor gravitational couplings and carry no momentum, only spin.

   The fully covariantly constant right-handed neutrinos with two possible helicities at given ME would define representation of SUSY at the limit of vanishing light-like momentum. At this limit the creation and annihilation operators creating the states would have vanishing anticommutator so that the oscillator operators would generate Grassmann algebra. Since creation and annihilation operators are hermitian conjugates, the states would have zero norm and the states generated by oscillator operators would be pure gauge and decouple from physics. This is the core of the earlier argument demonstrating that $\mathcal{N} = 1$ SUSY is not possible in TGD framework: LHC has given convincing experimental support for this belief.

*2. Could massless right-handed neutrinos covariantly constant in $CP_2$ degrees of freedom define $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SUSY?*

Consider next right-handed neutrinos, which are covariantly constant in $CP_2$ degrees of freedom but have a light-like four-momentum. In this case fermion number is conserved but this is consistent with $\mathcal{N} = 2$ SUSY at both MEs with fermion number conservation. $\mathcal{N} = 2$ SUSYs could emerge from $\mathcal{N} = 4$ SUSY when one half of SUSY generators annihilate the states, which is a basic phenomenon in supersymmetric theories.

1. At space-time level right-handed neutrinos couple to the space-time geometry - gravitation - although weak and color interactions are absent. One can say that this coupling forces them to move with light-like momentum parallel to that of ME. At the level of space-time surface right-handed neutrinos have a spectrum of excitations of four-dimensional analogs of conformal spinors at string world sheet (Hamilton-Jacobi structure).

   For MEs one indeed obtains massless solutions depending on longitudinal $M^2$ coordinates only since the induced metric in $M^2$ differs from the light-like metric only by a contribution which is light-like and contracts to zero with light-like momentum in the same direction. These solutions are analogs of (say) left movers of string theory. The dependence on $E^2$ degrees of freedom is holomorphic. That left movers are only possible would suggest that one has only single helicity and conservation of fermion number at given space-time sheet rather than 2 helicities and non-conserved fermion number: two real Majorana spinors combine to single complex Weyl spinor.

2. At imbedding space level one obtains a tensor product of ordinary representations of $\mathcal{N} = 2$ SUSY consisting of Weyl spinors with opposite helicities assigned with the ME. The state content is same as for a reduced $\mathcal{N} = 4$ SUSY with four $\mathcal{N} = 1$ Majorana spinors replaced by two complex $\mathcal{N} = 2$ spinors with fermion number conservation. This gives 4 states at both space-time sheets constructed from $\nu_R$ and its antiparticle. Altogether the two MEs give 8 states, which is one half of the 16 states of $\mathcal{N} = 4$ SUSY so that a degeneration of this symmetry forced by non-Majorana property is in question.

*3. Is the dynamics of $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM possible in right-handed neutrino sector?*

Could $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM be a part of quantum TGD? Could TGD be seen a fusion of a degenerate $\mathcal{N} = 4$ SYM describing the right-handed neutrino sector and string theory like theory describing the contribution of string world sheets carrying other leptonic and quark spinors? Or could one imagine even something simpler?

What is interesting that the net momenta assigned to the right handed neutrinos associated with a pair of MEs would correspond to the momenta assignable to the particles and obtained by p-adic mass calculations. It would seem that right-handed neutrinos provide a representation of the momenta of the elementary particles represented by wormhole contact structures. Does this mimircry generalize to a full duality so that all quantum numbers and even microscopic dynamics of defined by generalized Feynman diagrams (Euclidian space-time regions) would be represented by right-handed neutrinos and MEs? Could a generalization of $\mathcal{N} = 4$ SYM with non-trivial gauge group with proper choices of the ground states helicities allow to represent the entire microscopic dynamics?

Irrespective of the answer to this question one can compare the TGD based view about supersymmetric dynamics with what I have understood about $\mathcal{N} = 4$ SYM.

1. In the scattering of MEs induced by the dynamics of Kähler action the right-handed neutrinos play a passive role. Modified Dirac equation forces them to adopt the same direction of four-momentum as the MEs so that the scattering reduces to the geometric scattering for MEs as one indeed expects on basic of quantum classical correspondence. In $\nu_R$ sector the basic scattering vertex involves four MEs and could be a re-sharing of the right-handed neutrino content of the incoming two MEs between outgoing two MEs respecting fermion number conservation. Therefore $\mathcal{N} = 4$ SYM with fermion number conservation would represent the scattering of MEs at quantum level.

2. $\mathcal{N} = 4$ SUSY would suggest that also in the degenerate case one obtains the full scattering amplitude as a sum of permutations of external particles followed by projections to the directions of light-like momenta and that BCFW bridge represents the analog of fundamental braiding operation. The decoration of permutations means that each external line is effectively doubled. Could the scattering of MEs can be interpreted in terms of these decorated permutations? Could the doubling of permutations by decoration relate to the occurrence of pairs of MEs?

   One can also revert these questions. Could one construct massive states in $\mathcal{N} = 4$ SYM using pairs of momenta associated with particle with integer label $k$ and its decorated copy with label $k+n$? Massive external particles obtained in this manner as bound states of massless ones could solve the IR divergence problem of $\mathcal{N} = 4$ SYM.

3. The description of amplitudes in terms of leading singularities means picking up of the singular contribution by putting the fermionic propagators on mass shell. In the recent case it would give the inverse of massless Dirac propagator acting on the spinor at the end of the internal line annihilating it if it is a solution of Dirac equation.

   The only way out is a kind of cohomology theory in which solutions of Dirac equation represent exact forms. Dirac operator defines the exterior derivative d and virtual lines correspond to non-physical helicities with $d\Psi \neq 0$. Virtual fermions would be on mass-shell fermions with non-physical polarization satisfying $d^2\Psi = 0$. External particles would be those with physical polarization satisfying $d\Psi = 0$, and one can say that the Feynman diagrams containing physical helicities split into products of Feynman diagrams containing only non-physical helicities in internal lines.

4. The fermionic states at wormhole contacts should define the ground states of SUSY representation with helicity +1/2 and -1/2 rather than spin 1 or -1 as in standard realization of $\mathcal{N} = 4$ SYM used in the article. This would modify the theory but the twistorial and Grassmannian description would remain more or less as such since it depends on light-likeness and momentum conservation only.

*4. 3-vertices for sparticles are replaced with 4-vertices for MEs*

In $\mathcal{N} = 4$ SYM the basic vertex is on mass-shell 3-vertex which requires that for real light-like momenta all 3 states are parallel. One must allow complex momenta in order to satisfy energy conservation and light-likeness conditions. This is strange from the point of view of physics although number theoretically oriented person might argue that the extensions of rationals involving also imaginary unit are rather natural.

The complex momenta can be expressed in terms of two light-like momenta in 3-vertex with one real momentum. For instance, the three light-like momenta can be taken to be $p, k$, and $p - ka$ with $k = ap_R$. Here $p$ (incoming momentum) and $p_R$ are real light-like momenta satisfying $p \cdot p_R = 0$ but with opposite sign of energy, and $a$ is complex number. What is remarkable that also the negative sign of energy is necessary also now.

Should one allow complex light-like momenta in TGD framework? One can imagine two options.

1. Option I: no complex momenta. In zero energy ontology the situation is different due to the presence of a pair of MEs meaning replaced of 3-vertices with 4-vertices or 6-vertices, the allowance of negative energies in internal lines, and the fact that scattering is of sparticles is induced by that of MEs. In the simplest vertex a massive external particle with non-parallel MEs carrying non-parallel light-like momenta can decay to a pair of MEs with light-like momenta. This can be interpreted as 4-ME-vertex rather than 3-vertex (say) BFF so that complex momenta are not needed. For an incoming boson identified as wormhole contact the vertex can be seen as BFF vertex.

   To obtain space-like momentum exchanges one must allow negative sign of energy and one has strong conditions coming from momentum conservation and light-likeness which allow non-trivial solutions (real momenta in the vertex are not parallel) since basically the vertices are 4-vertices. This reduces dramatically the number of graphs. Note that one can also consider vertices in which three pairs of MEs join along their ends so that 6 MEs (analog of 3-boson vertex) would be involved.

2. Option II: complex momenta are allowed. Proceeding just formally, the $\sqrt{g_4}$ factor in Kähler action density is imaginary in Minkowskian and real in Euclidian regions. It is now clear that the formal approach is correct: Euclidian regions give rise to Kähler function and Minkowskian regions to the analog of Morse function. TGD as almost topological QFT inspires the conjecture about the reduction of Kähler action to boundary terms proportional to Chern-Simons term. This is guaranteed if the condition $j^\mu_K A_\mu = 0$ holds true: for the known extremals this is the case since Kähler current $j_K$ is light-like or vanishing for them. This would seem that Minkowskian and Euclidian regions provide dual descriptions of physics. If so, it would not be surprising if the real and complex parts of the four-momentum were parallel and in constant proportion to each other.

This argument suggests that also the conserved quantities implied by the Noether theorem have the same structure so that charges would receive an imaginary contribution from Minkowskian regions and a real contribution from Euclidian regions (or vice versa). Four-momentum would be complex number of form $P = P_M + iP_E$. Generalized light-likeness condition would give $P_M^2 = P_E^2$ and $P_M \cdot P_E = 0$. Complexified momentum would have 6 free components. A stronger condition would be $P_M^2 = 0 = P_E^2$ so that one would have two light-like momenta "orthogonal" to each other. For both relative signs energy $P_M$ and $P_E$ would be actually parallel: parametrization would be in terms of light-like momentum and scaling factor. This would suggest that complex momenta do not bring in anything new and Option II reduces effectively to Option I. If one wants a complete analogy with the usual twistor approach then $P_M^2 = P_E^2 \neq 0$ must be allowed.

*5. Is SUSY breaking possible or needed?*

It is difficult to imagine the breaking of the proposed kind of SUSY in TGD framework, and the first guess is that all these 4 super-partners of particle have identical masses. p-Adic thermodynamics does not distinguish between these states and the only possibility is that the p-adic primes differ for the spartners. But is the breaking of SUSY really necessary? Can one really distinguish between the 8 different states of a given elementary particle using the recent day experimental methods?

1. In electroweak and color interactions the spartners behave in an identical manner classically. The coupling of right-handed neutrinos to space-time geometry however forces the right-handed neutrinos to adopt the same direction of four-momentum as MEs has. Could some gravitational effect allow to distinguish between spartners? This would be trivially the case if the p-adic mass scales of spartners would be different. Why this should be the case remains however an open question.

2. In the case of unbroken SUSY only spin distinguishes between spartners. Spin determines statistics and the first naive guess would be that bosonic spartners obey totally different atomic physics allowing condensation of selectrons to the ground state. Very probably this is not true: the right-handed neutrinos are delocalized to 4-D MEs and other fermions correspond to wormhole contact structures and 2-D string world sheets.

   The coupling of the spin to the space-time geometry seems to provide the only possible manner to distinguish between spartners. Could one imagine a gravimagnetic effect with energy splitting proportional to the product of gravimagnetic moment and external gravimagnetic field B? If gravimagnetic moment is proportional to spin projection in the direction of B, a non-trivial effect would be possible. Needless to say this kind of effect is extremely small so that the unbroken SUSY might remain undetected.

3. If the spin of sparticle be seen in the classical angular momentum of ME as quantum classical correspondence would suggest then the value of the angular momentum might allow to distinguish between spartners. Also now the effect is extremely small.

*6. What can one say about scattering amplitudes?*

One expect that scattering amplitudes factorize with the only correlation between right-handed neutrino scattering and ordinary particle scattering coming from the condition that the four-momentum of the right-handed neutrino is parallel to that of massless extremal of more general preferred extremal having interpretation as a geometric counterpart of radiation quantum. This momentum is in turn equal to the massless four-momentum associated with the space-time sheet in question such that the sum of classical four-momenta associated with the space-time sheets equals to that for all wormhole throats involved. The right-handed neutrino amplitude itself would be simply constant. This certainly satisfies the SUSY constraint and it is actually difficult to find other candidates for the amplitude. The dynamics of right-handed neutrinos would be therefore that of spectator following the leader.

### Right-handed neutrino as inert neutrino?

There is a very interesting posting by Jester in Resonaances with title How many neutrinos in the sky? [C1]. Jester tells about the recent 9 years WMAP data [C3] and compares it with earlier 7 years data. In the earlier data the effective number of neutrino types was $N_{eff} = 4.34 \pm 0.87$ and in the recent data it is $N_{eff} = 3.26 \pm 0.35$. WMAP alone would give $N_{eff} = 3.89 \pm 0.67$ also in the recent data but also other data are used to pose constraings on $N_{eff}$.

To be precise, $N_{eff}$ could include instead of fourth neutrino species also some other weakly interacting particle. The only criterion for contributing to $N_{eff}$ is that the particle is in thermal equilibrium with other massless particles and thus contributes to the density of matter considerably during the radiation dominated epoch.

Jester also refers to the constraints on $N_{eff}$ from nucleosynthesis, which show that $N_{eff} \sim 4$ us slightly favored although the entire range $[3, 5]$ is consistent with data.

It seems that the effective number of neutrinos could be 4 instead of 3 although latest WMAP data combined with some other measurements favor 3. Later a corrected version of the eprint appeared [C3] telling that the original estimate of $N_{eff}$ contained a mistake and the correct estimate is $N_{eff} = 3.84 \pm 0.40$.

An interesting question is what $N_{eff} = 4$ could mean in TGD framework?

1. One poses to the modes of the modified Dirac equation the following condition: electric charge is conserved in the sense that the time evolution by modified Dirac equation does not mix a mode with a well-defined em charge with those with different em charge. The implication is that all modes except pure right handed neutrino are restricted at string world sheets. The first guess is that string world sheets are minimal surfaces of space-time surface (rather than those of imbedding space). One can also consider minimal surfaces of imbedding space but with effective metric defined by the anti-commutators of the modified gamma matrices. This would give a direct physical meaning for this somewhat mysterious effective metric.

   For the neutrino modes localized at string world sheets mixing of left and right handed modes takes place and they become massive. If only 3 lowest genera for partonic 2-surfaces are light, one has 3 neutrinos of this kind. The same applies to all other fermion species. The argument for why this could be the case relies on simple observation [K11]: the genera g=0,1,2 have the property that they allow for all values of conformal moduli $Z_2$ as a conformal symmetry (hyper-ellipticity). For $g > 2$ this is not the case. The guess is that this additional conformal symmetry is the reason for lightness of the three lowest genera.

2. Only purely right-handed neutrino is completely delocalized in 4-volume so that one cannot assign to it genus of the partonic 2-surfaces as a topological quantum number and it effectively gives rise to a fourth neutrino very much analogous to what is called sterile neutrino. Delocalized right-handed neutrinos couple only to gravitation and in case of massless extremals this forces them to have four-momentum parallel to that of ME: only massless modes are possible. Very probably this holds true for all preferred extremals to which one can assign massless longitudinal momentum direction which can vary with spatial position.

3. The coupling of $\nu_R$ is to gravitation alone and all electroweak and color couplings are absent. According to standard wisdom delocalized right-handed neutrinos cannot be in thermal equilibrium with other particles. This according to standard wisdom. But what about TGD?

   One should be very careful here: delocalized right-handed neutrinos is proposed to give rise to SUSY (not $\mathcal{N} = 1$ requiring Majorana fermions) and their dynamics is that of passive spectator who follows the leader. The simplest guess is that the dynamics of right handed neutrinos at the level of amplitudes is completely trivial and thus trivially supersymmetric. There are however correlations between four-momenta.

   (a) The four-momentum of $\nu_R$ is parallel to the light-like momentum direction assignable to the massless extremal (or more general preferred extremal). This direct coupling to the geometry is a special feature of the modified Dirac operator and thus of sub-manifold gravity.

   (b) On the other hand, the sum of massless four-momenta of two parallel pieces of preferred extremals is the - in general massive - four-momentum of the elementary particle defined by the wormhole contact structure connecting the space-time sheets (which are glued along their boundaries together since this is seems to be the only manner to get rid of boundary conditions requiring vacuum extremal property near the boundary). Could this direct coupling of the fouyr-momentum direction of right-handed neutrino to geometry and four-momentum directions of other fermions be enough for the right handed neutrinos to be counted as a fourth neutrino species in thermal equilibrium? This might be the case!

One cannot of course exclude the coupling of 2-D neutrino at string world sheets to 4-D purely right handed neutrinos analogous to the coupling inducing a mixing of sterile neutrino with ordinary neutrinos. Also this could help to achieve the thermal equilibrium with 2-D neutrino species.

## 6.6 $M^8-H$ duality, preferred extremals, criticality, and Mandelbrot fractals

$M^8-H$ duality [K54] represents an intriguing connection between number theory and TGD but the mathematics involved is extremely abstract and difficult so that I can only represent conjectures. In the following the basic duality is used to formulate a general conjecture for the construction of preferred extremals by iterative procedure. What is remarkable and extremely surprising is that the iteration gives rise to the analogs of Mandelbrot fractals and space-time surfaces can be seen as fractals defined as fixed sets of iteration. The analogy with Mandelbrot set can be also seen as a geometric correlate for quantum criticality.

### 6.6.1 $M^8-H$ duality briefly

$M^8-M^4\times CP_2$ duality [?]tates that certain 4-surfaces of $M^8$ regarded as a sub-space of complexified octonions can be mapped in a natural manner to 4-surfaces in $M^4\times CP_2$: this would mean that $M^4\times CP_2$ and therefore also the symmetries of standard model would have purely number theoretical meaning.

Consider a distribution of two planes $M^2(x)$ integrating to a 2-surface $\tilde{M}^2$ with the property that a fixed 1-plane $M^1$ defining time axis globally is contained in each $M^2(x)$ and therefore in $\tilde{M}^2$. $M^1$ defines real axis of octonionic plane $M^8$ and $M^2$(x) a local hyper-complex plane. Quaternionic subspaces with this property can be parameterized by points of $CP_2$: this leads to $M^8-H$ duality as can be shown by a simple argument.

1. Hyper-octonionic subspace of complexified octonions is obtained by multiplying octonionic imaginary units by commuting imaginary unit. This does not bring anything new as far as automorphisms are considered so that it is enough to consider octonions (so that $M^2$ is replaced with $C$). Octonionic frame consists of orthogonal octonionic units. The space of octonionic frames containing sub-frame spanning fixed $C$ is parameterized by $SU(3)$. The reason is that complexified octonionic units can be decomposed to the representations of $SU(3)\subset G_2$ as $1+1+3+\overline{3}$ and the sub-frame 1+1 spans the preferred $C$.

2. The quaternionic planes $H$ are represented by frames defined by four unit octonions spanning a quaternionic plane. Fixing $C\subset H$ means fixing the 1+1 part in the above decomposition. The sub-group of $SU(3)$ leaving the plane $H$ invariant can perform only a rotation in the plane defined by two quaternionic units in 3. This sub-group is $U(2)$ so that the space of quaternionic planes $H\supset C$ is parameterized by $SU(3)/U(2)=CP_2$.

3. Therefore quaternionic tangent plane $H\supset C$ can be mapped to a point of $CP_2$. In particular, any quaternionic surface in $E^8$, whose tangent plane at each point is quaternionic and contains $C$, can be mapped to $E^4\times CP_2$ by mapping the point $(e_1,e_2)\in E^4\times E^4$ to $(e_1,s)\in e^4\times CP_2$. The generalization from $E^8$ to $M^8$ is trivial. This is essentially what $M^8-H$ duality says.

This can be made more explicit. Define quaternionic surfaces in $M^8$ as 4-surfaces, whose tangent plane is quaternionic at each point $x$ and contains the local hyper-complex plane $M^2(x)$ and is therefore labelled by a point $s(x)\in CP_2$. One can write these surfaces as union over 2-D surfaces associated with points of $\tilde{M}^2$:

$$X^4=\cup_{x\in\tilde{M}^2}X^2(x)\subset E^6 \quad .$$

These surfaces can be mapped to surfaces of $M^4 \times CP_2$ via the correspondence $(m(x), e(x)) \rightarrow (m, s(T(X^4(x)))$. Also the image surface contains at given point $x$ the preferred plane $M^2(x) \supset M^1$. One can also write these surfaces as union over 2-D surfaces associated with points of $\tilde{M}^2$:

$$X^4 = \cup_{x \in \tilde{M}^2} X^2(x) \subset E^2 \times CP_2 \ .$$

One can also ask what are the conditions under which one can map surfaces $X^4 = \cup_{x \in \tilde{M}^2} X^2 \subset E^2 \times CP_2$ to 4-surfaces in $M^8$. The map would be given by $(m, s) \rightarrow (m, T^4(s)$ and the surface would be of the form as already described. The surface $X^4$ must be such that the distribution of 4-D tangent planes defined in $M^8$ is integrable and this gives complicated integrability conditions. One might hope that the conditions might hold true for preferred extremals satisfying some additional conditions.

One must make clear that the conditions discussed above do not allow most general possible surface.

1. The point is that for preferred extremals with Euclidian signature of metric the $M^4$ projection is 3-dimensional and involves light like projection. Here the fact that light-like line $L \subset M^2$ spans $M^2$ in the sense that the complement of its orthogonal complement in $M^8$ is $M^2$. Therefore one could consider also more general solution ansatz for which one has

$$X^4 = \cup_{x \in L(x) \subset \tilde{M}^2} X^3(x) \subset E^2 \times CP_2 \ .$$

2. One can also consider co-quaternionic surfaces as surfaces for which tangent space is in the dual of a quaternionic subspace. This says that the normal bundle rather than tangent bundle is quaternionic. The space-time regions with Euclidian signature of induced metric correspond naturally to co-quaternionic surfaces. Quaternionic surfaces are maximal associative sub-manifolds of octonionic space and one of the key ideas of the number theoretic vision about TGD is that associativity (co-associativity) defines the dynamics iof space-time surfaces. That this dynamics gives preferred extremals of Kähler action remains to be proven.

### 6.6.2 The integrability conditions

The integrability conditions are associated with the expression of tangent vectors of $T^(X^4)$ as a linear combination of coordinate gradients $\nabla m^k$, where $m^k$ denote the coordinates of $M^8$. Consider the 4 tangent vectors $e_{i)}$ for the quaternionic tangent plane (containing $M^2(x)$) regarded as vectors of $M^8$. $e_{i)}$ have components $e^k_{i)}$, $i = 1, .., 4$, $k = 1, ..., 8$. One must be able to express $e_{i)}$ as linear combinations of coordinate gradients $\nabla m^k$:

$$e^k_{i)} = e^\alpha_{i)} \partial_\alpha m^k \ .$$

Here $x^\alpha$ and $e^k$ denote coordinates for $X^4$ and $M^8$. By forming inner products of of $e_{i)}$ one finds that matrix $e^\alpha_{i)}$ represents the components of vierbein at $X^4$. One can invert this matrix to get $e^{i)}_\alpha$ satisfying $e^{i)}_\alpha e^\beta_{i)} = \delta^\beta_\alpha$ and $e^{i)}_\alpha e^\alpha_{j)} = \delta^i_j$. One can solve the coordinate gradients $\nabla m^k$ from above equation to get

$$\partial_\alpha m^k = e^{i)}_\alpha e^k_{i)} \equiv E^k_\alpha \ .$$

The integrability conditions follow from the gradient property and state

$$D_\alpha E^k_\beta = D_\beta E^k_\alpha \ .$$

One obtains $8 \times 6 = 48$ conditions in the general case. The slicing to a union of two-surfaces labeled by $M^2(x)$ reduces the number of conditions since the number of coordinates $m^k$ reduces from 8 to 6 and one has 36 integrability conditions but still them is much larger than the number of free variables- essentially the six transversal coordinates $m^k$.

For co-quaternionic surfaces one can formulate integrability conditions now as conditions for the existence of integrable distribution of orthogonal complements for tangent planes and it seems that the conditions are formally similar.

### 6.6.3 How to solve the integrability conditions and field equations for preferred extremals?

The basic idea has been that the integrability condition characterize preferred extremals so that they can be said to be quaternionic in a well-defined sense. Could one imagine solving the integrability conditions by some simple ansatz utilizing the core idea of $M^8-H$ duality? What comes in mind is that $M^8$ represents tangent space of $M^4\times CP_2$ so that one can assign to any point $(m,s)$ of 4-surface $X^4\subset M^4\times CP_2$ a tangent plane $T^4(x)$ in its tangent space $M^8$ identifiable as subspace of complexified octonions in the proposed manner. Assume that $s\in CP_2$ corresponds to a fixed tangent plane containing $M^2(x)$, and that all planes $M^2(x)$ are mapped to the same standard fixed hyper-octonionic plane $M^2\subset M^8$, which does not depend on $x$. This guarantees that $s$ corresponds to a unique quaternionic tangent plane for given $M^2(x)$.

Consider the map $T\circ s$. The map takes the tangent plane $T^4$ at point $(m,e)\in M^4\times E^4$ and maps it to $(m,s_1=s(T^4))\in M^4\times CP_2$. The obvious identification of quaternionic tangent plane at $(m,s_1)$ would be as $T^4$. One would have $T\circ s=Id$. One could do this for all points of the quaternion surface $X^4\subset E^4$ and hope of getting smooth 4-surface $X^4\subset H$ as a result. This is the case if the integrability conditions at various points $(m,s(T^4)(x))\in H$ are satisfied. One could equally well start from a quaternionic surface of $H$ and end up with integrability conditions in $M^8$ discussed above. The geometric meaning would be that the quaternionic surface in $H$ is image of quaternionic surface in $M^8$ under this map.

Could one somehow generalize this construction so that one could iterate the map $T\circ s$ to get $T\circ s=Id$ at the limit? If so, quaternionic space-time surfaces would be obtained as limits of iteration for rather arbitrary space-time surface in either $M^8$ or $H$. One can also consider limit cycles, even limiting manifolds with finite-dimension which would give quaternionic surfaces. This would give a connection with chaos theory.

1. One could try to proceed by discretizing the situation in $M^8$ and $H$. One does not fix quaternionic surface at either side but just considers for a fixed $m_2\in M^2(x)$ a discrete collection $X$ $\{(T^4_i\}\supset M^2(x)$ of quaternionic planes in $M^8$. The points $e_{2,i}\subset E^2\subset M2\times E^2=M^4$ are not fixed. One can also assume that the points $s_i=s(T^4_i)$ of $CP_2$ defined by the collection of planes form in a good approximation a cubic lattice in $CP_2$ but this is not absolutely essential. Complex Eguchi-Hanson coordinates $\xi^i$ are natural choice for the coordinates of $CP_2$. Assume also that the distances between the nearest $CP_2$ points are below some upper limit.

2. Consider now the iteration. One can map the collection $X$ to $H$ by mapping it to the set $s(X)$ of pairs $((m_2,s_i)$. Next one must select some candidates for the points $e_{2,i}\in E^2\subset M^4$ somehow. One can define a piece-wise linear surface in $M^4\times CP_2$ consisting of 4-planes defined by the nearest neighbors of given point $(m_2,e_{2,i},s_i)$. The coordinates $e_{2,i}$ for $E^2\subset M^4$ can be chosen rather freely. The collection $(e_{2,i},_i)$ defines a piece-wise linear surface in $H$ consisting of four-cubes in the simplest case. One can hope that for certain choices of $e_{2,i}$ the four-cubes are quaternionic and that there is some further criterion allowing to choose the points $e_{2,i}$ uniquely. The tangent planes contain by construction $M^2(x)$ so that the product of remaining two spanning tangent space vectors $(e_3,e_4)$ must give an element of $M^2$ in order to achieve quaternionicity. Another natural condition would be that the resulting tangent planes are not only quaternionic but also as near as possible to the planes $T^4_i$. These conditions allow to find $e_{2,i}$ giving rise to geometrically determined quaternionic tangent planes as near as possible to those determined by $s^i$.

3. What to do next? Should one replace the quaternionic planes $T^4_i$ with geometrically determined quaternionic planes as near as possible to them and map them to points $s_i$ slightly different from the original one and repeat the procedure? This would not add new points to the approximation, and this is an unsatisfactory feature.

4. Second possibility is based on the addition of the quaternionic tangent planes obtained in this manner to the original collection of quaternionic planes. Therefore the number of points in discretization increases and the added points of $CP_2$ are as near as possible to existing ones. One can again determine the points $e_{2,i}$ in such a manner that the resulting geometrically determined quaternionic tangent planes are as near as possible to the original ones. This guarantees that the algorithm converges.

5. The iteration can be stopped when desired accuracy is achieved: in other words the geometrically determined quaternionic tangent planes are near enough to those determined by the points $s_i$. Also limit cycles are possible and would be assignable to the transversal coordinates $e_{2i}$ varying periodically during iteration. One can quite well allow this kind of cycles, and they would mean that $e_2$ coordinate as a function of $CP_2$ coordinates characterizing the tangent plane is many-valued. This is certainly very probable for solutions representable locally as graphs $M^4 \rightarrow CP_2$. In this case the tangent planes associated with distant points in $E^2$ would be strongly correlated which must have non-trivial physical implications. The iteration makes sense also p-adically and it might be that in some cases only p-adic iteration converges for some value of $p$.

It is not obvious whether the proposed procedure gives rise to a smooth or even continuous 4-surface. The conditions for this are geometric analogs of the above described algebraic integrability conditions for the map assigning to the surface in $M^4 \times CP_2$ a surface in $M^8$. Therefore $M^8 - H$ duality could express the integrability conditions and preferred extremals would be 4-surfaces having counterparts also in the tangent space $M^8$ of $H$.

One might hope that the self-referentiality condition $s \circ T = Id$ for the $CP_2$ projection of $(m, s)$ or its fractal generalization could solve the complicated integrability conditions for the map $T$. The image of the space-time surface in tangent space $M^8$ in turn could be interpreted as a description of space-time surface using coordinates defined by the local tangent space $M^8$. Also the analogy for the duality between position and momentum suggests itself.

Is there any hope that this kind of construction could make sense? Or could one demonstrate that it fails? If $s$ would fix completely the tangent plane it would be probably easy to kill the conjecture but this is not the case. Same $s$ corresponds for different planes $M^2(x)$ to different point tangent plane. Presumably they are related by a local $G_2$ or $SO(7)$ rotation. Note that the construction can be formulated without any reference to the representation of the imbedding space gamma matrices in terms of octonions. Complexified octonions are enough in the tangent space of $M^8$.

### 6.6.4 Connection with Mandelbrot fractal and fractals as fixed sets for iteration

The occurrence of iteration in the construction of preferred extremals suggests a deep connection with the standard construction of 2-D fractals by iteration - about which Mandelbrot fractal [A56, A17] is the canonical example. $X^2(x)$ (or $X^3(x)$ in the case of light-like $L(x) \subset M^2(x)$) could be identified as a union of orbits for the iteration of $s \circ T$. The appearance of the iteration map in the construction of solutions of field equation would answer positively to a long standing question whether the extremely beautiful mathematics of 2-D fractals could have some application at the level of fundamental physics according to TGD.

$X^2$ (or $X^3$) would be completely analogous to Mandelbrot set in the sense that it would be boundary separating points in two different basis of attraction. In the case of Mandelbrot set iteration would take points at the other side of boundary to origin on the other side and to infinity. The points of Mandelbrot set are permuted by the iteration. In the recent case $s \circ T$ maps $X^2$ (or $X^3$) to itself. This map need not be diffeomorphism or even continuous map. The criticality of $X^2$ (or $X^3$) could be seen as a geometric correlate for quantum criticality.

In fact, iteration plays a very general role in the construction of fractals. Very general fractals can be defined as fixed sets of iteration and simple rules for iteration produce impressive representations for fractals appearing in Nature. The book of Michael Barnsley [A34] gives fascinating pictures about fractals appearing in Nature using this method. Therefore it would be highly satisfactory if space-time surfaces would be in a well-defined sense fixed sets of iteration. This would be also numerically beautiful aspect since fixed sets of iteration can be obtained as infinite limit of iteration for almost arbitrary initial set. This construction recipe would also give a concrete content for the notion measurement resolution at the level of construction of preferred extremals.

What is intriguing is that there are several very attractive approaches to the construction of preferred extremals. The challenge of unifying them still remains to be met.

## 6.7 Do geometric invariants of preferred extremals define topological invariants of space-time surface and code for quantum physics?

The recent progress in the understanding of preferred extremals [K5] led to a reduction of the field equations to conditions stating for Euclidian signature the existence of Kähler metric. The resulting conditions are a direct generalization of corresponding conditions emerging for the string world sheet and stating that the 2-metric has only non-diagonal components in complex/hypercomplex coordinates. Also energy momentum of Kähler action and has this characteristic (1,1) tensor structure. In Minkowskian signature one obtains the analog of 4-D complex structure combining hyper-complex structure and 2-D complex structure.

The construction lead also to the understanding of how Einstein's equations with cosmological term follow as a consistency condition guaranteeing that the covariant divergence of the Maxwell's energy momentum tensor assignable to Kähler action vanishes. This gives $T = kG + \Lambda g$. By taking trace a further condition follows from the vanishing trace of $T$:

$$R = \frac{4\Lambda}{k} . \quad (6.7.1)$$

That any preferred extremal should have a constant Ricci scalar proportional to cosmological constant is very strong prediction. Note that the accelerating expansion of the Universe would support positive value of $\Lambda$. Note however that both $\Lambda$ and $k \propto 1/G$ are both parameters characterizing one particular preferred extremal. One could of course argue that the dynamics allowing only constant curvature space-times is too simple. The point is however that particle can topologically condense on several space-time sheets meaning effective superposition of various classical fields defined by induced metric and spinor connection.

The following considerations demonstrate that preferred extremals can be seen as canonical representatives for the constant curvature manifolds playing central role in Thurston's geometrization theorem [A28] known also as hyperbolization theorem implying that geometric invariants of space-time surfaces transform to topological invariants. The generalization of the notion of Ricci flow to Maxwell flow in the space of metrics and further to Kähler flow for preferred extremals in turn gives a rather detailed vision about how preferred extremals organize to one-parameter orbits. It is quite possible that Kähler flow is actually discrete. The natural interpretation is in terms of dissipation and self organization.

Quantum classical correspondence suggests that this line of thought could be continued even further: could the geometric invariants of the preferred extremals could code not only for space-time topology but also for quantum physics? How to calculate the correlation functions and coupling constant evolution has remained a basic unresolved challenge of quantum TGD. Could the correlation functions be reduced to statistical geometric invariants of preferred extemals? The latest (means the end of 2012) and perhaps the most powerful idea hitherto about coupling constant evolution is quantum classical correspondence in statistical sense stating that the statistical properties of a preferred extremal in quantum superposition of them are same as those of the zero energy state in question. This principle would be quantum generalization of ergodic theorem stating that the time evolution of a single member of ensemble represents the ensemble statistically. This principle would allow to deduce correlation functions and S-matrix from the statistical properties of single preferred extremal alone using classical intuition. Also coupling constant evolution would be coded by the statistical properties of the representative preferred extremal.

### 6.7.1 Preferred extremals of Kähler action as manifolds with constant Ricci scalar whose geometric invariants are topological invariants

An old conjecture inspired by the preferred extremal property is that the geometric invariants of space-time surface serve as topological invariants. The reduction of Kähler action to 3-D Chern-Simons terms [K5] gives support for this conjecture as a classical counterpart for the view about TGD as almost topological QFT. The following arguments give a more precise content to this conjecture in terms of existing mathematics.

1. It is not possible to represent the scaling of the induced metric as a deformation of the space-time surface preserving the preferred extremal property since the scale of $CP_2$ breaks scale invariance. Therefore the curvature scalar cannot be chosen to be equal to one numerically. Therefore also the parameter $R = 4\Lambda/k$ and also $\Lambda$ and $k$ separately characterize the equivalence class of preferred extremals as is also physically clear.

   Also the volume of the space-time sheet closed inside causal diamond $CD$ remains constant along the orbits of the flow and thus characterizes the space-time surface. $\Lambda$ and even $k \propto 1/G$ can indeed depend on space-time sheet and p-adic length scale hypothesis suggests a discrete spectrum for $\Lambda/k$ expressible in terms of p-adic length scales: $\Lambda/k \propto 1/L_p^2$ with $p \simeq 2^k$ favored by p-adic length scale hypothesis. During cosmic evolution the p-adic length scale would increase gradually. This would resolve the problem posed by cosmological constant in GRT based theories.

2. One could also see the preferred extremals as 4-D counterparts of constant curvature 3-manifolds in the topology of 3-manifolds. An interesting possibility raised by the observed negative value of $\Lambda$ is that most 4-surfaces are constant negative curvature 4-manifolds. By a general theorem coset spaces $H^4/\Gamma$, where $H^4 = SO(1,4)/SO(4)$ is hyperboloid of $M^5$ and $\Gamma$ a torsion free discrete subgroup of $SO(1,4)$ [A11]. It is not clear to me, whether the constant value of Ricci scalar implies constant sectional curvatures and therefore hyperbolic space property. It could happen that the space of spaces with constant Ricci curvature contain a hyperbolic manifold as an especially symmetric representative. In any case, the geometric invariants of hyperbolic metric are topological invariants.

   By Mostow rigidity theorem [A18] finite-volume hyperbolic manifold is unique for $D > 2$ and determined by the fundamental group of the manifold. Since the orbits under the Kähler flow preserve the curvature scalar the manifolds at the orbit must represent different imbeddings of one and hyperbolic 4-manifold. In 2-D case the moduli space for hyperbolic metric for a given genus $g > 0$ is defined by Teichmueller parameters and has dimension $6(g-1)$. Obviously the exceptional character of $D = 2$ case relates to conformal invariance. Note that the moduli space in question plays a key role in p-adic mass calculations [K11].

   In the recent case Mostow rigidity theorem could hold true for the Euclidian regions and maybe generalize also to Minkowskian regions. If so then both "topological" and "geometro" in "Topological GeometroDynamics" would be fully justified. The fact that geometric invariants become topological invariants also conforms with "TGD as almost topological QFT" and allows the notion of scale to find its place in topology. Also the dream about exact solvability of the theory would be realized in rather convincing manner.

These conjectures are the main result independent of whether the generalization of the Ricci flow discussed in the sequel exists as a continuous flow or possibly discrete sequence of iterates in the space of preferred extremals of Kähler action. My sincere hope is that the reader could grasp how far reaching these result really are.

### 6.7.2 Is there a connection between preferred extremals and AdS$_4$/CFT correspondence?

The preferred extremals satisfy Einstein Maxwell equations with a cosmological constant and have negative scalar curvature for negative value of $\Lambda$. 4-D space-times with hyperbolic metric provide canonical representation for a large class of four-manifolds and an interesting question is whether these spaces are obtained as preferred extremals and/or vacuum extremals.

4-D hyperbolic space with Minkowski signature is locally isometric with AdS$_4$. This suggests at connection with AdS$_4$/CFT correspondence of M-theory. The boundary of AdS would be now replaced with 3-D light-like orbit of partonic 2-surface at which the signature of the induced metric changes. The metric 2-dimensionality of the light-like surface makes possible generalization of 2-D conformal invariance with the light-like coordinate taking the role of complex coordinate at light-like boundary. AdS could represent a special case of a more general family of space-time surfaces with constant Ricci scalar satistying Einstein-Maxwell equations and generalizing the AdS$_4$/CFT correspondence. There is however a strong objection from cosmology: the accelerated expansion of the Universe requires positive value of $\Lambda$ and favors De Sitter Space $dS_4$ instead of $AdS_4$.

These observations provide motivations for finding whether AdS$_4$ and/or $dS_4$ allows an imbedding as a vacuum extremal to $M^4 \times S^2 \subset M^4 \times CP_2$, where $S^2$ is a homologically trivial geodesic sphere of $CP_2$. It is easy to guess the general form of the imbedding by writing the line elements of, $M^4$, $S^2$, and AdS$_4$.

1. The line element of $M^4$ in spherical Minkowski coordinates $(m, r_M, \theta, \phi)$ reads as

$$ds^2 = dm^2 - dr_M^2 - r_M^2 d\Omega^2 \ . \tag{6.7.2}$$

2. Also the line element of $S^2$ is familiar:

$$ds^2 = -R^2(d\Theta^2 + sin^2(\theta)d\Phi^2) \ . \tag{6.7.3}$$

3. By visiting in Wikipedia one learns that in spherical coordinate the line element of AdS$_4/dS_4$ is given by

$$\begin{aligned} ds^2 &= A(r)dt^2 - \frac{1}{A(r)}dr^2 - r^2 d\Omega^2 \ , \\ A(r) &= 1 + \epsilon y^2 \ , \ y = \frac{r}{r_0} \ , \\ \epsilon &= 1 \ \text{for} \ AdS_4 \ , \ \epsilon = -1 \ \text{for} \ dS_4 \ . \end{aligned} \tag{6.7.4}$$

4. From these formulas it is easy to see that the ansatz is of the same general form as for the imbedding of Schwartschild-Nordstöm metric:

$$\begin{array}{ll} m = \Lambda t + h(y) \ , & r_M = r \ , \\ \Theta = s(y) \ , & \Phi = \omega(t + f(y)) \ . \end{array} \tag{6.7.5}$$

The non-trivial conditions on the components of the induced metric are given by

$$\begin{aligned} g_{tt} &= \Lambda^2 - x^2 sin^2(\Theta) = A(r) \ , \\ g_{tr} &= \frac{1}{r_0}\left[\Lambda\frac{dh}{dy} - x^2 sin^2(\theta)\frac{df}{dr}\right] = 0 \ , \\ g_{rr} &= \frac{1}{r_0^2}\left[(\frac{dh}{dy})^2 - 1 - x^2 sin^2(\theta)(\frac{df}{dy})^2 - R^2(\frac{d\Theta}{dy})^2\right] = -\frac{1}{A(r)} \ , \\ x &= R\omega \ . \end{aligned} \tag{6.7.6}$$

By some simple algebraic manipulations one can derive expressions for $sin(\Theta)$, $df/dr$ and $dh/dr$.

1. For $\Theta(r)$ the equation for $g_{tt}$ gives the expression

$$\begin{aligned} sin(\Theta) &= \pm\frac{P^{1/2}}{x} \ , \\ P &= \Lambda^2 - A = \Lambda^2 - 1 - \epsilon y^2 \ . \end{aligned} \tag{6.7.7}$$

The condition $0 \leq sin^2(\Theta) \leq 1$ gives the conditions

$$\begin{aligned}(\Lambda^2-x^2-1)^{1/2}\leq \quad y \quad &\leq (\Lambda^2-1)^{1/2} \quad \text{for } \epsilon=1\ (AdS_4)\ , \\ (-\Lambda^2+1)^{1/2}\leq \quad y \quad &\leq (x^2+1-\Lambda^2)^{1/2} \quad \text{for } \epsilon=-1\ (dS_4)\ .\end{aligned} \tag{6.7.8}$$

Only a spherical shell is possible in both cases. The model for the final state of star considered in [K58] predicted similar layer layer like structure and inspired the proposal that stars quite generally have an onionlike structure with radii of various shells characterize by p-adic length scale hypothesis and thus coming in some powers of $\sqrt{2}$. This brings in mind also Titius-Bode law.

2. From the vanishing of $g_{tr}$ one obtains

$$\frac{dh}{dy} = \frac{P}{\Lambda}\frac{df}{dy} \ . \tag{6.7.9}$$

3. The condition for $g_{rr}$ gives

$$(\frac{df}{dy})^2 = \frac{r_0^2}{AP}[A^{-1}-R^2(\frac{d\Theta}{dy})^2] \ . \tag{6.7.10}$$

Clearly, the right-hand side is positive if $P \geq 0$ holds true and $Rd\Theta/dy$ is small. One can express $d\Theta/dy$ using chain rule as

$$(\frac{d\Theta}{dy})^2 = \frac{x^2y^2}{P(P-x^2)} \ . \tag{6.7.11}$$

One obtains

$$(\frac{df}{dy})^2 = \Lambda r_0^2\frac{y^2}{AP}\left[\frac{1}{1+y^2}-x^2(\frac{R}{r_0})^2\frac{1}{P(P-x^2)}\right] \ . \tag{6.7.12}$$

The right hand side of this equation is non-negative for certain range of parameters and variable $y$. Note that for $r_0 \gg R$ the second term on the right hand side can be neglected. In this case it is easy to integrate $f(y)$.

The conclusion is that both AdS$_4$ and $dS^4$ allow a local imbedding as a vacuum extremal. Whether also an imbedding as a non-vacuum preferred extremal to $M^4\times S^2$, $S^2$ a homologically non-trivial geodesic sphere is possible, is an interesting question.

### 6.7.3 Generalizing Ricci flow to Maxwell flow for 4-geometries and Kähler flow for space-time surfaces

The notion of Ricci flow has played a key part in the geometrization of topological invariants of Riemann manifolds. I certainly did not have this in mind when I choose to call my unification attempt "Topological Geometrodynamics" but this title strongly suggests that a suitable generalization of Ricci flow could play a key role in the understanding of also TGD.

### Ricci flow and Maxwell flow for 4-geometries

The observation about constancy of 4-D curvature scalar for preferred extremals inspires a generalization of the well-known volume preserving Ricci flow [A22] introduced by Richard Hamilton. Ricci flow is defined in the space of Riemann metrics as

$$\frac{dg_{\alpha\beta}}{dt} = -2R_{\alpha\beta}+2\frac{R_{avg}}{D}g_{\alpha\beta} \ . \tag{6.7.13}$$

Here $R_{avg}$ denotes the average of the scalar curvature, and $D$ is the dimension of the Riemann manifold. The flow is volume preserving in average sense as one easily checks ($\langle g^{\alpha\beta}dg_{\alpha\beta}/dt\rangle = 0$). The volume preserving property of this flow allows to intuitively understand that the volume of a 3-manifold in the asymptotic metric defined by the Ricci flow is topological invariant. The fixed points of the flow serve as canonical representatives for the topological equivalence classes of 3-manifolds. These 3-manifolds (for instance hyperbolic 3-manifolds with constant sectional curvatures) are highly symmetric. This is easy to understand since the flow is dissipative and destroys all details from the metric.

What happens in the recent case? The first thing to do is to consider what might be called Maxwell flow in the space of all 4-D Riemann manifolds allowing Maxwell field.

1. First of all, the vanishing of the trace of Maxwell's energy momentum tensor codes for the volume preserving character of the flow defined as

   $$\frac{dg_{\alpha\beta}}{dt} = T_{\alpha\beta} \ . \tag{6.7.14}$$

   Taking covariant divergence on both sides and assuming that $d/dt$ and $D_\alpha$ commute, one obtains that $T^{\alpha\beta}$ is divergenceless.

   This is true if one assumes Einstein's equations with cosmological term. This gives

   $$\frac{dg_{\alpha\beta}}{dt} = kG_{\alpha\beta}+\Lambda g_{\alpha\beta} = kR_{\alpha\beta}+(-\frac{kR}{2}+\Lambda)g_{\alpha\beta} \ . \tag{6.7.15}$$

   The trace of this equation gives that the curvature scalar is constant. Note that the value of the Kähler coupling strength plays a highly non-trivial role in these equations and it is quite possible that solutions exist only for some critical values of $\alpha_K$. Quantum criticality should fix the allow value triplets $(G,\Lambda,\alpha_K)$ apart from overall scaling

   $$(G,\Lambda,\alpha_K)\rightarrow(xG,\Lambda/x,x\alpha_K \ .$$

   Fixing the value of $G$ fixes the values remaining parameters at critical points. The rescaling of the parameter $t$ induces a scaling by $x$.

2. By taking trace one obtains the already mentioned condition fixing the curvature to be constant, and one can write

   $$\frac{dg_{\alpha\beta}}{dt} = kR_{\alpha\beta}-\Lambda g_{\alpha\beta} \ . \tag{6.7.16}$$

   Note that in the recent case $R_{avg} = R$ holds true since curvature scalar is constant. The fixed points of the flow would be Einstein manifolds [A7, A35] satisfying

   $$R_{\alpha\beta} = \frac{\Lambda}{k}g_{\alpha\beta} \tag{6.7.17}$$

   .

3. It is by no means obvious that continuous flow is possible. The condition that Einstein-Maxwell equations are satisfied might pick up from a completely general Maxwell flow a discrete subset as solutions of Einstein-Maxwell equations with a cosmological term. If so, one could assign to this subset a sequence of values $t_n$ of the flow parameter $t$.

4. I do not know whether 3-dimensionality is somehow absolutely essential for getting the topological classification of closed 3-manifolds using Ricci flow. This ignorance allows me to pose some innocent questions. Could one have a canonical representation of 4-geometries as spaces with constant Ricci scalar? Could one select one particular Einstein space in the class four-metrics and could the ratio $\Lambda/k$ represent topological invariant if one normalizes metric or curvature scalar suitably. In the 3-dimensional case curvature scalar is normalized to unity. In the recent case this normalization would give $k = 4\Lambda$ in turn giving $R_{\alpha\beta} = g_{\alpha\beta}/4$. Does this mean that there is only single fixed point in local sense, analogous to black hole toward which all geometries are driven by the Maxwell flow? Does this imply that only the 4-volume of the original space would serve as a topological invariant?

**Maxwell flow for space-time surfaces**

One can consider Maxwell flow for space-time surfaces too. In this case Kähler flow would be the appropriate term and provides families of preferred extremals. Since space-time surfaces inside CD are the basic physical objects are in TGD framework, a possible interpretation of these families would be as flows describing physical dissipation as a four-dimensional phenomenon polishing details from the space-time surface interpreted as an analog of Bohr orbit.

1. The flow is now induced by a vector field $j^k(x,t)$ of the space-time surface having values in the tangent bundle of imbedding space $M^4 \times CP_2$. In the most general case one has Kähler flow without the Einstein equations. This flow would be defined in the space of all space-time surfaces or possibly in the space of all extremals. The flow equations reduce to

$$h_{kl} D_\alpha j^k(x,t) D_\beta h^l = \frac{1}{2} T_{\alpha\beta} \ . \tag{6.7.18}$$

   The left hand side is the projection of the covariant gradient $D_\alpha j^k(x,t)$ of the flow vector field $j^k(x,t)$ to the tangent space of the space-time surface. $D_{alpha}$ is covariant derivative taking into account that $j^k$ is imbedding space vector field. For a fixed point space-time surface this projection must vanish assuming that this space-time surface reachable. A good guess for the asymptotia is that the divergence of Maxwell energy momentum tensor vanishes and that Einstein's equations with cosmological constant are well-defined.

   Asymptotes corresponds to vacuum extremals. In Euclidian regions $CP_2$ type vacuum extremals and in Minkowskian regions to any space-time surface in any 6-D sub-manifold $M^4 \times Y^2$, where $Y^2$ is Lagrangian sub-manifold of $CP_2$ having therefore vanishing induced Kähler form. Symplectic transformations of $CP_2$ combined with diffeomorphisms of $M^4$ give new Lagrangian manifolds. One would expect that vacuum extremals are approached but never reached at second extreme for the flow.

   If one assumes Einstein's equations with a cosmological term, allowed vacuum extremals must be Einstein manifolds. For $CP_2$ type vacuum extremals this is the case. It is quite possible that these fixed points do not actually exist in Minkowskian sector, and could be replaced with more complex asymptotic behavior such as limit, chaos, or strange attractor.

2. The flow could be also restricted to the space of preferred extremals. Assuming that Einstein Maxwell equations indeed hold true, the flow equations reduce to

$$h_{kl} D_\alpha j^k(x,t) \partial_\beta h^l = \frac{1}{2}(k R_{\alpha\beta} - \Lambda g_{\alpha\beta}) \ . \tag{6.7.19}$$

   Preferred extremals would correspond to a fixed sub-manifold of the general flow in the space of all 4-surfaces.

3. One can also consider a situation in which $j^k(x,t)$ is replaced with $j^k(h,t)$ defining a flow in the entire imbedding space. This assumption is probably too restrictive. In this case the equations reduce to

$$(D_r j_l(x,t) + D_l j_r)\partial_\alpha h^r \partial_\beta h^l = kR_{\alpha\beta} - \Lambda g_{\alpha\beta} \ . \tag{6.7.20}$$

   Here $D_r$ denotes covariant derivative. Asymptotia is achieved if the tensor $D_k j_l + D_k j_l$ becomes orthogonal to the space-time surface. Note for that Killing vector fields of $H$ the left hand side vanishes identically. Killing vector fields are indeed symmetries of also asymptotic states.

It must be made clear that the existence of a continuous flow in the space of preferred extremals might be too strong a condition. Already the restriction of the general Maxwell flow in the space of metrics to solutions of Einstein-Maxwell equations with cosmological term might lead to discretization, and the assumption about reprentability as 4-surface in $M^4 \times CP_2$ would give a further condition reducing the number of solutions. On the other hand, one might consiser a possibility of a continuous flow in the space of constant Ricci scalar metrics with a fixed 4-volume and having hyperbolic spaces as the most symmetric representative.

### Dissipation, self organization, transition to chaos, and coupling constant evolution

A beautiful connection with concepts like dissipation, self-organization, transition to chaos, and coupling constant evolution suggests itself.

1. It is not at all clear whether the vacuum extremal limits of the preferred extremals can correspond to Einstein spaces except in special cases such as $CP_2$ type vacuum extremals isometric with $CP_2$. The imbeddability condition however defines a constraint force which might well force asymptotically more complex situations such as limit cycles and strange attractors. In ordinary dissipative dynamics an external energy feed is essential prerequisite for this kind of non-trivial self-organization patterns.

   In the recent case the external energy feed could be replaced by the constraint forces due to the imbeddability condition. It is not too difficult to imagine that the flow (if it exists!) could define something analogous to a transition to chaos taking place in a stepwise manner for critical values of the parameter $t$. Alternatively, these discrete values could correspond to those values of $t$ for which the preferred extremal property holds true for a general Maxwell flow in the space of 4-metrics. Therefore the preferred extremals of Kähler action could emerge as one-parameter (possibly discrete) families describing dissipation and self-organization at the level of space-time dynamics.

2. For instance, one can consider the possibility that in some situations Einstein's equations split into two mutually consistent equations of which only the first one is independent

$$\begin{aligned} xJ^{\alpha}_{\ \nu}J^{\nu\beta} &= R^{\alpha\beta} \ , \\ L_K &= xJ^{\alpha}_{\ \nu}J^{\nu\beta} = 4\Lambda \ , \\ x &= \frac{1}{16\pi\alpha_K} \ . \end{aligned} \tag{6.7.21}$$

   Note that the first equation indeed gives the second one by tracing. This happens for $CP_2$ type vacuum extremals.

   Kähler action density would reduce to cosmological constant which should have a continuous spectrum if this happens always. A more plausible alternative is that this holds true only asymptotically. In this case the flow equation could not lead arbitrary near to vacuum extremal, and one can think of situation in which $L_K = 4\Lambda$ defines an analog of limiting cycle or perhaps even strange attractor. In any case, the assumption would allow to deduce the asymptotic value of the action density which is of utmost importance from calculational point of view: action would be simply $S_K = 4\Lambda V_4$ and one could also say that one has minimal surface with $\Lambda$ taking the role of string tension.

3. One of the key ideas of TGD is quantum criticality implying that Kähler coupling strength is analogous to critical temperature. Second key idea is that p-adic coupling constant evolution represents discretized version of continuous coupling constant evolution so that each p-adic prime would correspond a fixed point of ordinary coupling constant evolution in the sense that the 4-volume characterized by the p-adic length scale remains constant. The invariance of the geometric and thus geometric parameters of hyperbolic 4-manifold under the Kähler flow would conform with the interpretation as a flow preserving scale assignable to a given p-adic prime. The continuous evolution in question (if possible at all!) might correspond to a fixed p-adic prime. Also the hierarchy of Planck constants relates to this picture naturally. Planck constant $\hbar_{eff} = n\hbar$ corresponds to a multi-furcation generating n-sheeted structure and certainly affecting the fundamental group.

4. One can of course question the assumption that a continuous flow exists. The property of being a solution of Einstein-Maxwell equations, imbeddability property, and preferred extremal property might allow allow only discrete sequences of space-time surfaces perhaps interpretable as orbit of an iterated map leading gradually to a fractal limit. This kind of discrete sequence might be also be selected as preferred extremals from the orbit of Maxwell flow without assuming Einstein-Maxwell equations. Perhaps the discrete p-adic coupling constant evolution could be seen in this manner and be regarded as an iteration so that the connection with fractality would become obvious too.

### Does a 4-D counterpart of thermodynamics make sense?

The interpretation of the Kähler flow in terms of dissipation, the constancy of $R$, and almost constancy of $L_K$ suggest an interpretation in terms of 4-D variant of thermodynamics natural in zero energy ontology (ZEO), where physical states are analogs for pairs of initial and final states of quantum event are quantum superpositions of classical time evolutions. Quantum theory becomes a "square root" of thermodynamics so that 4-D analog of thermodynamics might even replace ordinary thermodynamics as a fundamental description. If so this 4-D thermodynamics should be qualitatively consistent with the ordinary 3-D thermodynamics.

1. The first naive guess would be the interpretation of the action density $L_K$ as an analog of energy density $e = E/V_3$ and that of $R$ as the analog to entropy density $s = S/V_3$. The asymptotic states would be analogs of thermodynamical equilibria having constant values of $L_K$ and $R$.

2. Apart from an overall sign factor $\epsilon$ to be discussed, the analog of the first law $de = Tds - pdV/V$ would be

$$dL_K = kdR + \Lambda \frac{dV_4}{V_4} \quad .$$

One would have the correspondences $S \to \epsilon RV_4$, $e \to \epsilon L_K$ and $k \to T$, $p \to -\Lambda$. $k \propto 1/G$ indeed appears formally in the role of temperature in Einstein's action defining a formal partition function via its exponent. The analog of second law would state the increase of the magnitude of $\epsilon RV_4$ during the Kähler flow.

3. One must be very careful with the signs and discuss Euclidian and Minkowskian regions separately. Concerning purely thermodynamic aspects at the level of vacuum functional Euclidian regions are those which matter.

   (a) For $CP_2$ type vacuum extremals $L_K \propto E^2 + B^2$ , $R = \Lambda/k$, and $\Lambda$ are positive. In thermodynamical analogy for $\epsilon = 1$ this would mean that pressure is negative.

   (b) In Minkowskian regions the value of $R = \Lambda/k$ is negative for $\Lambda < 0$ suggested by the large abundance of 4-manifolds allowing hyperbolic metric and also by cosmological considerations. The asymptotic formula $L_K = 4\Lambda$ considered above suggests that also Kähler action is negative in Minkowskian regions for magnetic flux tubes dominating in TGD inspired cosmology: the reason is that the magnetic contribution to the action density $L_K \propto E^2 - B^2$ dominates.

Consider now in more detail the 4-D thermodynamics interpretation in Euclidian and Minkowskian regions assuming that the the evolution by quantum jumps has Kähler flow as a space-time correlate.

1. In Euclidian regions the choice $\epsilon = 1$ seems to be more reasonable one. In Euclidian regions $-\Lambda$ as the analog of pressure would be negative, and asymptotically (that is for $CP_2$ type vacuum extremals) its value would be proportional to $\Lambda \propto 1/GR^2$, where $R$ denotes $CP_2$ radius defined by the length of its geodesic circle.

   A possible interpretation for negative pressure is in terms of string tension effectively inducing negative pressure (note that the solutions of the modified Dirac equation indeed assign a string to the wormhole contact). The analog of the second law would require the increase of $RV_4$ in quantum jumps. The magnitudes of $L_K$, $R$, $V_4$ and $\Lambda$ would be reduced and approach their asymptotic values. In particular, $V_4$ would approach asymptotically the volume of $CP_2$.

2. In Minkowskian regions Kähler action contributes to the vacuum functional a phase factor analogous to an imaginary exponent of action serving in the role of Morse function so that thermodynamics interpretation can be questioned. Despite this one can check whether thermodynamic interpretation can be considered. The choice $\epsilon = -1$ seems to be the correct choice now. $-\Lambda$ would be analogous to a negative pressure whose gradually decreases. In 3-D thermodynamics it is natural to assign negative pressure to the magnetic flux tube like structures as their effective string tension defined by the density of magnetic energy per unit length. $-R \geq 0$ would entropy and $-L_K \geq 0$ would be the analog of energy density.

   $R = \Lambda/k$ and the reduction of $\Lambda$ during cosmic evolution by quantum jumps suggests that the larger the volume of CD and thus of (at least) Minkowskian space-time sheet the smaller the negative value of $\Lambda$.

   Assume the recent view about state function reduction explaining how the arrow of geometric time is induced by the quantum jump sequence defining experienced time [K4]. According to this view zero energy states are quantum superpositions over $CD$s of various size scales but with common tip, which can correspond to either the upper or lower light-like boundary of $CD$. The sequence of quantum jumps the gradual increase of the average size of $CD$ in the quantum superposition and therefore that of average value of $V_4$. On the other hand, a gradual decrease of both $-L_K$ and $-R$ looks physically very natural. If Kähler flow describes the effect of dissipation by quantum jumps in ZEO then the space-time surfaces would gradually approach nearly vacuum extremals with constant value of entropy density $-R$ but gradually increasing 4-volume so that the analog of second law stating the increase of $-RV_4$ would hold true.

3. The interpretation of $-R > 0$ as negentropy density assignable to entanglement is also possible and is consistent with the interpretation in terms of second law. This interpretation would only change the sign factor $\epsilon$ in the proposed formula. Otherwise the above arguments would remain as such.

### 6.7.4 Could correlation functions, S-matrix, and coupling constant evolution be coded the statistical properties of preferred extremals?

Quantum classical correspondence states that all aspects of quantum states should have correlates in the geometry of preferred extremals. In particular, various elementary particle propagators should have a representation as properties of preferred extremals. This would allow to realize the old dream about being able to say something interesting about coupling constant evolution although it is not yet possible to calculate the M-matrices and U-matrix. Hitherto everything that has been said about coupling constant evolution has been rather speculative arguments except for the general vision that it reduces to a discrete evolution defined by p-adic length scales. General first principle definitions are however much more valuable than ad hoc guesses even if the latter give rise to explicit formulas.

In quantum TGD and also at its QFT limit various correlation functions in given quantum state should code for its properties. By quantum classical correspondence these correlation functions should have counterparts in the geometry of preferred extremals. Even more: these classical counterparts for a given preferred extremal ought to be identical with the quantum correlation functions for the superposition of preferred extremals. This correspondence could be called quantum ergodicity by its analogy with ordinary ergodicity stating that the member of ensemble becomes representative of ensemble.

1. The marvelous implication of quantum ergodicity would be that one could calculate everything solely classically using the classical intuition - the only intuition that we have. Quantum ergodicity would also solve the paradox raised by the quantum classical correspondence for momentum eigenstates. Any preferred extremal in their superposition defining momentum eigenstate should code for the momentum characterizing the superposition itself. This is indeed possible if every extremal in the superposition codes the momentum to the properties of classical correlation functions which are identical for all of them.

2. The only manner to possibly achieve quantum ergodicity is in terms of the statistical properties of the preferred extremals. It should be possible to generalize the ergodic theorem stating that the properties of statistical ensemble are represented by single space-time evolution in the ensemble of time evolutions. Quantum superposition of classical worlds would effectively reduce to single classical world as far as classical correlation functions are considered. The notion of finite measurement resolution suggests that one must state this more precisely by adding that classical correlation functions are calculated in a given UV and IR resolutions meaning UV cutoff defined by the smallest CD and IR cutoff defined by the largest CD present.

3. The skeptic inside me immediately argues that TGD Universe is 4-D spin glass so that this quantum ergodic theorem must be broken. In the case of the ordinary spin classes one has not only statistical average for a fixed Hamiltonian but a statistical average over Hamiltonians. There is a probability distribution over the coupling parameters appearing in the Hamiltonian. Maybe the quantum counterpart of this is needed to predict the physically measurable correlation functions.

   Could this average be an ordinary classical statistical average over quantum states with different classical correlation functions? This kind of average is indeed taken in density matrix formalism. Or could it be that the square root of thermodynamics defined by ZEO actually gives automatically rise to this average? The eigenvalues of the "hermitian square root " of the density matrix would code for components of the state characterized by different classical correlation functions. One could assign these contributions to different "phases".

4. Quantum classical correspondence in statistical sense would be very much like holography (now individual classical state represents the entire quantum state). Quantum ergodicity would pose a rather strong constraint on quantum states. This symmetry principle could actually fix the spectrum of zero energy states to a high degree and fix therefore the M-matrices given by the product of hermitian square root of density matrix and unitary S-matrix and unitary U-matrix having M-matrices as its orthonormal rows.

5. In TGD inspired theory of consciousness the counterpart of quantum ergodicity is the postulate that the space-time geometry provides a symbolic representation for the quantum states and also for the contents of consciousness assignable to quantum jumps between quantum states. Quantum ergodicity would realize this strongly self-referential looking condition. The positive and negative energy parts of zero energy state would be analogous to the initial and final states of quantum jump and the classical correlation functions would code for the contents of consciousness like written formulas code for the thoughts of mathematician and provide a sensory feedback.

How classical correlation functions should be defined?

1. General Coordinate Invariance and Lorentz invariance are the basic constraints on the definition. These are achieved for the space-time regions with Minkowskian signature and 4-D $M^4$ projection if linear Minkowski coordinates are used. This is equivalent with the contraction of the indices of tensor fields with the space-time projections of $M^4$ Killing vector fields representing translations.

Accepting ths generalization, there is no need to restrict oneself to 4-D $M^4$ projection and one can also consider also Euclidian regions identifiable as lines of generalized Feynman diagrams.

Quantum ergodicity very probably however forces to restrict the consideration to Minkowskian and Euclidian space-time regions and various phases associated with them. Also $CP_2$ Killing vector fields can be projected to space-time surface and give a representation for classical gluon fields. These in turn can be contracted with $M^4$ Killing vectors giving rise to gluon fields as analogs of graviton fields but with second polarization index replaced with color index.

2. The standard definition for the correlation functions associated with classical time evolution is the appropriate starting point. The correlation function $G_{XY}(\tau)$ for two dynamical variables $X(t)$ and $Y(t)$ is defined as the average $G_{XY}(\tau) = \int_T X(t)Y(t+\tau)dt/T$ over an interval of length $T$, and one can also consider the limit $T \to \infty$. In the recent case one would replace $\tau$ with the difference $m_1 - m_2 = m$ of $M^4$ coordinates of two points at the preferred extremal and integrate over the points of the extremal to get the average. The finite time interval $T$ is replaced with the volume of causal diamond in a given length scale. Zero energy state with given quantum numbers for positive and negative energy parts of the state defines the initial and final states between which the fields appearing in the correlation functions are defined.

3. What correlation functions should be considered? Certainly one could calculate correlation functions for the induced spinor connection given electro-weak propagators and correlation functions for $CP_2$ Killing vector fields giving correlation functions for gluon fields using the description in terms of Killing vector fields. If one can uniquely separate from the Fourier transform uniquely a term of form $Z/(p^2 - m^2)$ by its momentum dependence, the coefficient $Z$ can be identified as coupling constant squared for the corresponding gauge potential component and one can in principle deduce coupling constant evolution purely classically. One can imagine of calculating spinorial propagators for string world sheets in the same manner. Note that also the dependence on color quantum numbers would be present so that in principle all that is needed could be calculated for a single preferred extremal without the need to construct QFT limit and to introduce color quantum numbers of fermions as spin like quantum numbers (color quantum numbers corresponds to $CP_2$ partial wave for the tip of the CD assigned with the particle).

4. What about Higgs field? TGD in principle allows scalar and pseudo-scalars which could be called Higgs like states. These states are however not necessary for particle massivation although they can represent particle massivation and must do so if one assumes that QFT limit exist. p-Adic thermodynamics however describes particle massivation microscopically.

The problem is that Higgs like field does not seem to have any obvious space-time correlate. The trace of the second fundamental form is the obvious candidate but vanishes for preferred extremals which are both minimal surfaces and solutions of Einstein Maxwell equations with cosmological constant. If the string world sheets at which all spinor components except right handed neutrino are localized for the general solution ansatz of the modified Dirac equation, the corresponding second fundamental form at the level of imbedding space defines a candidate for classical Higgs field. A natural expectation is that string world sheets are minimal surfaces of space-time surface. In general they are however not minimal surfaces of the imbedding space so that one might achieve a microscopic definition of classical Higgs field and its vacuum expectation value as an average of one point correlation function over the string world sheet.

Many detailed speculations about coupling constant evolution to be discussed in the sections below must be taken as innovative guesses doomed to have the eventual fate of guesses. The notion of quantum ergodicity could however be one of the really deep ideas about coupling constant evolution comparable to the notion of p-adic coupling constant evolution. Quantum Ergodicity (briefly QE) would also state something extremely non-trivial also about the construction of correlation functions and S-matrix. Because this principle is so new, the rest of the chapter does not yet contain any applications of QE. This should not lead the reader to under-estimate the potential power of QE.

## 6.8 Appendix: Hamilton-Jacobi structure

In the following the definition of Hamilton-Jacobi structure is discussed in detail.

### 6.8.1 Hermitian and hyper-Hermitian structures

The starting point is the observation that besides the complex numbers forming a number field there are hyper-complex numbers. Imaginary unit $i$ is replaced with $e$ satisfying $e^2 = 1$. One obtains an algebra but not a number field since the norm is Minkowskian norm $x^2 - y^2$, which vanishes at light-cone $x = y$ so that light-like hypercomplex numbers $x \pm e$) do not have inverse. One has "almost" number field.

Hyper-complex numbers appear naturally in 2-D Minkowski space since the solutions of a massless field equation can be written as $f = g(u = t - ex) + h(v = t + ex)$ whith $e^2 = 1$ realized by putting $e = 1$. Therefore Wick rotation relates sums of holomorphic and antiholomorphic functions to sums of hyper-holomorphic and anti-hyper-holomorphic functions. Note that $u$ and $v$ are hyper-complex conjugates of each other.

Complex n-dimensional spaces allow Hermitian structure. This means that the metric has in complex coordinates $(z_1, ...., z_n)$ the form in which the matrix elements of metric are nonvanishing only between $z_i$ and complex conjugate of $z_j$. In 2-D case one obtains just $ds^2 = g_{z\overline{z}}dzd\overline{z}$. Note that in this case metric is conformally flat since line element is proportional to the line element $ds^2 = dzd\overline{z}$ of plane. This form is always possible locally. For complex n-D case one obtains $ds^2 = g_{i\overline{j}}dz^i d\overline{z}^j$. $g_{i\overline{j}} = \overline{g_{j\overline{i}}}$ guaranteing the reality of $ds^2$. In 2-D case this condition gives $g_{z\overline{z}} = \overline{g_{z\overline{z}}}$.

How could one generalize this line element to hyper-complex n-dimensional case. In 2-D case Minkowski space $M^2$ one has $ds^2 = g_{uv}dudv$, $g_{uv} = 1$. The obvious generalization would be the replacement $ds^2 = g_{u_iv_j}du^idv^j$. Also now the analogs of reality conditions must hold with respect to $u_i \leftrightarrow v_i$.

### 6.8.2 Hamilton-Jacobi structure

Consider next the path leading to Hamilton-Jacobi structure.

4-D Minkowski space $M^4 = M^2 \times E^2$ is Cartesian product of hyper-complex $M^2$ with complex plane $E^2$, and one has $ds^2 = dudv + dzd\overline{z}$ in standard Minkowski coordinates. One can also consider more general integrable decompositions of $M^4$ for which the tangent space $TM^4 = M^4$ at each point is decomposed to $M^2(x) \times E^2(x)$. The physical analogy would be a position dependent decomposition of the degrees of freedom of massless particle to longitudinal ones ($M^2(x)$: light-like momentum is in this plane) and transversal ones ($E^2(x)$: polarization vector is in this plane). Cylindrical and spherical variants of Minkowski coordinates define two examples of this kind of coordinates (it is perhaps a good exercize to think what kind of decomposition of tangent space is in question in these examples). An interesting mathematical problem highly relevant for TGD is to identify all possible decompositions of this kind for empty Minkowski space.

The integrability of the decomposition means that the planes $M^2(x)$ are tangent planes for 2-D surfaces of $M^4$ analogous to Euclidian string world sheet. This gives slicing of $M^4$ to Minkowskian string world sheets parametrized by euclidian string world sheets. The question is whether the sheets are stringy in a strong sense: that is minimal surfaces. This is not the case: for spherical coordinates the Euclidian string world sheets would be spheres which are not minimal surfaces. For cylindrical and spherical coordinates hower $M^2(x)$ integrate to plane $M^2$ which is minimal surface.

Integrability means in the case of $M^2(x)$ the existence of light-like vector field $J$ whose flow lines define a global coordinate. Its existence implies also the existence of its conjugate and together these vector fields give rise to $M^2(x)$ at each point. This means that one has $J = \Psi\nabla\Phi$: $\Phi$ indeed defines the global coordinate along flow lines. In the case of $M^2$ either the coordinate $u$ or $v$ would be the coordinate in question. This kind of flows are called Beltrami flows. Obviously the same holds for the transversal planes $E^2$.

One can generalize this metric to the case of general 4-D space with Minkowski signature of metric. At least the elements $g_{uv}$ and $g_{z\overline{z}}$ are non-vanishing and can depend on both $u, v$ and $z, \overline{z}$ . They must satisfy the reality conditions $g_{z\overline{z}} = \overline{g_{z\overline{z}}}$ and $g_{uv} = \overline{g_{vu}}$ where complex conjugation in the argument involves also $u \leftrightarrow v$ besides $z \leftrightarrow \overline{z}$.

The question is whether the components $g_{uz}$, $g_{vz}$, and their complex conjugates are non-vanishing if they satisfy some conditions. They can. The direct generalization from complex 2-D space would be that one treats $u$ and $v$ as complex conjugates and therefore requires a direct generalization of the hermiticity condition

$$g_{uz} = \overline{g_{v\bar{z}}} \ \ , \ \ g_{vz} = \overline{g_{u\bar{z}}} \ \ .$$

This would give complete symmetry with the complex 2-D (4-D in real sense) spaces. This would allow the algebraic continuation of hermitian structures to Hamilton-Jacobi structures by just replacing $i$ with $e$ for some complex coordinates.

# Chapter 7

# Knots and TGD

## 7.1 Introduction

Witten has highly inspiring popular lecture about knots and quantum physics [A29] mentioning also his recent work with knots related to an attempt to understand Khovanov homology. Witten manages to explain in rather comprehensible manner both the construction recipe of Jones polynomial and the idea about how Jones polynomial emerges from topological quantum field theory as a vacuum expectation of so called Wilson loop defined by path integral with weighting coming from Chern-Simons action [A72] . Witten also tells that during the last year he has been working with an attempt to understand in terms of quantum theory the so called Khovanov polynomial associated with a much more abstract link invariant whose interpretation and real understanding remains still open. In particular, he mentions the approach of Gukov, Schwartz, and Vafa [A64, A64] as an attempt to understand Khovanov polynomial.

This kind of talks are extremely inspiring and lead to a series of questions unavoidably culminating to the frustrating "Why I do not have the brain of Witten making perhaps possible to answer these questions?". This one must just accept. In the following I summarize some thoughts inspired by the associations of the talk of Witten with quantum TGD and with the model of DNA as topological quantum computer. In my own childish manner I dare believe that these associations are interesting and dare also hope that some more brainy individual might take them seriously.

An idea inspired by TGD approach which also main streamer might find interesting is that the Jones invariant defined as vacuum expectation for a Wilson loop in 2+1-D space-time generalizes to a vacuum expectation for a collection of Wilson loops in 2+2-D space-time and could define an invariant for 2-D knots and for cobordisms of braids analogous to Jones polynomial. As a matter fact, it turns out that a generalization of gauge field known as gerbe is needed and that in TGD framework classical color gauge fields defined the gauge potentials of this field. Also topological string theory in 4-D space-time could define this kind of invariants. Of course, it might well be that this kind of ideas have been already discussed in literature.

Khovanov homology generalizes the Jones polynomial as knot invariant. The challenge is to find a quantum physical construction of Khovanov homology analous to the topological QFT defined by Chern-Simons action allowing to interpret Jones polynomial as vacuum expectation value of Wilson loop in non-Abelian gauge theory.

Witten's approach to Khovanov homology relies on fivebranes as is natural if one tries to define 2-knot invariants in terms of their cobordisms involving violent un-knottings. Despite the difference in approaches it is very useful to try to find the counterparts of this approach in quantum TGD since this would allow to gain new insights to quantum TGD itself as almost topological QFT identified as symplectic theory for 2-knots, braids and braid cobordisms. This comparison turns out to be extremely useful from TGD point of view.

1. A highly unique identification of string world sheets and therefore also of the braids whose ends carry quantum numbers of many particle states at partonic 2-surfaces emerges if one identifies the string word sheets as singular surfaces in the same manner as is done in Witten's approach. This identification need of course not be correct and later in the article a less ad hoc and unique identification is proposed. The conjectured slicings of preferred extremals by 3-D surfaces and

string world sheets central for quantum TGD can be identified uniquely if the identification is accepted. The slicing by 3-surfaces would be interpreted in gauge theory in terms of Higgs= constant surfaces with radial coordinate of $CP_2$ playing the role of Higgs. The slicing by string world sheets would be induced by different choices of $U(2)$ subgroup of $SU(3)$ leaving Higgs=constant surfaces invariant.

2. Also a physical interpretation of the operators Q, F, and P of Khovanov homology emerges. P would correspond to instanton number and F to the fermion number assignable to right handed neutrinos. The breaking of $M^4$ chiral invariance makes possible to realize Q physically. The finding that the generalizations of Wilson loops can be identified in terms of the gerbe fluxes $\int H_A J$ supports the conjecture that TGD as almost topological QFT corresponds essentially to a symplectic theory for braids and 2-knots.

The basic challenge of quantum TGD is to give a precise content to the notion of generalization Feynman diagram and the reduction to braids of some kind is very attractive possibility inspired by zero energy ontology. The point is that no $n > 2$-vertices at the level of braid strands are needed if bosonic emergence holds true.

1. For this purpose the notion of algebraic knot is introduced and the possibility that it could be applied to generalized Feynman diagrams is discussed. The algebraic structrures kei, quandle, rack, and biquandle and their algebraic modifications as such are not enough. The lines of Feynman graphs are replaced by braids and in vertices braid strands redistribute. This poses several challenges: the crossing associated with braiding and crossing occurring in non-planar Feynman diagrams should be integrated to a more general notion; braids are replaced with sub-manifold braids; braids of braids ....of braids are possible; the redistribution of braid strands in vertices should be algebraized. In the following I try to abstract the basic operations which should be algebraized in the case of generalized Feynman diagrams.

2. One should be also able to concretely identify braids and 2-braids (string world sheets) as well as partonic 2-surfaces and I have discussed several identifications during last years. Legendrian braids turn out to be very natural candidates for braids and their duals for the partonic 2-surfaces. String world sheets in turn could correspond to the analogs of Lagrangian sub-manifolds or two minimal surfaces of space-time surface satisfying the weak form of electric-magnetic duality. The latter opion turns out to be more plausible. This identification - if correct - would solve quantum TGD explicitly at string world sheet level which corresponds to finite measurement resolution.

3. Also a brief summary of generalized Feynman rules in zero energy ontology is proposed. This requires the identification of vertices, propagators, and prescription for integrating over al 3-surfaces. It turns out that the basic building blocks of generalized Feynman diagrams are well-defined.

## 7.2 Some TGD background

What makes quantum TGD [L4, L5, L8, L9, L6, L3, L7, L10] interesting concerning the description of braids and braid cobordisms is that braids and braid cobordisms emerge both at the level of generalized Feynman diagrams and in the model of DNA as a topological quantum computer [K17] .

### 7.2.1 Time-like and space-like braidings for generalized Feynman diagrams

1. In TGD framework space-times are 4-D surfaces in 8-D imbedding space. Basic objects are partonic 2-surfaces at the two ends of causal diamonds CD (intersections of future and past directed light-cones of 4-D Minkowski space with each point replaced with $CP_2$ ). The light-like orbits of partonic 2-surfaces define 3-D light-like 3-surfaces identifiable as lines of generalized Feynman diagrams. At the vertices of generalized Feynman diagrams incoming and outgoing light-like 3-surfaces meet. These diagrams are not direct generalizations of string diagrams since they are singular as 4-D manifolds just like the ordinary Feynman diagrams.

By strong form of holography one can assign to the partonic 2-surfaces and their tangent space data space-time surfaces as preferred exrtremals of Kähler action. This guarantees also general coordinate invariance and allows to interpret the extremals as generalized Bohr orbits.

2. One can assign to the partonic 2-surfaces discrete sets of points carrying quantum numbers. As a matter fact, these sets of points seem to emerge from the solutions of the Chern-Simons Dirac equation rather naturally. These points define braid strands as the partonic 2-surface moves and defines a light-like 3-surface as its orbit as a surface of 4-D space-time surface. In the generic case the strands get tangled in time direction and one has linking and knotting giving rise to a time-like braiding.

3. Also space-like braidings are possible. One can imagine that the partonic 2-surfaces are connected by space-like curves defining TGD counterparts for strings and that in the initial state these curves define space-like braids whose ends belong to different partonic 2-surfaces. Quite generally, the basic conjecture is that the preferred extremals define orbits of string-like objects with their ends at the partonic 2-surfaces. One would have slicing of space-time surfaces by string world sheets one one hand and by partonic 2-surface on one hand. This string model is very special due to the fact that the string orbits define what could be called braid cobordisms representing which could represent unknotting of braids. String orbits in higher dimensional space-times do not allow this topological interpretation.

### 7.2.2 Dance metaphor

Time like braidings induces space-like braidings and one can speak of time-like or dynamical braiding and even duality of time-like and space-like braiding. What happens can be understood in terms of dance metaphor.

1. One can imagine that the points carrying quantum numbers are like dancers at parquettes defined by partonic 2-surfaces. These parquettes are somewhat special in that it is moving and changing its shape.

2. Space-like braidings means that the feet of the dancers at different parquettes are connected by threads. As the dance continues, the threads connecting the feet of different dancers at different parquettes get tangled so that the dance is coded to the braiding of the threads. Time-like braiding induce space-like braiding. One has what might be called a cobordism for space-like braiding transforming it to a new one.

### 7.2.3 DNA as topological quantum computer

The model for topological quantum computation is based on the idea that time-like braidings defining topological quantum computer programs. These programs are robust since the topology of braiding is not affected by small deformations.

1. The first key idea in the model of DNA as topological quantum computer is based on the observation that the lipids of cell membrane form a 2-D liquid whose flow defines the dance in which dancers are lipids which define a flow pattern defining a topological quantum computation. Lipid layers assignable to cellular and nuclear membranes are the parquettes. This 2-D flow pattern can be induced by the liquid flow near the cell membrane or in case of nerve pulse transmission by the nerve pulses flowing along the axon. This alone defines topological quantum computation.

2. In DNA as topological quantum computer model one however makes a stronger assumption motivated by the vision that DNA is the brain of cell and that information must be communicated to DNA level wherefrom it is communicated to what I call magnetic body. It is assumed that the lipids of the cell membrane are connected to DNA nucleotides by magnetic flux tubes defining a space-like braiding. It is also possible to connect lipids of cell membrane to the lipids of other cell membranes, to the tubulins at the surfaces of microtubules, and also to the aminoadics of proteins. The spectrum of possibilities is really wide.

The space-like braid strands would correspond to magnetic flux tubes connecting DNA nucleotides to lipids of nuclear or cell membrane. The running of the topological quantum computer program defined by the time-like braiding induced by the lipid flow would be coded to a space-like braiding of the magnetic flux tubes. The braiding of the flux tubes would define a universal memory storage mechanism and combined with 4-D view about memory provides a very simple view about how memories are stored and how they are recalled.

## 7.3 Could braid cobordisms define more general braid invariants?

Witten says that one should somehow generalize the notion of knot invariant. The above described framework indeed suggests a very natural generalization of braid invariants to those of braid cobordisms reducing to braid invariants when the braid at the other end is trivial. This description is especially natural in TGD but allows a generalization in which Wilson loops in 4-D sense describe invariants of braid cobordisms.

### 7.3.1 Difference between knotting and linking

Before my modest proposal of a more general invariant some comments about knotting and linking are in order.

1. One must distinguish between internal knotting of each braid strand and linking of 2 strands. They look the same in the 3-D case but in higher dimensions knotting and linking are not the same thing. Codimension 2 surfaces get knotted in the generic case, in particular the 2-D orbits of the braid strands can get knotted so that this gives additional topological flavor to the theory of strings in 4-D space-time. Linking occurs for two surfaces whose dimension $d_1$ and $d_2$ satisfying $d_1 + d_2 = D - 1$, where $D$ is the dimension of the imbedding space.

2. 2-D orbits of strings do not link in 4-D space-time but do something more radical since the sum of their dimensions is $D = 4$ rather than only $D - 1 = 3$. They intersect and it is impossible to eliminate the intersection without a change of topology of the stringy 2-surfaces: a hole is generated in either string world sheet. With a slight deformation intersection can be made to occur generically at discrete points.

### 7.3.2 Topological strings in 4-D space-time define knot cobordisms

What makes the 4-D braid cobordisms interesting is following.

1. The opening of knot by using brute force by forcing the strands to go through each other induces this kind of intersection point for the corresponding 2-surfaces. From 3-D perspective this looks like a temporary cutting of second string, drawing the string ends to some distance and bringing them back and gluing together as one approaches the moment when the strings would go through each other. This surgical operation for either string produces a pair of non-intersecting 2-surfaces with the price that the second string world sheet becomes topologically non-trivial carrying a hole in the region were intersection would occur. This operation relates a given crossing of braid strands to its dual crossing in the construction of Jones polynomial in given step (string 1 above string 2 is transformed to string 2 above string 1).

2. One can also cut both strings temporarily and glue them back together in such a manner that end a/b of string 1 is glued to the end c/d of string 2. This gives two possibilities corresponding to two kinds of reconnections. Reconnections appears as the second operation in the construction of Jones invariant besides the operation putting the string above the second one below it or vice versa. Jones polynomial relates in a simple manner to Kauffman bracket allowing a recursive construction. At a given step a crossing is replaced with a weighted sum of the two reconnected terms [A2, A12] . Reconnection represents the analog of trouser vertex for closed strings replaced with braid strands.

3. These observations suggest that stringy diagrams describe the braid cobordisms and a kind of topological open string model in 4-D space-time could be used to construct invariants of braid cobordisms. The dynamics of strand ends at the partonic 2-surfaces would partially induce the dynamics of the space-like braiding. This dynamics need not induce the un-knotting of space-like braids and simple string diagrams for open strings are enough to define a cobordism leading to un-knotting. The holes needed to realize the crossover for braid strands would contribute to the Wilson loop an additional factor corresponding to the rotation of the gauge potential around the boundary of the hole (non-integrable phase factor). In abelian case this gives simple commuting phase factor.

Note that braids are actually much more closer to the real world than knots since a useful strand of knotted structure must end somewhere. The abstract closed loops of mathematician floating in empty space are not very useful in real life albeit mathematically very convenient as Witten notices. Also the braid cobordisms with ends of a collection of space-like braids at the ends of causal diamond are more practical than 2-knots in 4-D space. Mathematician would see these objects as analogous to surfaces in relative homology allowed to have boundaries if they located at fixed sub-manifolds. Homology for curves with ends fixed to be on some surfaces is a good example of this. Now these fixed sub-manifolds would correspond to space-like 3-surfaces at the ends CDs and light-like wormhole throats at which the signature of the induced metric changes and which are carriers of elementary particle quantum numbers.

## 7.4 Invariants 2-knots as vacuum expectations of Wilson loops in 4-D space-time?

The interpretation of string world sheets in terms of Wilson loops in 4-dimensional space-time is very natural. This raises the question whether Witten's a original identification of the Jones polynomial as vacuum expectation for a Wilson loop in 2+1-D space might be replaced with a vacuum expectation for a collection of Wilson loops in 3+1-D space-time and would characterize in the general case (multi-)braid cobordism rather than braid. If the braid at the lower or upped boundary is trivial, braid invariant is obtained. The intersections of the Wilson loops would correspond to the violent un-knotting operations and the boundaries of the resulting holes give an additional Wilson loop. An alternative interpretation would be as the analog of Jones polynomial for 2-D knots in 4-D space-time generalizing Witten's theory. This description looks completely general and does not require TGD at all.

The following considerations suggest that Wilson loops are not enough for the description of general 2-knots and that that Wilson loops must be replaced with 2-D fluxes. This requires a generalization of gauge field concept so that it corresponds to a 3-form instead of 2-form is needed. In TGD framework this kind of generalized gauge fields exist and their gauge potentials correspond to classical color gauge fields.

### 7.4.1 What 2-knottedness means concretely?

It is easy to imagine what ordinary knottedness means. One has circle imbedded in 3-space. One projects it in some plane and looks for crossings. If there are no crossings one knows that un-knot is in question. One can modify a given crossing by forcing the strands to go through each other and this either generates or removes knottedness. One can also destroy crossing by reconnection and this always reduces knottedness. Since knotting reduces to linking in 3-D case, one can find a simple interpretation for knottedness in terms of linking of two circles. For 2-knots linking is not what gives rise to knotting.

One might hope to find something similar in the case of 2-knots. Can one imagine some simple local operations which either increase of reduce 2-knottedness?

1. To proceed let us consider as simple situation as possible. Put sphere in 3-D time= constant section $E^3$ of 4-space. Add a another sphere to the same section $E^3$ such that the corresponding balls do not intersect. How could one build from these two spheres a knotted 2-sphere?

2. From two spheres one can build a single sphere in topological sense by connecting them with a small cylindrical tube connecting the boundaries of disks (circles) removed from the two spheres. If this is done in $E^3$, a trivial 2-knot results. One can however do the gluing of the cylinder in a more exotic manner by going temporarily to "hyper-space", in other words making a time travel. Let the cylinder leave the second sphere from the outer surface, let it go to future or past and return back to recent but through the interior. This is a good candidate for a knotted sphere since the attempts to deform it to self-non-intersecting sphere in $E^3$ are expected to fail since the cylinder starting from interior necessarily goes through the surface of sphere if wants to the exterior of the sphere.

3. One has actually $2\times 2$ manners to perform the connected sum of 2-spheres depending on whether the cylinders leave the spheres through exterior or interior. At least one of them (exterior-exterior) gives an un-knotted sphere and intuition suggests that all the three remaining options requiring getting out from the interior of sphere give a knotted 2-sphere. One can add to the resulting knotted sphere new spheres in the same manner and obtain an infinite number of them. As a matter fact, the proposed 1+3 possibilities correspond to different versions of connected sum and one could speak of knotting and non-knotting connected sums. If the addition of knotted spheres is performed by non-knotting connected sum, one obtains composites of already existing 2-knots. Connected sum composition is analogous to the composition of integer to a product of primes. One indeed speaks of prime knots and the number of prime knots is infinite. Of course, it is far from clear whether the connected sum operation is enough to build all knots. For instance it might well be that cobordisms of 1-braids produces knots not producible in this manner. In particular, the effects of time-like braiding induce braiding of space-like strands and this looks totally different from local knotting.

### 7.4.2 Are all possible 2-knots possible for stringy world sheets?

Whether all possible 2-knots are allowed for stringy world sheets, is not clear. In particular, if they are dynamically determined it might happen that many possibilities are not realized. For instance, the condition that the signature of the induced metric is Minkowskian could be an effective killer of 2-knottedness not reducing to braid cobordism.

1. One must start from string world sheets with Minkowskian signature of the induced metric. In other words, in the previous construction one must $E^3$ with 3-dimensional Minkowski space $M^3$ with metric signature 1+2 containing the spheres used in the construction. Time travel is replaced with a travel in space-like hyper dimension. This is not a problem as such. The spheres however have at least one two special points corresponding to extrema at which the time coordinate has a local minimum or maximum. At these points the induced metric is necesssarily degenerate meaning that its determinant vanishes. If one allows this kind of singular points one can have elementary knotted spheres. This liberal attitude is encouraged by the fact that the light-like 3-surfaces defining the basic dynamical objects of quantum TGD correspond to surfaces at which 4-D induced metric is degenerate. Otherwise 2-knotting reduces to that induced by cobordisms of 1-braids. If one allows only the 2-knots assignable to the slicings of the space-time surface by string world sheets and even restricts the consideration to those suggested by the duality of 2-D generalization of Wilson loops for string world sheets and partonic 2-surfaces, it could happen that the string world sheets reduce to braidings.

2. The time=constant intersections define a representation of 2-knots as a continuous sequence of 1-braids. For critical times the character of the 1-braids changes. In the case of braiding this corresponds to the basic operations for 1-knots having interpretation as string diagrams (reconnection and analog of trouser vertex). The possibility of genuine 2-knottedness brings in also the possibility that strings pop up from vacuum as points, expand to closed strings, are fused to stringy words sheet temporarily by the analog of trouser vertex, and eventually return to the vacuum. Essentially trouser diagram but second string open and second string closed and beginning from vacuum and ending to it is in question. Vacuum bubble interacting with open string would be in question. The believer in string model might be eager to accept this picture but one must be cautious.

### 7.4.3 Are Wilson loops enough for 2-knots?

Suppose that the space-like braid strands connecting partonic 2-surfaces at given boundary of $CD$ and light-like braids connecting partonic 2-surfaces belonging to opposite boundaries of $CD$ form connected closed strands. The collection of closed loops can be identified as boundaries of Wilson loops and the expectation value is defined as the product of traces assignable to the loops. The definition is exactly the same as in 2+1-D case [A72] .

Is this generalization of Wilson loops enough to describe 2-knots? In the spirit of the proposed philosophy one could ask whether there exist two-knots not reducible to cobordisms of 1-knots whose knot invariants require cobordisms of 2-knots and therefore 2-braids in 5-D space-time. Could it be that dimension $D = 4$ is somehow very special so that there is no need to go to $D = 5$? This might be the case since for ordinary knots Jones polynomial is very faithful invariant.

Innocent novice could try to answer the question in the following manner. Let us study what happens locally as the 2-D closed surface in 4-D space gets knotted.

1. In 1-D case knotting reduces to linking and means that the first homotopy group of the knot complement is changed so that the imbedding of first circle implies that the there exists imbedding of the second circle that cannot be transformed to each other without cutting the first circle temporarily. This phenomenon occurs also for single circle as the connected sum operation for two linked circles producing single knotted circle demonstrates.

2. In 2-D case the complement of knotted 2-sphere has a non-trivial second homotopy group so that 2-balls have homotopically non- equivalent imbeddings, which cannot be transformed to each other without intersection of the 2-balls taking place during the process. Therefore the description of 2-knotting in the proposed manner would require cobordisms of 2-knots and thus 5-D space-time surfaces. However, since 3-D description for ordinary knots works so well, one could hope that the generalization the notion of Wilson loop could allow to avoid 5-D description altogether. The generalized Wilson loops would be assigned to 2-D surfaces and gauge potential $A$ would be replaced with 2-gauge potential $B$ defining a three-form $F = dB$ as the analog of gauge field.

3. This generalization of bundle structure known as gerbe structure has been introduced in algebraic geometry [A8, A49] and studied also in theoretical physics [A58] . 3-forms appear as analogs of gauge fields also in the QFT limit of string model. Algebraic geometer would see gerbe as a generalization of bundle structure in which gauge group is replaced with a gauge groupoid. Essentially a structure of structures seems to be in question. For instance, the principal bundles with given structure group for given space defines a gerbe. In the recent case the space of gauge fields in space-time could be seen as a gerbe. Gerbes have been also assigned to loop spaces and WCW can be seen as a generalization of loop space. Lie groups define a much more mundane example about gerbe. The 3-form $F$ is given by $F(X, Y, Z) = B(X, [Y, Z])$ , where $B$ is Killing form and for $U(n)$ reduces to $(g^{-1}dg)^3$. It will be found that classical color gauge fields define gerbe gauge potentials in TGD framework in a natural manner.

## 7.5 TGD inspired theory of braid cobordisms and 2-knots

In the sequel the considerations are restricted to TGD and to a comparison of Witten's ideas with those emerging in TGD framework.

### 7.5.1 Weak form of electric-magnetic duality and duality of space-like and time-like braidings

Witten notices that much of his work in physics relies on the assumption that magnetic charges exist and that rather frustratingly, cosmic inflation implies that all traces of them disappear. In TGD Universe the non-trivial topology of $CP_2$ makes possible Kähler magnetic charge and inflation is replaced with quantum criticality. The recent view about elementary particles is that they correspond to string like objects with length of order electro-weak scale with Kähler magnetically charged wormhole throats at their ends. Therefore magnetic charges would be there and LHC might be able to detect their signatures if LHC would get the idea of trying to do this.

Witten mentions also electric-magnetic duality. If I understood correctly, Witten believes that it might provide interesting new insights to the knot invariants. In TGD framework one speaks about weak form of elecric magnetic duality. This duality states that Kähler electric fluxes at space-like ends of the space-time sheets inside CDs and at wormhole throats are proportional to Kähler magneic fluxes so that the quantization of Kähler electric charge quantization reduces to purely homological quantization of Kähler magnetic charge.

The weak form of electric-magnetic duality fixes the boundary conditions of field equations at the light-like and space-like 3-surfaces. Together with the conjecture that the Kähler current is proportional to the corresponding instanton current this implies that Kähler action for the preferred extremal sof Kähler action reduces to 3-D Chern-Simons term so that TGD reduces to almost topological QFT. This means an enormous mathematical simplification of the theory and gives hopes about the solvability of the theory. Since knot invariants are defined in terms of Abelian Chern-Simons action for induced Kähler gauge potential, one might hope that TGD could as a by-product define invariants of braid cobordisms in terms of the unitary U-matrix of the theory between zero energy states and having as its rows the non-unitary M-matrices analogous to thermal S-matrices.

Electric magnetic duality is 4-D phenomenon as is also the duality between space-like and time like braidings essential also for the model of topological quantum computation. Also this suggests that some kind of topological string theory for the space-time sheets inside CDs could allow to define the braid cobordism invariants.

### 7.5.2 Could Kähler magnetic fluxes define invariants of braid cobordisms?

Can one imagine of defining knot invariants or more generally, invariants of knot cobordism in this framework? As a matter fact, also Jones polynomial describes the process of unknotting and the replacement of unknotting with a general cobordism would define a more general invariant. Whether the Khovanov invariants might be understood in this more general framework is an interesting question.

1. One can assign to the 2-dimensional stringy surfaces defined by the orbits of space-like braid strands Kähler magnetic fluxes as flux integrals over these surfaces and these integrals depend only on the end points of the space-like strands so that one deform the space-like strands in an arbitrarily manner. One can in fact assign these kind of invariants to pairs of knots and these invariants define the dancing operation transforming these knots to each other. In the special case that the second knot is un-knot one obtains a knot-invariant (or link- or braid- invariant).

2. The objection is that these invariants depend on the orbits of the end points of the space-like braid strands. Does this mean that one should perform an averaging over the ends with the condition that space-like braid is not affected topologically by the allowed deformations for the positions of the end points?

3. Under what conditions on deformation the magnetic fluxes are not affect in the deformation of the braid strands at 3-D surfaces? The change of the Kähler magnetic flux is magnetic flux over the closed 2-surface defined by initial non-deformed and deformed stringy two-surfaces minus flux over the 2-surfaces defined by the original time-like and space-like braid strands connected by a thin 2-surface to their small deformations. This is the case if the deformation corresponds to a U(1) gauge transformation for a Kähler flux. That is diffeomorphism of $M^4$ and symplectic transformation of $CP_2$ inducing the U(1) gauge transformation.

   Hence a natural equivalence for braids is defined by these transformations. This is quite not a topological equivalence but quite a general one. Symplectic transformations of $CP_2$ at light-like and space-like 3-surfaces define isometries of the world of classical worlds so that also in this sense the equivalence is natural. Note that the deformations of space-time surfaces correspond to this kind of transformations only at space-like 3-surfaces at the ends of CDs and at the light-like wormhole throats where the signature of the induced metric changes. In fact, in quantum TGD the sub-spaces of world of classical worlds with constant values of zero modes (non-quantum fluctuating degrees of freedom) correspond to orbits of 3-surfaces under symplectic transformations so that the symplectic restriction looks rather natural also from the point of view of quantum dynamics and the vacuum expectation defined by Kähler function be defined for physical states.

4. A further possibility is that the light-like and space-like 3-surfaces carry vanishing induced Kähler fields and represent surfaces in $M^4 \times Y^2$, where $Y^2$ is Lagrangian sub-manifold of $CP_2$ carrying vanishing Kähler form. The interior of space-time surface could in principle carry a non-vanishing Kähler form. In this case weak form of self-duality cannot hold true. This however implies that the Kähler magnetic fluxes vanish identically as circulations of Kähler gauge potential. The non-integrable phase factors defined by electroweak gauge potentials would however define non-trivial classical Wilson loops. Also electromagnetic field would do so. It would be therefore possible to imagine vacuum expectation value of Wilson loop for given quantum state. Exponent of Kähler action would define for non-vacuum extremals the weighting. For 4-D vacuum extremals this exponent is trivial and one might imagine of using imaginary exponent of electroweak Chern-Simons action. Whether the restriction to vacuum extremals in the definition of vacuum expectations of electroweak Wilson loops could define general enough invariants for braid cobordisms remains an open question.

5. The quantum expectation values for Wilson loops are non-Abelian generalizations of exponentials for the expectation values of Kähler magnetic fluxes. The classical color field is proportional to the induced Kähler form and its holonomy is Abelian which raises the question whether the non-Abelian Wilson loops for classical color gauge field could be expressible in terms of Kähler magnetic fluxes.

### 7.5.3 Classical color gauge fields and their generalizations define gerbe gauge potentials allowing to replace Wilson loops with Wilson sheets

As already noticed, the description of 2-knots seems to necessitate the generalization of gauge field to 3-form and the introduction of a gerbe structure. This seems to be possible in TGD framework.

1. Classical color gauge fields are proportional to the products $B_A = H_A J$ of the Hamiltonians of color isometries and of Kähler form and the closed 3-form $F_A = dB_A = dH_A \wedge J$ could serve as a colored 3-form defining the analog of U(1) gauge field. What would be interesting that color would make F non-vanishing. The "circulation" $h_A = \oint H_A J$ over a closed partonic 2-surface transforms covariantly under symplectic transformations of $CP_2$, whose Hamiltonians can be assigned to irreps of SU(3): just the commutator of Hamiltonians defined by Poisson bracket appears in the infinitesimal transformation. One could hope that the expectation values for the exponents of the fluxes of $B_A$ over 2-knots could define the covariants able to catch 2-knotted-ness in TGD framework. The exponent defining Wilson loop would be replaced with $exp(iQ^A h_A)$, where $Q^A$ denote color charges acting as operators on particles involved.

2. Since the symplectic group acting on partonic 2-surfaces at the boundary of $CD$ replaces color group as a gauge group in TGD, one can ask whether symplectic SU(3) should be actually replaced with the entire symplectic group of $\cup_{\pm}\delta M^4_{\pm} \times CP_2$ with Hamiltonians carrying both spin and color quantum numbers. The symplectic fluxes $\oint H_A J$ are indeed used in the construction of both quantum states and of WCW geometry. This generalization is indeed possible for the gauge potentials $B_A J$ so that one would have infinite number of classical gauge fields having also interpretation as gerbe gauge potentials.

3. The objection is that symplectic transformations are not symmetries of Kähler action. Therefore the action of symplectic transformation induced on the space-time surface reduces to a symplectic transformation only at the partonic 2-surfaces. This spoils the covariant transformation law for the 2-fluxes over stringy world sheets unless there exist preferred stringy world sheets for which the action is covariant. The proposed duality between the descriptions based on partonic 2-surfaces and stringy world sheets realized in terms of slicings of space-time surface by string world sheets and partonic 2-surfaces suggests that this might be the case.

   This would mean that one can attach to a given partonic 2-surface a unique collection string world sheets. The duality suggests even stronger condition stating that the total exponents $exp(iQ^A h_A)$ of fluxes are the same irrespective whether $h_A$ evaluated for partonic 2-surfaces or for string world sheets defining the analog of 2-knot. This would mean an immense calculational simplification! This duality would correspond very closely to the weak form of electric magnetic duality whose various forms I have pondered as a must for the geometry of WCW. Partonic 2-surfaces indeed correspond to magnetic monopoles at least for elementary particles and stringy world sheets to surfaces carrying electric flux (note that in the exponent magnetic charges do not make themselves visible so that the identity can make sense also for $H_A = 1$).

4. Quantum expectation means in TGD framework a functional integral over the symplectic orbits of partonic 2-surfaces plus 4-D tangent space data assigned to the upper and lower boundaries of $CD$. Suppose that holography fixes the space-like 3-surfaces at the ends of $CD$ and light-like orbits of partonic 2-surfaces. In completely general case the braids and the stringy space-time sheets could be fixed using a representation in terms of space-time coordinates so that the representation would be always the same but the imbedding varies as also the values of the exponent of Kähler function, of the Wilson loop, and of its 2-D generalization. The functional integral over symplectic transforms of 3-surfaces implies that Wilson loop and its 2-D generalization varies.

   The proposed duality however suggests that both Wilson loop and its 2-D generalization are actually fixed by the dynamics. One can ask whether the presence of 2-D analog of Wilson loop has a direct physical meaning bringing into almost topological stringy dynamics associated with color quantum numbers and coding explicit information about space-time interior and topology of field lines so that color dynamics would also have interpretation as a theory of 2-knots. If the proposed duality suggested by holography holds true, only the data at partonic 2-surfaces would be needed to calculate the generalized Wilson loops.

This picture is very speculative and sounds too good to be true but follows if one consistently applies holography.

### 7.5.4 Summing sup the basic ideas

Let us summarize the ideas discussed above.

1. Instead of knots, links, and braids one could study knot and link cobordisms, that is their dynamical evolutions concretizable in terms of dance metaphor and in terms of interacting string world sheets. Each space-like braid strand can have purely internal knotting and braid strands can be linked. TGD could allow to identify uniquely both space-like and time-like braid strands and thus also the stringy world sheets more or less uniquely and it could be that the dynamics induces automatically the temporary cutting of braid strands when knot is opened violently so that a hole is generated. Gerbe gauge potentials defined by classical color gauge fields could make also possible to characterize 2-knottedness in symplectic invariant manner in terms of color gauge fluxes over 2-surfaces.

   The weak form of electric-magnetic duality would reduce the situation to almost topological QFT in general case with topological invariance replaced with symplectic one which corresponds to the fixing of the values of non-quantum fluctuating zero modes in quantum TGD. In the vacuum sector it would be possible to have the counterparts of Wilson loops weighted by 3-D electroweak Chern-Simons action defined bythe induced spinor connection.

2. One could also leave TGD framework and define invariants of braid cobordisms and 2-D analogs of braids as vacuum expectations of Wilson loops using Chern-Simons action assigned to 3-surfaces at which space-like and time-like braid strands end. The presence of light-like and space-like 3-surfaces assignable to causal diamonds could be assumed also now.

I checked whether the article of Gukov, Scwhartz, and Vafa entitled "Khovanov-Rozansky Homology and Topological Strings" [A64, A64] relies on the primitive topological observations made above. This does not seem to be the case. The topological strings in this case are strings in 6-D space rather than 4-D space-time.

What is interesting that twistorial considerations lead to a conjecture that 4-D space-time surfaces in 8-D imbedding space have a dual description in terms of certain 6-D homomorphic surfaces which are sphere bundles in 12-D $CP_3 \times CP_3$ and effectively 4-D. This suggests a connection between descriptions based on topological strings in 6-D space and Wilson loops in 4-D space-time. Could it really be that these completely trivial observations are not a standard part of knot theory?

There is also an article by Dror Bar-Natan with title "Khovanov's homology for tangles and cobordisms" [A33] . The article states that the Khovanov homology theory for knots and links generalizes to tangles, cobordisms and 2-knots. The article does not say anything explicit about Wilson loops but talks about topological QFTs.

An article of Witten about his physical approach to Khovanov homology has appeared in arXiv [A73] . The article is more or less abracadabra for anyone not working with M-theory but the basic idea is simple. Witten reformulates 3-D Chern-Simons theory as a path integral for $\mathcal{N} = 4$ SYM in the 4-D half space $W\times;R$. This allows him to use dualities and bring in the machinery of M-theory and 6-branes. The basic structure of TGD forces a highly analogous appproach: replace 3-surfaces with 4-surfaces, consider knot cobordisms and also 2-knots, introduce gerbes, and be happy with symplectic instead of topological QFT, which might more or less be synonymous with TGD as almost topological QFT. Symplectic QFT would obviously make possible much more refined description of knots.

## 7.6 Witten's approach to Khovanov homology from TGD point of view

Witten's approach to Khovanov comohology [A73] relies on fivebranes as is natural if one tries to define 2-knot invariants in terms of their cobordisms involving violent un-knottings. Despite the difference in approaches it is very useful to try to find the counterparts of this approach in quantum TGD since this would allow to gain new insights to quantum TGD itself as almost topological QFT identified as symplectic theory for 2-knots, braids and braid cobordisms.

An essentially unique identification of string world sheets and therefore also of the braids whose ends carry quantum numbers of many particle states at partonic 2-surfaces emerges if one identifies the string word sheets as singular surfaces in the same manner as is done in Witten's approach [A73] .

Also a physical interpretation of the operators $Q$, $F$, and $P$ of Khovanov homology emerges. $P$ would correspond to instanton number and $F$ to the fermion number assignable to right handed neutrinos. The breaking of $M^4$ chiral invariance makes possible to realize $Q$ physically. The finding that the generalizations of Wilson loops can be identified in terms of the gerbe fluxes $\int H_A J$ supports the conjecture that TGD as almost topological QFT corresponds essentially to a symplectic theory for braids and 2-knots.

### 7.6.1 Intersection form and space-time topology

The violent unknotting corresponds to a sequence of steps in which braid or knot becomes trivial and this very process defines braid invariants in TGD approach in nice concordance with the basic recipe for the construction of Jones polynomial. The topological invariant characterizing this process as a dynamics of 2-D string like objects defined by braid strands becomes knot invariant or more generally, invariant depending on the initial and final braids.

The process is describable in terms of string interaction vertices and also involves crossings of braid strands identifiable as self-intersections of the string world sheet. Hence the intersection form for the 2-surfaces defining braid strand orbits becomes a braid invariant. This intersection form is also a central invariant of 4-D manifolds and Donaldson's theorem [A6] says that for this invariant characterizes simply connected smooth 4-manifold completely. Rank, signature, and parity of this form in the basis defined by the generators of 2-homology (excluding torsion elements) characterize smooth closed and orientable 4-manifold. It is possible to diagonalize this form for smoothable 4-surfaces. Although the situation in the recent case differs from that in Donaldson theory in that the 4-surfaces have boundary and even fail to be manifolds, there are reasons to believe that the theory of braid cobordisms and 2-knots becomes part of the theory of topological invariants of 4-surfaces just as knot theory becomes part of the theory of 3-manifolds. The representation of 4-manifolds as space-time surfaces might also bring in physical insights.

This picture leads to ideas about string theory in 4-D space-time as a topological QFT. The string world sheets define the generators of second relative homology group. "Relative" means that closed surfaces are replaced with surfaces with boundaries at wormhole throats and ends of $CD$. These string world sheets, if one can fix them uniquely, would define a natural basis for homology group defining the intersection form in terms of violent unbraiding operations (note that also reconnections are involved).

Quantum classical correspondence encourages to ask whether also physical states must be restricted in such a manner that only this minimum number of strings carrying quantum numbers at their ends ending to wormhole throats should be allowed. One might hope that there exists a unique identification of the topological strings implying the same for braids and allowing to identify various symplectic invariants as Hamiltonian fluxes for the string world sheets.

### 7.6.2 Framing anomaly

In 3-D approach to knot theory the framing of links and knots represents an unavoidable technical problem [A73] . Framing means a slight shift of the link so that one can define self-linking number as a linking number for the link and its shift. The problem is that this framing of the link - or trivialization of its normal bundle in more technical terms- is not topological invariant and one obtains a large number of framings. For links in $S^3$ the framing giving vanishing self-linking number is the unique option and Atyiah has shown that also in more general case it is possible to identify a unique framing.

For 2-D surfaces self-linking is replaced with self-intersection. This is well-defined notion even without framing and indeed a key invariant. One might hope that framing is not needed also for string world sheets. If needed, this framing would induce the framing at the space-like and light-like 3-surfaces. The restriction of the section of the normal bundle of string world sheet to the 3-surfaces must lie in the tangent space of 3-surfaces. It is not clear whether this is enough to resolve the non-uniqueness problem.

### 7.6.3 Khovanov homology briefly

Khovanov homology involves three charges $Q$, $F$, and $P$. $Q$ is analogous to super charge and satisfies $Q^2 = 0$ for the elements of homology. The basic commutation relations between the charges are $[F,Q] = Q$ and $[P,Q] = 0$. One can show that the Khovanov homology $\kappa(L)$ for link can be expressed as a bi-graded direct sum of the eigen-spaces $V_{m,n}$ of $F$ and $P$, which have integer valued spectra. Obviously $Q$ increases the eigenvalue of $F$ and maps $V_{m,n}$ to $V_{m+1,n}$ just as exterior derivative in de-Rham comology increases the degree of differential form. $P$ acts as a symmetry allowing to label the elements of the homology by an integer valued charge $n$.

Jones polynomial can be expressed as an index assignable to Khovanov homology:

$$\mathcal{J}(q|L) = Tr((-1)^F q^P \quad . \tag{7.6.1}$$

Here $q$ defining the argument of Jones polynomial is root of unity in Chern-Simons theory but can be extended to complex numbers by extending the positive integer valued Chern-Simons coupling $k$ to a complex number. The coefficients of the resulting Laurent polynomial are integers: this result does not follow from Chern-Simons approach alone. Jones polynomial depends on the spectrum of $F$ only modulo 2 so that a lot of information is lost as the homology is replaced with the polynomial.

Both the need to have a more detailed characterization of links and the need to understand why the Wilson loop expectation is Laurent polynomial with integer coefficients serve as motivations of Witten for searching a physical approach to Khovanov polynomial.

The replacement of $D = 2$ in braid group approach to Jones polynomial with $D = 3$ for Chern-Simons approach replaced by something new in $D = 4$ would naturally correspond to the dimensional hierarchy of TGD in which partonic 2-surfaces plus their 2-D tangent space data fix the physics. One cannot quite do with partonic 2-surfaces and the inclusion of 2-D tangent space-data leads to holography and unique space time surfaces and perhaps also unique string world sheets serving as duals for partonic 2-surfaces. This would realize the weak form of electric magnetic duality at the level of homology much like Poincare duality relates cohomology and homology.

### 7.6.4 Surface operators and the choice of the preferred 2-surfaces

The choice of preferred 2-surfaces and the identification of surface operators in $\mathcal{N} = 4$ YM theory is discussed in [A60]. The intuitive picture is that preferred 2-surfaces- now string world sheets defining braid cobordisms and 2-knots- correspond to singularities of classical gauge fields. Surface operator can be said to create this singularity. In functional integral this means the presence of the exponent defining the analog of Wilson loop.

1. In [A60] the 2-D singular surfaces are identified as poles for the magnitude $r$ of the Higgs field. One can assign to the 2-surface fractional magnetic charges defined for the Cartan algebra part $A_C$ of the gauge connection as circulations $\oint A_C$ around a small circle around the axis of singularity at $r = \infty$. What happens that 3-D $r = constant$ surface reduces to a 2-D surface at $r = \infty$ whereas $A_C$ and entire gauge potential behaves as $A = A_C = \alpha d\phi$ near singularity. Here $\phi$ is coordinate analogous to angle of cylindrical coordinates when t-z plane represents the singular 2-surface. $\alpha$ is a linear combination of Cartan algebra generators.

2. The phase factor assignable to the circulation is essentially $exp(i2\pi\alpha)$ and for non-fractional magnetic charges it differs from unity. One might perhaps say that string word sheets correspond to singularities for the slicing of space-time surface with 3-surfaces at which 3-surfaces reduce to 2-surfaces.

Consider now the situation in TGD framwork.

1. The gauge group is color gauge group and gauge color gauge potentials correspond to the quantities $H_A J$. One can also consider a generalization by allowing all Hamiltonians generating symplectic transformations of $CP_2$. Kähler gauge potential is in essential role since color gauge field is proportional to Kähler form.

2. The singularities of color gauge fields can be identified by studing the theory locally as a field theory from $CP_2$ to $M^4$. It is quite possible to have space-time surfaces for which $M^4$ coordinates are many-valued functions of $CP_2$ coordinates so that one has a covering of $CP_2$ locally. For singular 2-surfaces this covering becomes singular in the sense that separate sheets coincide. These coverings do not seem to correspond to those assignable to the hierarchy of Planck constants implied by the many-valuedness of the time derivatives of the imbedding space coordinates as functions of canonical momentum densities but one must be very cautious in making too strong conclusions here.

3. To proceed introduce the Eguchi-Hanson coordinates

$$(\xi^1, \xi^2) = [rcos(\theta/2)exp(i(\Psi + \Phi)/2), rsin(\theta/2)exp(i(-\Psi + \Phi)/2]$$

   for $CP_2$ with the defining property that the coordinates transform linearly under $U(2) \subset SU(3)$. In QFT context these coordinates would be identified as Higgs fields. The choice of these coordinates is unique apart from the choice of the $U(2)$ subgroup and rotation by element of $U(2)$ once this choice has been made. In TGD framework the definition of $CD$ involves the fixing of these coordinates and the interpretation is in terms of quantum classical correspondence realizing the choice of quantization axes of color at the level of the WCW geometry.

   $r$ has a natural identification as the magnitude of Higgs field invariant under $U(2) \subset SU(3)$. The $SU(2) \times U(1)$ invariant 3-sphere reduces to a homologically non-trivial geodesic 2-sphere at $r = \infty$ so that for this choice of coordinates this surface defines in very natural manner the counterpart of singular 2-surface in $CP_2$ geometry. At this sphere the second phase associated with $CP_2$ coordinates- $\Phi$ - becomes a redundant coordinate just like the angle $\Phi$ at the poles of sphere. There are two other similar spheres and these three spheres are completely analogous to North and South poles of 2-sphere.

4. One possibility is that the singular surfaces correspond to the inverse images for the projection of the imbedding map to $r = \infty$ geodesic sphere of $CP_2$ for a $CD$ corresponding to a given choice of quantization axes. Also the inverse images of all homological non-trivial geodesic spheres defining the three poles of $CP_2$ can be considered. The inverse images of this geodesic 2-sphere under the imbedding-projection map would naturally correspond to 2-D string world sheets for the preferred extremals for a generic space-time surface. For cosmic strings and massless extremals the inverse image would be 4-dimensional but this problem can be circumvented easily. The identification turned out to be somewhat ad hoc and later a much more convicing unique identification of string world sheets emerged and will be discussed in the sequel. Despite this the general aspects of the proposal deserves a discussion.

5. The existence of dual slicings of space-time surface by 3-surfaces and lines on one hand and by string world sheets $Y^2$ and 2-surfaces $X^2$ with Euclidian signature of metric on one hand, is one of the basic conjectures about the properties of preferred extremals of Kähler action. A stronger conjecture is that partonic 2-surfaces represent particular instances of $X^2$. The proposed picture suggests an amazingly simple and physically attractive identification of these slicings.

   (a) The slicing induced by the slicing of $CP_2$ by $r = constant$ surfaces defines an excellent candidate for the slicing by 3-surfaces. Physical the slices would correspond to equivalence classes of choices of the quantization axes for color group related by $U(2)$. In gauge theory context they would correspond to different breakings of $SU(3)$ symmetry labelled by the vacuum expectation of the Higgs field $r$ which would be dynamical for $CP_2$ projections and play the role of time coordinate.

   (b) The slicing by string world sheets would naturally correspond to the slicing induced by the 2-D space of homologically non-trivial geodesic spheres (or triplets of them) and could be called "$CP_2/S^2$". One has clearly bundle structure with $S^2$ as base space and "$CP_2/S^2$" as fiber. Partonic 2-surfaces could be seen locally as sections of this bundle like structure assigning a point of "$CP_2/S^2$" to each point of $S^2$. Globally this does not make sense for partonic 2-surfaces with genus larger than $g = 0$.

6. In TGD framework the Cartan algebra of color gauge group is the natural identification for the Cartan algebra involved and the fluxes defining surface operators would be the classical fluxes $\int H_A J$ over the 2-surfaces in question restricted to Cartan algebra. What would be new is the interpretation as gerbe gauge potentials so that flux becomes completely analogous to Abelian circulation.

   If one accepts the extension of the gauge algebra to a symplectic algebra, one would have the Cartan algebra of the symplectic algebra. This algebra is defined by generators which depend on the second half $P_i$ or $Q_i$ of Darboux coordinates. If $P_i$ are chosen to be functions of the coordinates $(r, \theta)$ of $CP_2$ coordinates whose Poisson brackets with color isospin and hyper charge generators inducing rotations of phases $(\Psi, \Phi)$ of $CP_2$ complex coordinates vanish, the symplectic Cartan algebra would correspond to color neutral Hamiltonians. The spherical harmonics with non-vanishing angular momentum vanish at poles and one expects that same happens for $CP_2$ spherical harmonics at the three poles of $CP_2$. Therefore Cartan algebra is selected automatically for gauge fluxes.

   This subgroup leaves the ends of the points of braids at partonic 2-surfaces invariant so that symplectic transformations do not induce braiding.

If this picture -resulting as a rather straightforward translation of the picture applied in QFT context- is correct, TGD would predict uniquely the preferred 2-surfaces and therefore also the braids as inverse images of $CP_2$ geodesic sphere for the imbedding of space-time surface to $CD \times CP_2$. Also the conjecture slicings by 3-surfaces and string world sheets could be identified. The identification of braids and slicings has been indeed one of the basic challenges in quantum TGD since in quantum theory one does not have anymore the luxury of topological invariance and I have proposed several identifications. If one accepts only these space-time sheets then the stringy content for a given space-time surface would be uniquely fixed.

The assignment of singularities to the homologically non-trivial geodesic sphere suggests that the homologically non-trivial space-time sheets could be seen as 1-dimensional idealizations of magnetic flux tubes carrying Kähler magnetic flux playing key role also in applications of TGD, in particular biological applications such as DNA as topological quantum computer and bio-control and catalysis.

### 7.6.5 The identification of charges $Q$, $P$ and $F$ of Khovanov homology

The challenge is to identify physically the three operators $Q$, $F$, and $P$ appearing in Khovanov homology. Taking seriously the proposal of Witten [A73] and looking for its direct counterpart in TGD leads to the identification and physical interpretation of these charges in TGD framework.

1. In Witten's approach $P$ corresponds to instanton number assignable to the classical gauge field configuration in space-time. In TGD framework the instanton number would naturally correspond to that assignable to $CP_2$ Kähler form. One could consider the possibility of assigning

this charge to the deformed $CP_2$ type vacuum extremals assigned to the space-like regions of space-time representing the lines of generalized Feynman diagrams having elementary particle interpretation. $P$ would be or at least contain the sum of unit instanton numbers assignable to the lines of generalized Feynman diagrams with sign of the instanton number depending on the orientation of $CP_2$ type vacuum extremal and perhaps telling whether the line corresponds to positive or negative energy state. Note that only pieces of vacuum extremals defined by the wormhole contacts are in question and it is somewhat questionable whether the rest of them in Minkowskian regions is included.

2. $F$ corresponds to $U(1)$ charge assignable to $R$-symmetry of $N = 4$ SUSY in Witten's theory. The proposed generalization of twistorial approach in TGD framework suggests strongly that this identification generalizes to TGD. In TGD framework all solutions of modified Dirac equation at wormhole throats define super-symmetry generators but the supersymmetry is badly broken. The covariantly constant right handed neutrino defines the minimally broken supersymmetry since there are no direct couplings to gauge fields. This symmetry is however broken by the mixing of right and left handed $M^4$ chiralities present for both Dirac actions for induced gamma matrices and for modified Dirac equations defined by Kähler action and Chern-Simons action. It is caused by the fact that both the induced and modified gamma matrices are combinations of $M^4$ and $CP_2$ gamma matrices. $F$ would therefore correspond to the net fermion number assignable to right handed neutrinos and antineutrinos. $F$ is not conserved because of the chirality mixing and electroweak interactions respecting only the conservation of lepton number.

   Note that the mixing of $M^4$ chiralities in sub-manifold geometry is a phenomenon characteristic for TGD and also a direct signature of particle massivation and SUSY breaking. It would be nice if it would allow the physical realization of $Q$ operator of Khovanov homology.

3. Witten proposes an explicit formula for $Q$ in terms of 5-dimensional time evolutions interpolating between two 4-D instantons and involving sum of sign factors assignable to Dirac determinants. In TGD framework the operator $Q$ should increase the right handed neutrino number by one unit and therefore transform one right-handed neutrino to a left handed one in the minimal situation. In zero energy ontology $Q$ should relate to a time evolution either between ends of $CD$ or between the ends of single line of generalized Feynman diagram. If instanton number can be assigned solely to the wormhole contacts, this evolution should increase the number of $CP_2$ type extremals by one unit. 3-particle vertex in which right handed neutrino assignable to a partonic 2-surface transforms to a left handed one is thus a natural candidate for defining the action of $Q$. In TGD framework Dirac determinant of 3-D Chern-Simons Dirac operator is conjectured to define exponent of Kähler function reducing to the exponent of Chern-Simons Kähler form. Maybe the sign factor could relate to this determinant.

4. Note that the almost topological QFT property of TGD together with the weak form of electric-magnetic duality implies that Kähler action reduces to Abelian Chern-Simons term. Ordinary Chern-Simons theory involves imaginary exponent of this term but in TGD the exponent would be real. Should one replace the real exponent of Kähler function with imaginary exponent? If so, TGD would be very near to topological QFT also in this respect. This would also force the quantization of the coupling parameter $k$ in Chern-Simons action. On the other hand, the Chern-Simons theory makes sense also for purely imaginary $k$ [A73] .

### 7.6.6 What does the replacement of topological invariance with symplectic invariance mean?

One interpretation for the symplectic invariance is as an analog of diffeo-invariance. This would imply color confinement. Another interpretation would be based on the identification of symplectic group as a color group. Maybe the first interpretation is the proper restriction when one calculates invariants of braids and 2-knots.

The replacement of topological symmetry with symplectic invariance means that TGD based invariants for braids carry much more refined information than topological invariants. In TGD approach $M^4$ diffeomorphisms act freely on partonic 2-surfaces and 4-D tangent space data but the action in $CP_2$ degrees of freedom reduces to symplectic transformations. One could of course consider also the restriction to symplectic transformations of the light-cone boundary and this would give additional refinements.

It is is easy to see what symplectic invariance means by looking what it means for the ends of braids at a given partonic 2-surface.

1. Symplectic transformations respect the Kähler magnetic fluxes assignable to the triangles defined by the finite number of braid points so that these fluxes defining symplectic areas define some minimum number of coordinates parametrizing the moduli space in question. For topological invariance all $n$-point configurations obtained by continuous or smooth transformations are equivalent braid end configurations. These finite-dimensional moduli spaces would be contracted with point in topological QFT.

2. This picture led to a proposal of what I call symplectic QFT [K8] in which the associativity condition for symplectic fusion rules leads the hierarchy of algebras assigned with symplectic triangulations and forming a structures known as operad in category theory. The ends of braids at partonic 2-surfaces would would define unique triangulation of this kind if one accepts the identification of string like 2-surfaces as inverse images of homologically non-trivial geodesic sphere.

Note that both diffeomorphisms and symplectic transformations can in principle induce braiding of the braid strands connecting two partonic 2-surfaces. Should one consider the possibility that the allow transformations are restricted so that they do not induce braiding?

1. These transformations induce a transformation of the space-time surface which however is not a symplectic transformation in the interior in general. An attractive conjecture is that for the preferred extremals this is the case at the inverse images of the homologically non-trivial geodesic sphere. This would conform with the proposed duality between partonic 2-surfaces and string world sheets inspired by holography and also with quantum classical correspondence suggesting that at string world sheets the transformations induced by symplectic transformations at partonic 2-surfaces act like symplectic transformations.

2. If one allows only the symplectic transformations in Cartan algebra leaving the homologically non-trivial geodesic sphere invariant, the infinitesimal symplectic transformations would affect neither the string word sheets nor braidings but would modify the partonic 2-surfaces at all points except at the intersections with string world sheets.

## 7.7 Algebraic braids, sub-manifold braid theory, and generalized Feynman diagrams

Ulla send me a link to an article by Sam Nelson about very interesting new-to-me notion known as algebraic knots [A61, A40], which has initiated a revolution in knot theory. This notion was introduced 1996 by Louis Kauffmann [A53] so that it is already 15 year old concept. While reading the article I realized that this notion fits perfectly the needs of TGD and leads to a progress in attempts to articulate more precisely what generalized Feynman diagrams are.

In the following I will summarize briefly the vision about generalized Feynman diagrams, introduce the notion of algebraic knot, and after than discuss in more detail how the notion of algebraic knot could be applied to generalized Feynman diagrams. The algebraic structrures kei, quandle, rack, and biquandle and their algebraic modifications as such are not enough. The lines of Feynman graphs are replaced by braids and in vertices braid strands redistribute. This poses several challenges: the crossing associated with braiding and crossing occurring in non-planar Feynman diagrams should be integrated to a more general notion; braids are replaced with sub-manifold braids; braids of braids ....of braids are possible; the redistribution of braid strands in vertices should be algebraized. In the following I try to abstract the basic operations which should be algebraized in the case of generalized Feynman diagrams.

One should be also able to concretely identify braids and 2-braids (string world sheets) as well as partonic 2-surfaces and I have discussed several identifications during last years. Legendrian braids turn out to be very natural candidates for braids and their duals for the partonic 2-surfaces. String world sheets in turn could correspond to the analogs of Lagrangian sub-manifolds or to minimal surfaces of space-time surface satisfying the weak form of electric-magnetic duality. The latter option turns out to be more plausible. Finite measurement resolution would be realized as symplectic invariance with respect to the subgroup of the symplectic group leaving the end points of braid strands invariant. In accordance with the general vision TGD as almost topological QFT would mean symplectic QFT. The identification of braids, partonic 2-surfaces and string world sheets - if correct - would solve quantum TGD explicitly at string world sheet level in other words in finite measurement resolution.

Irrespective of whether the algebraic knots are needed, the natural question is what generalized Feynman diagrams are. It seems that the basic building bricks can be identified so that one can write rather explicit Feynman rules already now. Of course, the rules are still far from something to be burned into the spine of the first year graduate student.

### 7.7.1 Generalized Feynman diagrams, Feynman diagrams, and braid diagrams

#### How knots and braids a la TGD differ from standard knots and braids?

TGD approach to knots and braids differs from the knot and braid theories in given abstract 3-manifold (4-manifold in case of 2-knots and 2-braids) is that space-time is in TGD framework identified as 4-D surface in $M^4 \times CP_2$ and preferred 3-surfaces correspond to light-like 3-surfaces defined by wormhole throats and space-like 3-surfaces defined by the ends of space-time sheets at the two light-like boundaries of causal diamond $CD$.

The notion of finite measurement resolution effectively replaces 3-surfaces of both kinds with braids and space-time surface with string world sheets having braids strands as their ends. The 4-dimensionality of space-time implies that string world sheets can be knotted and intersect at discrete points (counterpart of linking for ordinary knots). Also space-time surface can have self-intersections consisting of discrete points.

The ordinary knot theory in $E^3$ involves projection to a preferred 2-plane $E^2$ and one assigns to the crossing points of the projection an index distinguishing between two cases which are transformed to each other by violently taking the first piece of strand through another piece of strand. In TGD one must identify some physically preferred 2-dimensional manifold in imbedding space to which the braid strands are projected. There are many possibilities even when one requires maximal symmetries. An obvious requirement is however that this 2-manifold is large enough.

1. For the braids at the ends of space-time surface the 2-manifold could be large enough sphere $S^2$ of light-cone boundary in coordinates in which the line connecting the tips of $CD$ defines a preferred time direction and therefore unique light-like radial coordinate. In very small knots it could be also the geodesic sphere of $CP_2$ (apart from the action of isometries there are two geodesic spheres in $CP_2$).

2. For light-like braids the preferred plane would be naturally $M^2$ for which time direction corresponds to the line connecting the tips of $CD$ and spatial direction to the quantization axis of spin. Note that these axes are fixed uniquely and the choices of $M^2$ are labelled by the points of projective sphere $P^2$ telling the direction of space-like axis. Preferred plane $M^2$ emerges naturally also from number theoretic vision and corresponds in octonionic pictures to hyper-complex plane of hyper-octonions. It is also forced by the condition that the choice of quantization axes has a geometric correlate both at the level of imbedding space geometry and the geometry of the "world of classical worlds".

The braid theory in TGD framework could be called sub-manifold braid theory and certainly differs from the standard one.

1. If the first homology group of the 3-surface is non-trivial as it when the light-like 3-surfaces represents an orbit of partonic 2-surface with genus larger than zero, the winding of the braid strand (wrapping of branes in M-theory) meaning that it represents a homologically non-trivial curve brings in new effects not described by the ordinary knot theory. A typical new situation is the one in which 3-surface is locally a product of higher genus 2-surface and line segment so

that knot strand can wind around the 2-surface. This gives rise to what are called non-planar braid diagrams for which the projection to plane produces non-standard crossings.

2. In the case of 2-knots similar exotic effects could be due to the non-trivial 2-homology of space-time surface. Wormhole throats assigned with elementary particle wormhole throats are homologically non-trivial 2-surfaces and might make this kind of effects possible for 2-knots if they are possible.

The challenge is to fnd a generalization of the usual knot and braid theories so that they apply in the case of braids (2-braids) imbedded in 3-D (4-D) surfaces with preferred highly symmetry sub-manifold of $M^4 \times CP_2$ defining the analog of plane to which the knots are projected. A proper description of exotic crossings due to non-trivial homology of 3-surface (4-surface) is needed.

#### Basic questions

The questions are following.

1. How the mathematical framework of standard knot theory should be modified in order to cope with the situation encountered in TGD? To my surprise I found that this kind of mathematical framework exists: so called algebraic knots [A61, A40] define a generalization of knot theory very probably able to cope with this kind of situation.

2. Second question is whether the generalized Feynman diagrams could be regarded as braid diagrams in generalized sense. Generalized Feynman diagrams are generalizations of ordinary Feynman diagrams. The lines of generalized Feynman diagrams correspond to the orbits of wormhole throats and of wormhole contacts with throats carrying elementary particle quantum numbers.

   The lines meet at vertices which are partonic 2-surfaces. Single wormhole throat can describe fermion whereas bosons have wormhole contacts with fermion and antifermion at the opposite throats as building bricks. It seems however that all fermions carry Kähler magnetic charge so that physical particles are string like objects with magnetic charges at their ends.

   The short range of weak interactions results from the screening of the axial isospin by neutrinos at the other end of string like object and also color confinement could be understood in this manner. One cannot exclude the possibility that the length of magnetic flux tube is of order Compton length.

3. Vertices of the generalized Feynman diagrams correspond to the partonic 2-surfaces along which light-like 3-surfaces meet and this is certainly a challenge for the required generalization of braid theory. The basic objection against the reduction to algebraic braid diagrams is that reaction vertices for particles cannot be described by ordinary braid theory: the splitting of braid strands is needed.

   The notion of bosonic emergence [K39] however suggests that 3-vertex and possible higher vertices correspond to the splitting of braids rather than braid strands. By allowing braids which come from both past and future and identifying free fermions as wormhole throats and bosons as wormhole contacts consisting of a pair of wormhole throats carrying fermion and antifermion number, one can understand boson excanges as recombinations without anyneed to have splitting of braid strands. Strictly and technically speaking, one would have tangles like objects instead of braids. This would be an enormous simplification since $n > 2$-vertices which are the source of divergences in QFT:s would be absent.

4. Non-planar Feynman diagrams are the curse of the twistor approach and I have already earlier proposed that the generalized Feynman amplitudes and perhaps even twistorial amplitudes could be constructed as analogs of knot invariants by recursively transforming non-planar Feynman diagrams to planar ones for which one can write twistor amplitudes. This forces to answer two questions.

   (a) Does the non-nonplanarity of Feynman diagrams - completely combinatorial objects identified as diagrams in plane - have anything to do with the non-planarity of algebraic knot diagrams and with the non-planarity of generalized Feynman diagrams which are purely geometric objects?

(b) Could these two kind of non-planarities be fused to together by identifying the projection 2-plane as preferred $M^2 \subset M^4$. This would mean that non-planarity in QFT sense is defined for entire braids: braid A can have virtual crossing with B. Non-planarity in the sense of knot theory would be defined for braid strands inside the braids. At vertices braid strands are redistributed between incoming lines and the analog of virtual crossing be identifiable as an exchange of braid strand between braids. Several kinds of non-planarities would be present and the idea about gradual unknotting of a non-planar diagram so that a planar diagram results as the final outcome might make sense and allow to generalize the recursion recipe for the twistorial amplitudes.

(c) One might consider the possibility that inside orbits of wormhole throats defining the lines of Feynman diagrams the $R$-matrix for integrable QFT in $M^2$ (only permutations of momenta are allowed) describes the dynamics so that one obtains just a permutation of momenta assigned to the braid strands. Ordinary braiding would be described by existing braid theories. The core problem would be the representation of the exchange of a strand between braids algebraically.

### 7.7.2 Brief summary of algebraic knot theory

#### Basic ideas of algebraic knot theory

In ordinary knot theory one takes as a starting point the representation of knots of $E^3$ by their plane plane projections to which one attach a "color" to each crossing telling whether the strand goes over or under the strand it crosses in planar projection. These numbers are fixed uniquely as one traverses through the entire knot in given direction.

The so called Reidermeister moves are the fundamental modifications of knot leaving its isotopy equivalence class unchanged and correspond to continuous deformations of the knot. Any algebraic invariant assignable to the knot must remain unaffected under these moves. Reidermeister moves as such look completely trivial and the non-trivial point is that they represent the minimum number of independent moves which are represented algebraically.

In algebraic knot theory topological knots are replaced by typographical knots resulting as planar projections. This mapping of topology to algebra and this is always fascinating. It turns out that the existing knot invariants generalize and ordinary knot theory can be seen as a special case of the algebraic knot theory. In a loose sense one can say that the algebraic knots are to the classical knot theory what algebraic numbers are to rational numbers.

Virtual crossing is the key notion of the algebraic knot theory. Virtual crossing and their rules of interaction were introduced 1996 by Louis Kauffman as basic notions [A2]. For instance, a strand with only virtual crossings should be replaceable by any strand with the same number of virtual crossings and same end points. Reidermeister moves generalize to virtual moves. One can say that in this case crossing is self-intersection rather than going under or above. I cannot be eliminated by a small deformation of the knot. There are actually several kinds of non-standard crossings: examples listed in figure 7 of [A61]) are virtual, flat, singular, and twist bar crossings.

Algebraic knots have a concrete geometric interpretation.

1. Virtual knots are obtained if one replaces $E^3$ as imbedding space with a space which has non-trivial first homology group. This implies that knot can represent a homologically non-trivial curve giving an additional flavor to the unknottedness since homologically non-trivial curve cannot be transformed to a curve which is homologically non-trivial by any continuous deformation.

2. The violent projection to plane leads to the emergence of virtual crossings. The product $(S^1 \times S^1) \times D$, where $(S^1 \times S^1)$ is torus $D$ is finite line segment, provides the simplest example. Torus can be identified as a rectangle with opposite sides identified and homologically non-trivial knots correspond to curves winding $n_1$ times around the first $S^1$ and $n_2$ times around the second $S^1$. These curves are not continuous in the representation where $S^1 \times S^1$ is rectangle in plane.

3. A simple geometric visualization of virtual crossing is obtained by adding to the plane a handle along which the second strand traverses and in this manner avoids intersection. This visualization allows to understand the geometric motivation for the the virtual moves.

This geometric interpretation is natural in TGD framework where the plane to which the projection occurs corresponds to $M^2 \subset M^4$ or is replaced with the sphere at the boundary of $S^2$ and 3-surfaces can have arbitrary topology and partonic 2-surfaces defining as their orbits light-like 3-surfaces can have arbitrary genus.

In TGD framework the situation is however more general than represented by sub-manifold braid theory. Single braid represents the line of generalized Feynman diagram. Vertices represent something new: in the vertex the lines meet and the braid strands are redistributed but do not disappear or pop up from anywhere. That the braid strands can come both from the future and past is also an important generalization. There are physical argments suggesting that there are only 3-vertices for braids but not higher ones [K11]. The challenge is to represent algebraically the vertices of generalized Feynman diagrams.

### Algebraic knots

The basic idea in the algebraization of knots is rather simple. If $x$ and $y$ are the crossing portions of knot, the basic algebraic operation is binary operation giving "the result of $x$ going under $y$", call it $x \triangleright y$ telling what happens to $x$. "Portion of knot" means the piece of knot between two crossings and $x \triangleright y$ denotes the portion of knot next to $x$. The definition is asymmetrical in $x$ and $y$ and the dual of the operation would be $y \triangleleft x$ would be "the result of $y$ going above $x$". One can of course ask, why not to define the outcome of the operation as a pair $(x \triangleleft y, y \triangleright x)$. This operation would be bi-local in a well-defined sense. One can of course do this: in this case one has binary operation from $X \times X \to X \times X$ mapping pairs of portions to pairs of portions. In the first case one has binary operation $X \times X \to X$.

The idea is to abstract this basic idea and replace $X$ with a set endowed with operation $\triangleright$ or $\triangleleft$ or both and formukate the Reidermeister conditions given as conditions satisfied by the algebra. One ends up to four basic algebraic structures kei, quandle, rack, and biquandle.

1. In the case of non-oriented knots the kei is the algebraic structure. Kei - or invontary quandle-is a set $X$ with a map $X \times X \to X$ satisfying the conditions

   (a) $x \triangleright x = x$ (idenpotency, one of the Reidemeister moves)

   (b) $(x \triangleright y) \triangleright y$ =x (operation is its own right inverse having also interpretation as Reidemeister move)

   (c) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ (self-distributivity)

   $Z([t])/(t^2)$ module with $x \triangleright y = tx + (1-t)y$ is a kei.

2. For orientable knot diagram there is preferred direction of travel along knot and one can distinguish between $\triangleright$ and its right inverse $\triangleright^{-1}$. This gives quandle satisfying the axios

   (a) $x \triangleright x = x$

   (b) $(x \triangleright y) \triangleright^{-1} y = (x \triangleright^{-1} y) \triangleright y = x$

   (c) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

   $Z[t^{\pm 1}]$ nodule with $x \triangleright y = tx + (1-t)y$ is a quandle.

3. One can also introduce framed knots: intuitively one attaches to a knot very near to it. More precise formulation in terms of a section of normal bundle of the knot. This makes possible to speak about self-linking. Reidermeister moves must be modified appropriately. In this case rack is the appropriate structure. It satisfied the axioms of quandle except the first axiom since corresponding operation is not a move anymore. Rack axioms are eqivalent with the requirement that functions $f_y : X \to X$ defined by $f_y(x)x \triangleright y)$ are automorphisms of the structure. Therefore the elements of rack represent its morphisms. The modules over $Z[t^{\pm 1}, s]/s(t+s-1)$ are racks. Coxeter racks are inner product spaces with $x \triangleright y$ obtained by reflecting $x$ across $y$.

4. Biquandle consists of arcs connecting the subsequent crossings (both under- and over-) of oriented knot diagram. Biquandle operation is a map $B : X \times X \rightarrow X \times X$ of order pairs satisfying certain invertibility conditions together with set theoretic Yang-Baxter equation:

$$(B \times I)(I \times B)(B \times I) = (I \times B)(B \times I)(I \times B) \ .$$

Here $I : X \rightarrow X$ is the identity map. The three conditions to which Yang-Baxter equation decomposes gives the counterparts of the above discussed axioms. Alexander biquandle is the module $Z(t^{\pm 1}, s^{\pm}1$ with $B(x,y) = (ty + (1-ts)x, sx)$ where one has $s \neq 1$. If one includes virtual, flat and singular crossings one obtains virtual/singular aundles and semiquandles.

### 7.7.3 Generalized Feynman diagrams as generalized braid diagrams?

Zero energy ontology suggests the interpretation of the generalized Feynman diagrams as generalized braid diagrams so that there would be no need for vertices at the fundamental braid strand level. The notion of algebraic braid (or tangle) might allow to formulate this idea more precisely.

#### Could one fuse the notions of braid diagram and Feynman diagram?

The challenge is to fuse the notions of braid diagram and Feynman diagram having quite different origin.

1. All generalized Feynman diagrams are reduced to sub-manifold braid diagrams at microscopic level by bosonic emergence (bosons as pairs of fermionic wormhole throats). Three-vertices appear only for entire braids and are purely topological whereas braid strands carrying quantum numbers are just re-distributed in vertices. No 3-vertices at the really microscopic level! This is an additional nail to the coffin of divergences in TGD Universe.

2. By projecting the braid strands of generalized Feynman diagrams to preferred plane $M^2 \subset M^4$ (or rather 2-D causal diamond), one could achieve a unified description of non-planar Feynman diagrams and braid diagrams. For Feynman diagrams the intersections have a purely combinatorial origin coming from representations as 2-D diagrams.

   For braid diagrams the intersections have different origin and non-planarity has different meaning. The crossings of entire braids analogous to those appearing in non-planar Feynman diagrams should define one particular exotic crossing besides virtual crossings of braid strands due to non-trivial first homology of 3-surfaces.

3. The necessity to choose preferred plane $M^2$ looks strange from QFT point of view. In TGD framework it is forced by the number theoretic vision in which $M^2$ represents hyper-complex plane of sub-space of hyper-octonions which is subspace of complexified octonions. The choice of $M^2$ is also forced by the condition that the choice of quantization axes has a geometric correlate both at the level of imbedding space geometry and the geometry of the "world of classical worlds".

4. Also 2-braid diagrams defined as projections of string world sheets are suggestive and would be defined by a projections to the 3-D boundary of $CD$ or to $M^3 \subset M^4$. They would provide a more concrete stringy illustration about generalized Feynman diagram as analog of string diagram. Another attractive illustration is in terms of dance metaphor with the boundary of $CD$ defining the 3-D space-like parquette. The duality between space-like and light-like braids is expected to be of importance.

The obvious conjecture is that Feynman amplitudes are a analogous to knot invariants constructible by gradually reducing non-planar Feynman diagrams to planar ones after which the already existing twistor theoretical machinery of $\mathcal{N} = 4$ SYMs would apply [K61].

### Does 2-D integrable QFT dictate the scattering inside the lines of generalized Feynman diagrams

The preferred plane $M^2$ (more precisely, 2-D causal diamond having also interpretation as Penrose diagram) plays a key role as also the preferred sphere $S^2$ at the boundary of $CD$. It is perhaps not accident that a generalization of braiding was discovered in integrable quantum field theories in $M^2$. The S-matrix of this theory is rather trivial looking: particle moving with different velocities cross each other and suffer a phase lag and permutation of 2-momenta which has physical effects only in the case of non-identical particles. The $R$-matrix describing this process reduces to the $R$-matrix describing the basic braiding operation in braid theories at the static limit.

I have already earlier conjectured that this kind of integrable QFT is part of quantum TGD [K13]. The natural guess is that it describes what happens for the projections of 4-momenta in $M^2$ in scattering process inside lines of generalized Feynman diagrams. If integrable theories in $M^2$ control this scattering, it would cause only phase changes and permutation of the $M^2$ projections of the 4-momenta. The most plausible guess is that $M^2$ QFT characterized by $R$-matrix describes what happens to the braid momenta during the free propagation and the remaining challenge would be to understand what happens in the vertices defined by 2-D partonic surfaces at which re-distribution of braid strands takes place.

### How quantum TGD as almost topological QFT differs from topological QFT for braids and 3-manifolds

One must distinguish between two topological QFTs. These correspond to topological QFT defining braid invariants and invariants of 3-manifolds respectively. The reason is that knots are an essential element in the procedure yielding 3-manifolds. Both 3-manifold invariants and knot invariants would be defined as Wilson loops involving path integral over gauge connections for a given 3-manifold with exponent o non-Abelkian f Chern-Simons action defining the weight.

1. In TGD framework the topological QFT producing braid invariants for a given 3-manifold is replaced with sub-manifold braid theory. Kähler action reduces Chern-Simons terms for preferred extremals and only these contribute to the functional integral. What is the counterpart of topological invariance in this framework? Are general isotopies allowed or should one allow only sub-group of symplectic group of $CD$ boundary leaving the end points of braids invariant? For this option Reidermeister moves are undetectable in the finite measurement resolution defined by the subgroup of the symplectic group. Symplectic transformations would not affect 3-surfaces as the analogs of abstract contact manifold since induced Kähler form would not be affected and only the imbedding would be changed.

   In the approach based on inclusions of HFFs gauge invariance or its generalizations would represent finite measurement resolution (the action of included algebra would generate states not distiguishable from the original one).

2. There is also ordinary topological QFT allowing to construct topological invariants for 3-manifold. In TGD framework the analog of topological QFT is defined by Chern-Simons-Kähler action in the space of preferred 3-surfaces. Now one sums over small deformations of 3-surface instead of gauge potentials. If extremals of Chern-Simons-Kähler action are in question, symplectic invariance is the most that one can hope for and this might be the situation quite generally. If all light-like 3-surfaces are allowed so that only weak form of electric-magnetic duality at them would bring metric into the theory, it might be possible to have topological invariance at 3-D level but not at 4-D level. It however seems that symplectic invariance with respect to subgroup leaving end points of braids invariant is the realistic expectation.

### Could the allowed braids define Legendrian sub-manifolds of contact manifolds?

The basic questions concern the identification of braids and 2-braids. In quantum TGD they cannot be arbitrary but determined by dynamics providing space-time correlates for quantum dynamics. The deformations of braids should mean also deformations of 3-surfaces which as topological manifolds would however remain as such. Therefore topological QFT for given 3-manifold with path integral over gauge connections would in TGD correspond to functional integral of 3-surfaces corresponding to same topology even symplectic structure. The quantum fluctuating degrees of freedom indeed correspond to symplectic group divided by its subgroup defining measurement resolution.

What is the dynamics defining the braids strands? What selects them? I have considered this problem several times. Just two examples is enough here.

1. Could they be some special light-like curves? Could the condition that the end points of the curves correspond to rational points in some preferred coordinates allow to select these light-like curves? But what about light-like curves associated with the ends of the space-time surface?

2. The solutions of modified Dirac equation [K19] are localized to curves by using the analog of periodic boundary conditions: the length of the curve is quantized in the effective metric defined by the modified gamma matrices. Here one however introcuced a coordinate along light-like 3-surface and it is not clear how one should fix this preferred coordinate.

*1. Legendrian and Lagrangian sub-manifolds*

A hint about what is missing comes from the observation that a non-vanishing Chern-Simons-Kähler form $A$ defines a contact structure [A5] at light-like 3-surfaces if one has $A \wedge dA \neq 0$. This condition states complete non-intebrability of the distribution of 2-planes defined by the condition $A_\mu t^\mu = 0$, where $t$ is tangent vector in the tangent bundle of light-like 3-surface. It also states that the flow lines of $A$ do *not* define global coordinate varying along them.

1. It is however possible to have 1-dimensional curves for which $A_\mu t^\mu = 0$ holds true at each point. These curves are known as Legendrian sub-manifolds to be distinguished from Lagrangian manifolds for which the projection of symplectic form expressible locally as $J = dA$ vanishes. The set of this curves is discrete so that one obtains braids. Legendrian knots are the simplest example of Legendrian sub-manifolds and the question is whether braid strands could be identified as Legendrian knots. For Legendrian braids symplectic invariance replaces topological invariance and Legendrian knots and braids can be trivial in topological sense. In some situations the property of being Legendrian implies un-knottedness.

2. For Legendrian braid strands the Kähler gauge potential vanishes. Since the solutions of the modified Dirac equation are localized to braid strands, this means that the coupling to Kähler gauge potential vanishes. From physics point of view a generalization of Legendre braid strand by allowing gauge transformations $A \rightarrow A + d\Phi$ looks natural since it means that the coupling of induced spinors is pure gauge terms and can be eliminated by a gauge transformation.

*2. 2-D duals of Legendrian sub-manifolds*

One can consider also what might be called 2-dimensional duals of Legendrian sub-manifolds.

1. Also the one-form obtained from the dual of Kähler magnetic field defined as $B^\mu = \epsilon^{\mu\nu\gamma} J_{\nu\nu}$ defines a distribution of 2-planes. This vector field is ill-defined for light-like surfaces since contravariant metric is ill-defined. One can however multiply $B$ with the square root of metric determining formally so that metric would disappear completely just as it disappears from Chern-Simons action. This looks however somewhat tricky mathematically. At the 3-D space-like ends of space-time sheets at boundaries of $CD$ $B^\mu$ is however well-defined as such.

2. The distribution of 2-planes is integrable if one has $B \wedge dB = 0$ stating that one has Beltrami field: physically the conditions states that the current $dB$ feels no Lorentz force. The geometric content is that $B$ defines a global coordinate varying along its flow lines. For the preferred extremals of Kähler action Beltrami condition is satisfied by isometry currents and Kähler current in the interior of space-time sheets. If this condition holds at 3-surfaces, one would have an global time coordinate and integrable distribution of 2-planes defining a slicing of the 2-surface. This would realize the conjecture that space-time surface has a slicing by partonic 2-surfaces. One could say that the 2-surfaces defined by the distribution are orthogonal to $B$. This need not however mean that the projection of $J$ to these 2-surfaces vanishes. The condition $B \wedge dB = 0$ on the space-like 3-surfaces could be interpreted in terms of effective 2-dimensionality. The simplest option posing no additional conditions would allow two types of braids at space-like 3-surfaces and only Legendrian braids at light-like 3-surfaces.

These observations inspire a question. Could it be that the conjectured dual slicings of space-time sheets by space-like partonic 2-surfaces and by string world sheets are defined by $A_\mu$ and $B^\mu$ respectively associated with slicings by light-like 3-surfaces and space-like 3-surfaces? Could partonic 2-surfaces be identified as 2-D duals of 1-D Legendrian sub-manifolds?

The identification of braids as Legendrian braids for light-like 3-surfaces and with Legendrian braids or their duals for space-like 3-surfaces would in turn imply that topological braid theory is replaced with a symplectic braid theory in accordance with the view about TGD as almost topological QFT. If finite measurement resolution corresponds to the replacement of symplectic group with the coset space obtained by dividing by a subgroup, symplectic subgroup would take the role of isotopies in knot theory. This symplectic subgroup could be simply the symplectic group leaving the end points of braids invariant.

### An attempt to identify the constraints on the braid algebra

The basic problems in understanding of quantum TGD are conceptual. One must proceed by trying to define various concepts precisely to remove the many possible sources of confusion. With this in mind I try collect essential points about generalized Feynman diagrams and their relation to braid diagrams and Feynman diagrams and discuss also the most obvious constraints on algebraization.

Let us first summarize what generalized Feynman diagrams are.

1. Generalized Feynman diagrams are 3-D (or 4-D, depends on taste) objects inside $CD \times CP_2$. Ordinary Feynman diagrams are in plane. If finite measurement resolution has as a space-time correlate discretization at the level of partonic 2-surfaces, both space-like and light-like 3-surfaces reduce to braids and the lines of generalized Feynman diagrams correspond to braids. It is possible to obtain the analogs of ordinary Feynman diagrams by projection to $M^2 \subset M^4$ defined uniquely for given $CD$. The resulting apparent intersections would represent ne particular kind of exotic intersection.

2. Light-like 3-surfaces define the lines of generalized Feynman diagrams and the braiding results naturally. Non-trivial first homology for the orbits of partonic 2-surfaces with genus $g > 0$ could be called homological virtual intersections.

3. It zero energy ontology braids must be characterized by time orientation. Also it seems that one must distinguish in zero energy ontology between on mass shell braids and off mass shell braid pairs which decompose to pairs of braids with positive and negative energy massless on mass shell states. In order to avoid confusion one should perhaps speak about tangles insie $CD$ rather than braids. The operations of the algebra are same except that the braids can end either to the upper or lower light-like boundary of $CD$. The projection to $M^2$ effectively reduces the $CD$ to a 2-dimensional causal diamond.

4. The vertices of generalized Feynman diagrams are partonic 2-surfaces at which the light-like 3-surfaces meet. This is a new element. If the notion of bosonic emergence is accepted no $n > 2$-vertices are needed so that braid strands are redistributed in the reaction vertices. The redistribution of braid strands in vertices must be introduced as an additional operation somewhat analogous to $\triangleright$ and the challenge is to reduce this operation to something simple. Perhaps the basic operation reduces to an exchange of braid strand between braids. The process can be seen as a decay of of braid with the conservation of braid strands with strands from future and past having opposite strand numbers. Also for this operation the analogs of Reidermeister moves should be identified. In dance metaphor this operation corresponds to a situation in which the dancer leaves the group to which it belongs and goes to a new one.

5. A fusion of Feynman diagrammatic non-planarity and braid theoretic non-planarity is needed and the projection to $M^2$ could provide this fusion when at least two kinds of virtual crossings are allowed. The choice of $M^2$ could be global. An open question is whether the choice of $M^2$ could characterize separately each line of generalized Feynman diagram characterized by the four-momentum associated with it in the rest system defined by the tips of $CD$. Somehow the theory should be able to fuse the braiding matrix for integrable QFT in $M^2$ applying to entire braids with the braiding matrix for braid theory applying at the level of single braid.

Both integral QFTs in $M^2$ and braid theories suggest that biquandle structure is the structure that one should try to generalized.

1. The representations of resulting bi-quandle like structure could allow abstract interesting information about generalized Feynman diagrams themselves but the dream is to construct generalized Feynman diagrams as analogs of knot invariants by a recursive procedure analogous to un-knotting of a knot.

2. The analog of bi-quandle algebra should have a hierarchical structure containing braid strands at the lowest level, braids at next level, and braids of braids...of braids at higher levels. The notion of operad would be ideal for formulating this hierarchy and I have already proposed that this notion must be essential for the generalized Feynman diagrammatics. An essential element is the vanishing of total strand number in the vertex (completely analogous to conserved charged such as fermion number). Again a convenient visualization is in terms of dancers forming dynamical groups, forming groups of groups forming .....

   I have already earlier suggested [K13] that the notion of operad [A20] relying on permutation group and its subgroups acting in tensor products of linear spaces is central for understanding generalized Feynman diagrams. $n \to n_1 + n_2$ decay vertex for n-braid would correspond to "symmetry breaking" $S_n \to S_{n_1} \times S_{n_2}$. Braid group represents the covering of permutation group so that braid group and its subgroups permuting braids would suggest itself as the basic group theoretical notion. One could assign to each strand of $n$-braid decaying to $n_1$ and $n_2$ braids a two-valued color telling whether it becomes a strand of $n_1$-braid or $n_2$-braid. Could also this "color" be interpreted as a particular kind of exotic crossing?

3. What could be the analogs of Reidermaster moves for braid strands?

   (a) If the braid strands are dynamically determined, arbitrary deformations are not possible. If however all isotopy classes are allowed, the interpretation would be that a kind of gauge choice selecting one preferred representation of strand among all possible ones obtained by continuous deformations is in question.

   (b) Second option is that braid strands are dynamically determined within finite measurement resolution so that one would have braid theory in given length scale resolution.

   (c) Third option is that topological QFT is replaced with symplectic QFT: this option is suggested by the possibility to identify braid strands as Legendrian knots or their duals. Subgroup of the symplectic group leaving the end points of braids invariant would act as the analog of continous transformations and play also the role of gauge group. The new element is that symplectic transformations affect partonic 2-surfaces and space-time surfaces except at the end points of braid.

4. Also 2-braids and perhaps also 2-knots could be useful and would provide string theory like approach to TGD. In this case the projections could be performed to the ends of $CD$ or to $M^3$, which can be identified uniquely for a given $CD$.

5. There are of course many additional subtleties involved. One should not forget loop corrections, which naturally correspond to sub-$CD$s. The hierarchy of Planck constants and number theoretical universality bring in additional complexities.

All this looks perhaps hopelessly complex but the Universe around is complex even if the basic principles could be very simple.

### 7.7.4 About string world sheets, partonic 2-surfaces, and two-knots

String world sheets and partonic 2-surfaces provide a beatiful visualization of generalized Feynman diagrams as braids and also support for the duality of string world sheets and partonic 2-surfaces as duality of light-like and space-like braids. Dance metaphor is very helpful here.

1. The projection of string world sheets and partonic 2-surfaces to 3-D space replaces knot projection. In TGD context this 3-D of space could correspond to the 3-D light-like boundary of $CD$ and 2-knot projection would correspond to the projection of the braids associated with the lines of generalized Feynman diagram. Another identification would be as $M^1 \times E^2$, where $M^1$ is the line connecting the tips of $CD$ and $E^2$ the orthogonal complement of $M^2$.

2. Using dance metaphor for light-like braiding, braids assignable to the lines of generalized Feynman diagrams would correspond to groups of dancers. At vertices the dancing groups would exchange members and completely new groups would be formed by the dancers . The number of dancers (negative for those dancing in the reverse time direction) would be conserved. Dancers would be connected by threads representing strings having braid points at their ends. During the dance the light-like braiding would induce space-like braiding as the threads connecting the dancers would get entangled. This would suggest that the light-like braids and space-like braidings are equivalent in accordance with the conjectured duality between string-world sheets and partonic 2-surfaces. The presence of genuine 2-knottedness could spoil this equivalence unless it is completely local.

Can string world sheets and partonic 2-surfaces get knotted?

1. Since partonic 2-surfaces (wormhole throats) are imbedded in light-cone boundary, the preferred 3-D manifolds to which one can project them is light-cone boundary (boundary of $CD$). Since the projection reduces to inclusion these surfaces cannot get knotted. Only if the partonic 2-surfaces contains in its interior the tip of the light-cone something non-trivial identifiable as virtual 2-knottedness is obtained.

2. One might argue that the conjectured duality between the descriptions provided by partonic 2-surfaces and string world sheets requires that also string world sheets represent trivial 2-braids. I have shown earlier that nontrivial local knots glued to the string world sheet require that $M^4$ time coordinate has a local maximum. Does this mean that 2-knots are excluded? This is not obvious: TGD allows also regions of space-time surface with Euclidian signature and generalized Feynman graphs as 4-D space-time regions are indeed Euclidian. In these regions string world sheets could get knotted.

What happens for knot diagrams when the dimension of knot is increased to two? According to the articles of Nelson [A61] and Carter [A40] the crossings for the projections of braid strands are replaced with more complex singularities for the projections of 2-knots. One can decompose the 2-knots to regions surrounded by boxes. Box can contain just single piece of 2-D surface; it can contain two intersection pieces of 2-surfaces as the counterpart of intersecting knot strands and one can tell which of them is above which; the box can contain also a discrete point in the intersection of projections of three disjoint regions of knot which consists of discrete points; and there is also a box containing so called cone point. Unfortunately, I failed to understand the meaning of the cone point.

For 2-knots Reidemeister moves are replaced with Roseman moves. The generalization would allow virtual self intersections for the projection and induced by the non-trivial second homology of 4-D imbedding space. In TGD framework elementary particles have homologically non-trivial partonic 2-surfaces (magnetic monpoles) as their building bricks so that even if 2-knotting in standard sense might be not allowed, virtual 2-knotting would be possible. In TGD framework one works with a subgroup of symplectic transformations defining measurement resolution instead of isotopies and this might reduce the number of allowed mov

### The dynamics of string world sheets and the expression for Kähler action

The dynamics of string world sheets is an open question. Effective 2-dimensionality suggests that Kähler action for the preferred extremal should be expressible using 2-D data but there are several guesses for what the explicit expression could be, and one can only make only guesses at this moment and apply internal consistency conditions in attempts to kill various options.

*1. Could weak form of electric-magnetic duality hold true for string world sheets?*

If one believes on duality between string world sheets and partonic 2-surfaces, one can argue that string world sheets are most naturally 2-surfaces at which the weak form of electric magnetic duality holds true. One can even consider the possibility that the weak form of electric-magnetic duality holds true only at the the string world sheets and partonic 2-surfaces but not at the preferred 3-surfaces.

1. The weak form of electric magnetic duality would mean that induced Kähler form is non-vanishing at them and Kähler magnetic flux over string world sheet is proportional to Kähler electric flux.

2. The flux of the induced Kähler form of $CP_2$ over string world sheet would define a dimensionless "area". Could Kähler action for preferred extremals reduces to this flux apart from a proportionality constant. This "area" would have trivially extremum with respect to symplectic variations if the braid strands are Legendrian sub-manifolds since in this case the projection of Kähler gauge potential on them vanishes. This is a highly non-trivial point and favors weak form of electric-magnetic duality and the identification of Kähler action as Kähler magnetic flux. This option is also in spirit with the vision about TGD as almost topological QFT meaning that induced metric appears in the theory only via electric-magnetic duality.

3. Kähler magnetic flux over string world sheet has a continuous spectrum so that the identification as Kähler action could make sense. For partonic 2-surfaces the magnetic flux would be quantized and give constant term to the action perhaps identifiable as the contribution of $CP_2$ type vacuum extremals giving this kind of contribution.

The change of space-time orientation by changing the sign of permutation symbol would change the sign in electric-magnetic duality condition and would not be a symmetry. For a given magnetic charge the sign of electric charge changes when orientation is changed. The value of Kähler action does not depend on space-time orientation but weak form of electric-magnetic duality as boundary condition implies dependence of the Kähler action on space-time orientation. The change of the sign of Kähler electric charge suggests the interpretation of orientation change as one aspect of charge conjugation. Could this orientation dependence be responsible for matter antimatter asymmetry?

*2. Could string world sheets be Lagrangian sub-manifolds in generalized sense?*

Legendrian sub-manifolds can be lifted to Lagrangian sub-manifolds [A5] Could one generalize this by replacing Lagrangian sub-manifold with 2-D sub-manifold of space-times surface for which the projection of the induced Kähler form vanishes? Could string world sheets be Lagrangian sub-manifolds?

I have also proposed that the inverse image of homologically non-trivial sphere of $CP_2$ under imbedding map could define counterparts of string world sheets or partonic 2-surfaces. This conjecture does not work as such for cosmic strings, massless extremals having 2-D projection since the inverse image is in this case 4-dimensional. The option based on homologically non-trivial geodesic sphere is not consistent with the identification as analog of Lagrangian manifold but the identification as the inverse image of homologically trivial geodesic sphere is.

The most general option suggested is that string world sheet is mapped to 2-D Lagrangian sub-manifold of $CP_2$ in the imbedding map. This would mean that theory is exactly solvable at string world sheet level. Vacuum extremals with a vanishing induced Kähler form would be exceptional in this framework since they would be mapped as a whole to Lagrangian sub-manifolds of $CP_2$. The boundary condition would be that the boundaries of string world sheets defined by braids at preferred 3-surfaces are Legendrian sub-manifolds. The generalization would mean that Legendrian braid strands could be continued to Lagrangian string world sheets for which induced Kähler form vanishes. The physical interpretation would be that if particle moves along this kind of string world sheet, it feels no covariant Lorentz-Kähler force and contra variant Lorentz forces is orthogonal to the string world sheet.

There are however serious objections.

1. This proposal does not respect the proposed duality between string world sheets and partonic 2-surfaces which as carries of Kähler magnetic charges cannot be Lagrangian 2-manifolds.

2. One loses the elegant identification of Kähler action as Kähler magnetic flux since Kähler magnetic flux vanishes. Apart from proportionality constant Kähler electric flux

$$\int_{Y^2} *J$$

is as a dimensionless scaling invariant a natural candidate for Kähler action but need not be extremum if braids are Legendrian sub-manifolds whereas for Kähler magnetic flux this is the case. There is however an explicit dependence on metric which does not conform with the idea that almost topological QFT is symplectic QFT.

3. The sign factor of the dual flux which depends on the orientation of the string world sheet and thus changes sign when the orientation of space-time sheet is changed by changing that of the string world sheet. This is in conflict with the independence of Kähler action on orientation. One can however argue that the orientation makes itself actually physically visible via the weak form of electric-magnetic duality. If the above discussed duality holds true, the net contribution to Kähler action would vanish as the total Kähler magnetic flux for partonic 2-surfaces. Therefore the duality cannot hold true if Kähler action reduces to dual flux.

4. There is also a purely formal counter argument. The inverse images of Lagrangian sub-manifolds of $CP_2$ can be 4-dimensional (cosmic strings and massless extremals) whereas string world sheets are 2-dimensonal.

#### String world sheets as minimal surfaces

Effective 2-dimensionality suggests a reduction of Kähler action to Chern-Simons terms to the area of minimal surfaces defined by string world sheets holds true [K24]. Skeptic could argue that the expressibility of Kähler action involving no dimensional parameters except $CP_2$ scaled does not favor this proposal. The connection of minimal surface property with holomorphy and conformal invariance however forces to take the proposal seriously and it is easy to imagine how string tension emerges since the size scale of $CP_2$ appears in the induced metric [K24].

One can ask whether the mimimal surface property conforms with the proposal that string worlds sheets obey the weak form of electric-magnetic duality and with the proposal that they are generalized Lagrangian sub-manifolds.

1. The basic answer is simple: minimal surface property and possible additional conditions (Lagrangian sub-manifold property or the weak form of electric magnetic duality) poses only additional conditions forcing the space-time sheet to be such that the imbedded string world sheet is a minimal surface of space-time surface: minimal surface property is a condition on space-time sheet rather than string world sheet. The weak form of electric-magnetic duality is favored because it poses conditions on the first derivatives in the normal direction unlike Lagrangian sub-manifold property.

2. Any proposal for 2-D expression of Kähler action should be consistent with the proposed real-octonion analytic solution ansatz for the preferred extremals [K5]. The ansatz is based on real-octonion analytic map of imbedding space to itself obtained by algebraically continuing real-complex analytic map of 2-D sub-manifold of imbedding space to another such 2-D sub-manifold. Space-time surface is obtained by requiring that the "imaginary" part of the map vanishes so that image point is hyper-quaternion valued. Wick rotation allows to formulate the conditions using octonions and quaternions. Minimal surfaces (of space-time surface) are indeed objects for which the imbedding maps are holomorphic and the real-octonion analyticity could be perhaps seen as algebraic continuation of this property.

3. Does Kähler action for the preferred exremals reduce to the area of the string world sheet or to Kähler magnetic flux or are the representations equivalent so that the induced Kähler form would effectively define area form? If the Kähler form form associated with the induced metric on string world sheet is proportional to the induced Kähler form the Kähler magnetic flux is proportional to the area and Kähler action reduces to genuine area. Could one pose this condition as an additional constraint on string world sheets? For Lagrangian sub-manifolds Kähler electric field should be proportional to the area form and the condition involves information about space-time surface and is therefore more complex and does not look plausible.

### Explicit conditions expressing the minimal surface property of the string world sheet

It is instructive to write explicitly the condition for the minimal surface property of the string world sheet and for the reduction of the area Kähler form to the induced Kähler form. For string world sheets with Minkowskian signature of the induced metric Kähler structure must be replaced by its hyper-complex analog involving hyper-complex unit $e$ satisfying $e^2 = 1$ but replaced with real unit at the level hyper-complex coordinates. $e$ can be represented as antisymmetric Kähler form $J_g$ associated with the induced metric but now one has $J_g^2 = g$ instead of $J_g^2 = -g$. The condition that the signed area reduces to Kähler electric flux means that $J_g$ must be proportional to the induced Kähler form: $J_g = kJ$, $k = constant$ in a given space-time region.

One should make an educated guess for the imbedding of the string world sheet into a preferred extremal of Kähler action. To achieve this it is natural to interpret the minimal surface property as a condition for the preferred Kähler extremal in the vicinity of the string world sheet guaranteing that the sheet is a minimal surface satisfying $J_g = kJ$. By the weak form of electric-magnetic duality partonic 2-surfaces represent both electric and magnetic monopoles. The weak form of electric-magnetic duality requires for string world sheets that the Kähler magnetic field at string world sheet is proportional to the component of the Kähler electric field parallel to the string world sheet. Kähler electric field is assumed to have component only in the direction of string world sheet.

*1. Minkowskian string world sheets*

Let us try to formulate explicitly the conditions for the reduction of the signed area to Kähler electric flux in the case of Minkowskian string world sheets.

1. Let us assume that the space-time surface in Minkowskian regions has coordinates coordinates $(u, v, w, \overline{w})$ [K5]. The pair $(u, v)$ defines light-like coordinates at the string world sheet having identification as hyper-complex coordinates with hyper-complex unit satisfying $e = 1$. $u$ and $v$ need not - nor cannot as it turns out - be light-like with respect to the metric of the space-time surface. One can use $(u, v)$ as coordinates for string world sheet and assume that $w = x^1 + ix^2$ and $\overline{w}$ are constant for the string world sheet. Without a loss of generality one can assume $w = \overline{w} = 0$ at string world sheet.

2. The induced Kähler structure must be consistent with the metric. This implies that the induced metric satisfies the conditions

$$g_{uu} = g_{vv} = 0 \ . \tag{7.7.1}$$

   The analogs of these conditions in regions with Euclidian signature would be $g_{zz} = g_{\overline{z}\overline{z}} = 0$.

3. Assume that the imbedding map for space-time surface has the form

$$s^m = s^m(u, v) + f^m(u, v, x^m)_{kl} x^k x^l \ , \tag{7.7.2}$$

   so that the conditions

$$\partial_l k s^m = 0 \ , \ \partial_k \partial_u s^m = 0, \ \ \partial_k \partial_v s^m = 0 \tag{7.7.3}$$

   are satisfies at string world sheet. These conditions imply that the only non-vanishing components of the induced $CP_2$ Kähler form at string world sheet are $J_{uv}$ and $J_{w\overline{w}}$. Same applies to the induced metric if the metric of $M^4$ satisfies these conditions (no non-vanishing components of form $m_{uk}$ or $m_{vk}$).

4. Also the following conditions hold true for the induced metric of the space-time surface

$$\partial_k g_{uv} = 0 \ , \ \partial_u g_{kv} = 0 \ , \ \partial_v g_{ku} = 0 \ . \tag{7.7.4}$$

at string world sheet as is easy to see by using the ansatz.

Consider now the minimal surface conditions stating that the trace of the four components of the second fundamental form whose components are labelled by the coordinates $\{x^\alpha\} \equiv (u, v, w, \overline{w})$ vanish for string world sheet.

1. Since only $g_{uv}$ is non-vanishing, only the components $H^k_{uv}$ of the second fundamental form appear in the minimal surface equations. They are given by the general formula

$$\begin{aligned} H^\alpha_{uv} &= H^\gamma P^\alpha_\gamma \ , \\ H^\alpha &= \left(\partial_u\partial_v x^\alpha + \left({}_\beta{}^\alpha{}_\gamma\right) \partial_u x^\beta \partial_v x^\gamma\right) \ . \end{aligned} \tag{7.7.5}$$

   Here $P^\alpha_\gamma$ is the projector to the normal space of the string world sheet. Formula contains also Christoffel symbols $\left({}_\beta{}^\alpha{}_\gamma\right)$.

2. Since the imbedding map is simply $(u, v) \to (u, v, 0, 0)$ all second derivatives in the formula vanish. Also $H^k = 0, k \in \{w, \overline{w}\}$ holds true. One has also $\partial_u x^\alpha = \delta^\alpha_u$ and $\partial_v x^\beta = \delta^\beta_v$. This gives

$$H^\alpha = \left({}_u{}^\alpha{}_v\right) \ . \tag{7.7.6}$$

   All these Christoffel symbols however vanish if the assumption $g_{uu} = g_{vv} = 0$ and the assumptions about imbedding ansatz hold true. Hence a minimal surface is in question.

Consider now the conditions on the induced metric of the string world sheet

1. The conditions reduce to

$$g_{uu} = g_{vv} = 0 \ . \tag{7.7.7}$$

   The conditions on the diagonal components of the metric are the analogs of Virasoro conditions fixing the coordinate choices in string models. The conditions state that the coordinate lines for $u$ and $v$ are light-like curves in the induced metric.

2. The conditions can be expressed directly in terms of the induced metric and read

$$\begin{aligned} m_{uu} + s_{kl}\partial_u s^k \partial_u s^l &= 0 \ , \\ m_{vv} + s_{kl}\partial_v s^k \partial_v s^l &= 0 \ . \end{aligned} \tag{7.7.8}$$

   The $CP_2$ contribution is negative for both equations. The conditions make sense only for $(m_{uu} > 0, m_{vv} > 0)$. Note that the determinant condition $m_{uu}m_{vv} - m_{uv}m_{vu} < 0$ expresses the Minkowskian signature of the $(u, v)$ coordinate plane in $M^4$.

The additional condition states

$$J^g_{uv} = kJ_{uv} \ . \tag{7.7.9}$$

It reduces signed area to Kähler electric flux. If the weak form of electric-magnetic duality holds true one can interpret the area as magnetic flux defined as the flux of the dual of induced Kähler form over space-like surface and defining electric charge. A further condition is that the boundary of string world sheet is Legendrean manifold so that the flux and thus area is extremized also at the boundaries.

*2.Conditions for the Euclidian string world sheets*

One can do the same calculation for string world sheet with Euclidian signature. The only difference is that $(u,v)$ is replaced with $(z,\overline{z})$. The imbedding map has the same form assuming that space-time sheet with Euclidian signature allows coordinates $(z,\overline{z},w,\overline{w})$ and the local conditions on the imbedding are a direct generalization of the above described conditions. In this case the vanishing for the diagonal components of the string world sheet metric reads as

$$\begin{aligned} h_{kl}\partial_z s^k \partial_z s^l &= 0 \ , \\ h_{kl}\partial_{\overline{z}} s^k \partial_{\overline{z}} s^l &= 0 \ . \end{aligned} \tag{7.7.10}$$

The natural ansatz is that complex $CP_2$ coordinates are holomorphic functions of the complex coordinates of the space-time sheet.

*3. Wick rotation for Minkowskian string world sheets leads to a more detailed solution ansatz*

Wick rotation is a standard trick used in string models to map Minkowskian string world sheets to Euclidian ones. Wick rotation indeed allows to define what one means with real-octonion analyticity. Could one identify string world sheets in Minkowskian regions by using Wick rotation and does this give the same result as the direct approach?

Wick rotation transforms space-time surfaces in $M^4 \times CP_2$ to those in $E^4 \times CP_2$. In $E^4 \times CP_2$ octonion real-analyticity is a well-defined notion and one can identify the space-time surfaces surfaces at which the imaginary part of of octonion real-analytic function vanishes: imaginary part is defined via the decomposition of octonion to two quaternions as $o = q_1 + Iq_2$ where $I$ is a preferred octonion unit. The reverse of the Wick rotation maps the quaternionic surfaces to what might be called hyper-quaternionic surfaces in $M^4 \times CP_2$.

In this picture string world sheets would be hyper-complex surfaces defined as inverse imagines of complex surfaces of quaternionic space-time surface obtained by the inverse of Wick rotation. For this approach to be equivalent with the above one it seems necessary to require that the the treatment of the conditions on metric should be equivalent to that for which hyper-complex unit $e$ is not put equal to 1. This would mean that the conditions reduce to independent conditions for the real and imaginary parts of the real number formally represented as hyper-complex number with $e = 1$.

Wick rotation allows to guess the form of the ansatz for $CP_2$ coordinates as functions of space-time coordinates In Euclidian context holomorphich functions of space-time coordinates are the natural ansatz. Therefore the natural guess is that one can map the hypercomplex number $t \pm ez$ to complex coordinate $t \pm iz$ by the analog of Wick rotation and assume that $CP_2$ complex coordinates are analytic functions of the complex space-time coordinates obtained in this manner.

The resulting induced metric could be obtained directly using real coordinates $(t,z)$ for string world sheet or by calculating the induced metric in complex coordinates $t \pm iz$ and by mapping the expressions to hyper-complex numbers by Wick rotation (by replacing $i$ with $e = 1$). If the diagonal components of the induced metric vanish for $t \pm iz$ they vanish also for hyper-complex coordinates so that this approach seem to make sense.

### Electric-magnetic duality for flux Hamiltonians and the existence of Wilson sheets

One must distinguish between two conjectured dualities. The weak form of electric-magnetic duality and the duality between string world sheets and partonic 2-surfaces. Could the first duality imply equivalence of not only electric and magnetic flux Hamiltonians but also electric and magnetic Wilson sheets? Could the latter duality allow two different representations of flux Hamiltonians?

1. For electric-magnetic duality holding true at string world sheets one would have non-vanishing Kähler form and the fluxes would be non-vanishing. The Hamiltonian fluxes

$$Q_{m,A} = \int_{X^2} JH_A dx^1 dx^2 = \int_{X^2} H_A J_{\alpha\beta} dx^\alpha \wedge dx^\beta \qquad (7.7.11)$$

   for partonic 2-surfaces $X^2$ define WCW Hamiltonians playing a key role in the definition of WCW Kähler geometry. They have also interpretation as a generalization of Wilson loops to Wilson 2-surfaces.

2. Weak form of electric magnetic duality would imply both at partonic 2-surfaces and string world sheets the proportionality

$$Q_{m,A} = \int_{X^2} JH_A dx^1 \wedge dx^2 \propto Q^*_{m,A} = \int_{X^2} H_A * J_{\alpha\beta} dx^\alpha \wedge dx^\beta \quad . \qquad (7.7.12)$$

   Thefore the electric-magnetic duality would have a concrete meaning also at the level of WCW geometry.

3. If string world sheets are Lagrangian sub-manifolds Hamiltonian fluxes would vanish identically so that the identification as Wilson sheets does not make sense. One would lose electric-magnetic duality for flux sheets. The dual fluxes

$$*Q_A = \int_{Y^2} *JH_A dx^1 \wedge dx^2 = \int_{Y^2} \epsilon_{\alpha\beta}{}^{\gamma\delta} J_{\gamma\delta} = \int_{Y^2} \frac{\sqrt{det(g_4)}}{det(g_2^\perp)} J^\perp_{34} dx^1 \wedge dx^2$$

   for string world sheets $Y^2$ are however non-vanishing. Unlike fluxes, the dual fluxes depend on the induced metric although they are scaling invariant.

Under what conditions the conjectured duality between partonic 2-surface and string world sheets hold true at the level of WCW Hamiltonians?

1. For the weak form of electric-magnetic duality at string world sheets the duality would mean that the sum of the fluxes for partonic 2-surfaces and sum of the fluxes for string world sheets are identical apart from a proportionality constant:

$$\sum_i Q_A(X_i^2) \propto \sum_i Q_A(Y_i^2) \quad . \qquad (7.7.13)$$

   Note that in zero ontology it seems necessary to sum over all the partonic surfaces (at both ends of the space-time sheet) and over all string world sheets.

2. For Lagrangian sub-manifold option the duality can hold true only in the form

$$\sum_i Q_A(X_i^2) \propto \sum_i Q^*_A(Y_i^2) \quad . \qquad (7.7.14)$$

   Obviously this option is less symmetric and elegant.

#### Summary

There are several arguments favoring weak form of electric-magnetic duality for both string world sheets and partonic 2-surfaces. Legendrian sub-manifold property for braid strands follows from the assumption that Kähler action for preferred extremals is proportional to the Kähler magnetic flux associated with preferred 2-surfaces and is stationary with respect to the variations of the boundary. What is especially nice is that Legendrian sub-manifold property implies automatically unique braids. The minimal option favored by the idea that 3-surfaces are basic dynamical objects is the one for which weak form of electric-magnetic duality holds true only at partonic 2-surfaces and string world sheets. A stronger option assumes it at preferred 3-surfaces. Duality between string world sheets and partonic 2-surfaces suggests that WCW Hamiltonians can be defined as sums of Kähler magnetic fluxes for either partonic 2-surfaces or string world sheets.

### 7.7.5 What generalized Feynman rules could be?

After all these explanations the skeptic reader might ask whether this lengthy discussion gives any idea about what the generalized Feynman rules might look like. The attempt to answer this question is a good manner to make a map about what is understood and what is not understood. The basic questions are simple. What constraints does zero energy ontology (ZEO) pose? What does the necessity to projecti the four-momenta to a preferred plane $M^2$ mean? What mathematical expressions one should assign to the propagator lines and vertices? How does one perform the functional integral over 3-surfaces in finite measurement resolution? The following represents tentatative answers to these questions but does not say much about exact role of algebraic knots.

#### Zero energy ontology

ZEO poses very powerful constraints on generalized Feynman diagrams and gives hopes that both UV and IR divergences cancel.

1. ZEO predicts that the fermions assigned with braid strands associated with the virtual particles are on mass shell massless particles for which the sign of energy can be also negative: in the case of wormhole throats this can give rise to a tachyonic exchange.

2. The on mass shell conditions for each wormhole throat in the diagram involving loops are very stringent and expected to eliminate very large classes of diagrams. If however given diagonal diagram leading from n-particle state to the same n-particle state -completely analogous to self energy diagram- is possible then the ladders form by these diagrams are also possible and one one obtains infinite of this kind of diagrams as generalized self energy correction and is excellent hopes that geometric series gives a closed algebraic function.

3. IR divergences plaguing massless theories are cancelled if the incoming and outgoing particles are massive bound states of massless on mass shell particles. In the simplest manner this is achieved when the 3-momenta are in opposite direction. For internal lines the massive on-mass shell-condition is not needed at all. Therefore there is an almost complete separation of the problem how bound state masses are determined from the problem of constructing the scattering amplitudes.

4. What looks like a problematic aspect ZEO is that the massless on-mass-shell propagators would diverge for wormhole throats. The solution comes from the projection of 4-momenta to $M^2$. In the generic the projection is time-like and one avoids the singularity. The study of solutions of the modified Dirac equation [K19] and number theoretic vision [K52] indeed suggests that the four-momenta are obtained by rotating massless $M^2$ momenta and their projections to $M^2$ are in general integer multiples of hyper-complex primes or light-like. The light-like momenta would be treated like in the case of ordinary Feynman diagrams using $i\epsilon$-prescription of the propagator and would also give a finite contributions corresponding to integral over physical on mass shell states. This guarantees also the vanishing of the possible IR divergences coming from the summation over different $M^2$ momenta.

   There is a strong temptation to identify - or at least relate - the $M^2$ momenta labeling the solutions of the modified Dirac equation with the region momenta of twistor approach [K63].

The reduction of the region momenta to $M^2$ momenta could dramatically simplify the twistorial description. It does not seem however plausible that $\mathcal{N} = 4$ super-symmetric gauge theory could allow the identification of $M^2$ projections of 4-momenta as region momenta. On the other hand, there is no reason to expect the reduction of TGD certainly to a gauge theory containing QCD as part. For instance, color magnetic flux tubes in many-sheeted space-time are central for understanding jets, quark gluon plasma, hadronization and fragmentation [L11] but cannot be deduced from QCD. Note also that the splitting of parton momenta to their $M^2$ projections and transversal parts is an ad hoc assumption motivated by parton model rather than first principle implication of QCD: in TGD framework this splitting would emerge from first principles.

5. Zero energy ontology strongly suggests that all particles (including photons, gluons, and gravitons) have mass which can be arbitrarily small and can be see as being due to the fact that particle "eats" Higgs like states giving it the otherwise lacking polarization states. This would mean a generalization of the notion of Higgs particle to a Higgs like particle with spin. It would also mean rearrangmenet of massless states at wormhole throat level to massives physical states.

   The projection of the momenta to $M^2$ is consistent with this vision. The natural generalization of the gauge condition $p{\cdot}\epsilon = 0$ is obtained by replacing $p$ with the projection of the total momentum of the boson to $M^2$ and $\epsilon$ with its polarization so that one has $p_{||} \cdot \epsilon$. If the projection to $M^2$ is light-like, three polarization states are possible in the generic case, so that massivation is required by internal consistency. Note that if intermediate states in the unitary condition were states with light-like $M^2$-momentum one could have a problematic situation.

6. A further natural assumption is that the $M^2$ projections of all momenta assignable to braid strands are parallel. Only the projections of the momenta to the orthogonal complement $E^2$ of $M^2$ can be non-parallel and for massive wormhole throats they must be non-parallel. This assumption does not break Lorentz invariance since in the full amplitude one must integrate over possible choices of $M^2$. It also interpret the gauge conditions either at the level of braid strands or of partons. Quantum classical correspondence in strong form would actually suggests that quantum 4-momenta should co-incide with the classical ones. The restriction to $M^2$ projections is however necessary and seems also natural. For instance, for massless extremals only $M^2$ projection of wave-vector can be well-defined: in transversal degrees of freedom there is a superposition over Fourier components with diffrent transversal wave-vectors. Also the partonic description of hadrons gives for the $M^2$ projections of the parton momenta a preferred role. It is highly encouraging that this picture emerged first from the modified Dirac equation and purely number theoretic vision based on the identification of $M^2$ momenta in terms of hyper-complex primes.

   The number theoretical approach also suggests a number theoretical quantization of the transversal parts of the momenta [K52]: four-momenta would be obtained by rotating massless $M^2$ momenta in $M^4$ in such a manner that the components of the resulting 3-momenta are integer valued. This leads to a classical problem of number theory which is to deduce the number of 3-vectors of fixed length with integer valued components. One encounters the n-dimensional generalization of this problem in the construction of discrete analogs of quantum groups (these "classical" groups are analogous to Bohr orbits) and emerge in quantum arithmetics [K65], which is a deformation of ordinary arithmetics characterized by p-adic prime and giving rigorous justification for the notion of canonical identification mapping p-adic numbers to reals.

7. The real beauty of Feynman rules is that they guarantee unitarity automatically. In fact, unitarity reduces to Cutkosky rules which can be formulated in terms of cut obtained by putting certain subset of interal lines on mass shell so that it represents on mass shell state. Cut analyticity implies the usual $iDisc(T) = TT^\dagger$. In the recent context the cutting of the internal lines by putting them on-mass-shell requires a generalization.

   (a) The first guess is that on mass shell property means that $M^2$ projection for the momenta is light-like. This would mean that also these momenta contribute to the amplitude but the contribution is finite just like in the usual case. In this formulation the real particles would be the massless wormhole throats.

(b) Second possibility is that the internal lines on on mass shell states corresponding to massive on mass-shell-particles. This would correspond to the experimental meaning of the unitary conditions if real particles are the massive on mass shell particles. Mathematically it seems possible to pick up from the amplitude the states which correspond to massive on mass shell states but one should understand why the discontinuity should be associated with physical net masses for wormhole contacts or many-particle states formed by them. General connection with unitarity and analyticity might allow to understand this.

8. $CD$s are labelled by various moduli and one must integrate over them. Once the tips of the $CD$ and therefore a preferred $M^1$ is selected, the choice of angular momentum quantization axis orthogonal to $M^1$ remains: this choice means fixing $M^2$. These choices are parameterized by sphere $S^2$. It seems that an integration over different choices of $M^2$ is needed to achieve Poincare invariance.

**How the propagators are determined?**

In accordance with previous sections it will be assumed that the braid are Legendrian braids and therefore completely well-defined. One should assign propagator to the braid. A good guess is that the propagator reduces to a product of three terms.

1. A multi-particle propagator which is a product of collinear massless propagators for braid strands with fermionin number $F = 0, 1-1$. The constraint on the momenta is $p_i = \lambda_i p$ with $\sum_i \lambda_i = 1$. So that the fermionic propagator is $\frac{1}{\prod_i \lambda_i} p^k \gamma_k$. If one gas $p = nP$, where $P$ is hyper-complex prime, one must sum over combinations of $\lambda_i = n_i$ satisfying $\sum_i n_i = n$.

2. A unitary $S$-matrix for integrable QFT in $M^2$ in which the velocities of particles assignable to braid strands appear for which fixed by $R$-matrix defines the basic 2-vertex representing the process in which a particle passes through another one. For this $S$-matrix braids are the basic units. To each crossing appearing in non-planar Feynman diagram one would have an $R$-matrix representing the effect of a reconnection the ends of the lines coming to the crossing point. In this manner one could gradually transform the non-planar diagram to a planar diagram. One can ask whether a formulation in terms of a suitable R-matrix could allow to generalize twistor program to apply in the case of non-planar diagrams.

3. An S-matrix predicted by topological QFT for a given braid. This S-matrix should be constructible in terms of Chern-Simons term defining a sympletic QFT.

There are several questions about quantum numbers assignable to the braid strands.

1. Can braid strands be only fermionic or can they also carry purely bosonic quantum numbers corresponding to WCW Hamiltonians and therefore to Hamiltonians of $\delta M^4_\pm \times CP_2$? Nothing is lost if one assumes that both purely bosonic and purely fermionic lines are possible and looks whether this leads to inconsistencies. If virtual fermions correspond to single wormhole throat they can have only time-like $M^2$-momenta. If virtual fermions correspond to pairs of wormhole throats with second throat carrying purely bosonic quantum numbers, also fermionic can have space-like net momenta. The interpretation would be in terms of topological condensation. This is however not possible if all strands are fermionic. Situation changes if one identifies physical fermions wormhole throats at the ends of Kähler magnetic flux tube as one indeed does: in this case virtual net momentum can be space-like if the sign of energy is opposite for the ends of the flux tube.

2. Are the 3-momenta associated with the wormholes of wormhole contact parallel so that only the sign of energy could distinguish between them for space-like total momentum and $M^2$ mass squared would be the same? This assumption simplifies the situation but is not absolutely necessary.

3. What about the momentum components orthogonal to $M^2$? Are they restricted only by the massless mass shell conditions on internal lines and quantization of the $M^2$ projection of 4-momentum?

4. What braids do elementary particles correspond? The braids assigned to the wormhole throat lines can have arbitrary number $n$ of strands and for $n = 1, 2$ the treatment of braiding is almost trivial. A natural assumption is that propagator is simply a product of massless collinear propagators for $M^2$ projection of momentum [K20]. Collinearity means that propagator is product of a multifermion propagator $\frac{1}{\lambda_i p_k \gamma_k}$, znd multiboson propagator $\frac{1}{\mu_i p_k \gamma_k}$, $\sum \lambda_i + \sum_i \mu_i = 1$. There are also quantization conditions on $M^2$ projections of momenta from modified Dirac equation implying that multiplies of hyper-complex prime are in question in suitable units. Note however that it is not clear whether purely bosonic strands are present.

5. For ordinary elementary particles with propagators behaving like $\prod_i \lambda_i^{-1} 1 p^{-n}$, only $n \leq 2$ is possible. The topologically really interesting states with more than two braid strands are something else than what we have used to call elementary particles. The proposed interpretation is in terms of anyonic states [K40]. One important implication is that $\mathcal{N} = 1$ SUSY generated by right-handed neutrino or its antineutrino is SUSY for which all members of the multiplet assigned to a wormhole throat have braid number smaller than 3. For $\mathcal{N} = 2$ SUSY generated by right-handed neutrino and its antiparticle the states containing fermion and neutrino-antineutrino pair have three braid strands and SUSY breaking is expected to be strong.

### Vertices

Conformal invariance raises the hope that vertices can be deduced from super-conformal invariance as n-point functions. Therefore lines would come from integrable QFT in $M^2$ and topological braid theory and vertices from confofmal field theory: both theories are integrable.

The basic questions is how the vertices are defined by the 2-D partonic surfaces at which the ends of lines meet. Finite measurement resolution reduces the lines to braids so that the vertices reduces to the intersection of braid strands with the partonic 2-surface.

1. Conformal invariance is the basic symmetry of quantum TGD. Does this mean that the vertices can be identified as $n$-point functions for points of the partonic 2-surface defined by the incoming and outgoing braid strands? How strong constraints can one pose on this conformal field theory? Is this field theory free and fixed by anticommutation relations of induced spinor fields so that correlation function would reduce to product of fermionic two points functions with standard operator in the vertices represented by strand ends. If purely bosonic vertices are present, their correlation functions must result from the functional integral over WCW.

2. For the fermionic fields associated with each incoming braid the anticommutators of fermions and antifermions are trivial just as the usual equal time anticommutation relations. This means that the vertex reduces to sum of products of fermionic correlation functions with arguments belonging to different incoming and outgoing lines. How can one calculate the correlators?

   (a) Should one perform standard second quantization of fermions at light-like 3-surface allowing infinite number of spinor modes, apply a finite measurement resolution to obtain braids, for each partonic 2-surface, and use the full fermion fields to calculate the correlators? In this case braid strands would be discontinuous in vertices. A possible problem might be that the cutoff in spinor modes seems to come from the theory itself: finite measurement resolution is a property of quantum state itself.

   (b) Could finite measurement resolution allow to approximate the braid strands with continuous ones so that the correlators between strands belonging to different lines are given by anticommutation relations? This would simplify enormously the situation and would conform with the idea of finite measurement resolution and the vision that interaction vertices reduce to braids. This vision is encouraged by the previous considerations and would mean that replication of braid strands analogous to replication of DNA strands can be seen as a fundamental process of Nature. This of course represents an important deviation from the standard picture.

3. Suppose that one accepts the latter option. What can happen in the vertex, where line goes from one braid to another one?

(a) Can the direction of momentum changed as visual intuition suggests? Is the total braid momentum conservation the only constraint so that the velocities assignable braid strands in each line would be constrained by the total momentum of the line.

(b) What kind of operators appear in the vertex? To get some idea about this one can look for the simplest possible vertex, namely FFB vertex which could in fact be the only fundamental vertex as the arguments of [K11] suggest. The propagator of spin one boson decomposes to product of a projection operator to the polarization states divited by $p^2$ factor. The projection operator sum over products $\epsilon_i^k \gamma_k$ at both ends where $\gamma_k$ acts in the spinor space defined by fermions. Also fermion lines have spinor and its conjugate at their ends. This gives rise to $p^k \gamma_k / p^2$. $p^k \gamma_k$ is the analog of the bosonic polarization tensor factorizing into a sum over products of fermionic spinors and their conjugates. This gives the BFF vertex $\epsilon_i^k \gamma_k$ slashed between the fermionic propagators which are effectively 2-dimensional.

(c) Note that if H-chiralities are same at the throats of the wormhole contact, only spin one states are possible. Scalars would be leptoquarks in accordance with general view about lepton and quark number conservation. One particular implication is that Higgs in the standard sense is not possible in TGD framework. It can appear only as a state with a polarization which is in $CP_2$ direction. In any case, Higgs like states would be eaten by massless state so that all particles would have at least a small mass.

### Functional integral over 3-surfaces

The basic question is how one can functionally integrate over light-like 3-surfaces or space-like 3-surfaces.

1. Does effective 2-dimensionality allow to reduce the functional integration to that over partonic 2-surfaces assigned with space-time sheet inside $CD$ plus radiative corrections from the hierarchy of sub-$CD$s?

2. Does finite measurement resolution reduce the functional integral to a ordinary integral over the positions of the end points of braids and could this integral reduce to a sum? Symplectic group of $\delta M^4_\pm \times CP_2$ basically parametrizes the quantum fluctuating degrees of freedom in WCW. Could finite measurement resolution reduce the symplectic group of $\delta M^4_\pm \times CP_2$ to a coset space obtained by dividing with symplectic transformations leaving the end points invariant and could the outcome be a discrete group as proposed? Functional integral would reduce to sum.

3. If Kähler action reduces to Chern-Simons-Kähler terms to surface area terms in the proposed manner, the integration over WCW would be very much analogous to a functional integral over string world sheets and the wisdom gained in string models might be of considerable help.

### Summary

What can one conclude from these argument? To my view the situation gives rise to a considerable optimism. I believe that on basis of the proposed picture it should be possible to build a concrete mathematical models for the generalized Feynman graphics and the idea about reduction to generalized braid diagrams having algebraic representations could pose additional powerful constraints on the construction. Braid invariants could also be building bricks of the generalized Feynman diagrams. In particular, the treatment of the non-planarity of Feynman diagrams in terms of $M^2$ braiding matrix would be something new.

## 7.8 Electron as a trefoil or something more general?

The possibility that electron, and also other elementary particles could correspond to knot is very interesting. The video model [B21] was so fascinating (I admire the skills of the programmers) that I started to question my belief that all related to knots and braids represents new physics (say anyons), [K40] and that it is hopeless to try to reduce standard model quantum numbers with purely group theoretical explanation (except family replication) to topological quantum numbers.

Electroweak and color quantum numbers should by quantum classical correspondence have geometric correlates in space-time geometry. Could these correlates be topological? As a matter of fact, the correlates existing if the present understanding of the situation is correct but they are not topological.

Despite this, I played with various options and found that in TGD Universe knot invariants do not provide plausible space-time correlates for electroweak quantum numbers. The knot invariants and many other topological invariants are however present and mean new physics. As following arguments try to show, elementary particles in TGD Universe are characterized by extremely rich spectrum of topological quantum numbers, in particular those associated with knotting and linking: this is basically due to the 3-dimensionality of 3-space.

For a representation of trefoil knot by R.W. Gray see
http://www.rwgrayprojects.com/Lynn/Presentation20070926/p008.html. The homepage of Louis Kauffman [A9] is a treasure trove for anyone interested in ideas related to possible applications of knots to physics. One particular knotty idea is discussed in the article Emergent Braided Matter of Quantum Geometry by Bilson-Thompson, Hackett, and Kauffman [B39].

### 7.8.1 Space-time as 4-surface and the basic argument

Space-time as a 4-surface in $M^4 \times CP_2$ is the key postulate. The dynamics of space-time surfaces is determined by so called Kähler action - essentially Maxwell action for the Kähler form of $CP_2$ induced to $X^4$ in induced metric. Only so called preferred extremals are accepted and one can in very loose sense say that general coordinate invariance is realized by assigning to a given 3-surface a unique 4-surface as a preferred extremal analogous to Bohr orbit for a particle identified as 3-D surface rather than point-like object.

One ends up with a radical generalization of space-time concept to what I call many-sheeted space-time. The sheets of many-sheeted space-time are at distance of $CP_2$ size scale ($10^4$ Planck lengths as it turns out) and can touch each other which means formation of wormhole contact with wormhole throats as its ends. At throats the signature of the induced metric changes from Minkowskian to Euclidian. Euclidian regions are identified as 4-D analogs of lines of generalized Feynman diagrams and the $M^4$ projection of wormhole contact can be arbitrarily large: macroscopic, even astrophysical. Macroscopic object as particle like entity means that it is accompanied by Euclidian region of its size.

Elementary particles are identified as wormhole contacts. The wormhole contacts born in mere touching are not expected to be stable. The situation changes if there is a monopole magnetic flux ($CP_2$ carries self dual purely homological monopole Kähler form defining Maxwell field, this is not Dirac monopole) since one cannot split the contact. The lines of the Kähler magnetic field must be closed, and this requires that there is another wormhole contact nearby. The magnetic flux from the upper throat of contact A travels to the upper throat of contact B along "upper space-time sheet", goes to "lower" space-time sheet along contact B and returns back to the wormhole contact A so that closed loop results.

In principle, wormhole throat can have arbitrary orientable topology characterized by the number $g$ of handles attached to sphere and known as genus. The closed flux tube corresponds to topology $X_g^2 \times S^1$, g=0,1,2,... Genus-generation correspondence [K11] states that electron, muon, and tau lepton and similarly quark generations correspond to $g = 0, 1, 2$ in TGD Universe and CKM mixing is induced by topological mixing.

Suppose that one can assign to this flux tube a closed string: this is indeed possible but I will not bother reader with details yet. What one can say about the topology of this string?

1. $X_g^2$ has homology $Z^{2g}$ and $S^1$ homology $S^1$. The entire homology is $Z^{2g+1}$ so that there are $2g + 1$ additional integer valued topological quantum numbers besides genus. $Z^{2g+1}$ obviously breaks topologically universality stating that fermion generations are exact copies of each other apart from mass. This would be new physics. If the size of the flux loop is of order Compton length, the topological excitations need not be too heavy. One should however know how to excite them.

2. The circle $S^1$ is imbedded in 3-surface and can get knotted. This means that all possible knots characterize the topological states of the the fermion. Also this means extremely rich spectrum of new physics.

### 7.8.2 What is the origin of strings going around the magnetic flux tube?

What is then the origin of these knotted strings? The study of the modified Dirac equation [K67] determining the dynamics of induced spinor fields at space-time surface led to a considerable insight here. This requires however additional notions such as zero energy ontology (ZEO), and causal diamond ($CD$) defined as intersection of future and past directed light-cones (double 4-pyramid is the $M^4$ projection. Note that $CD$ has $CP_2$ as Cartesian factor and is analogous to Penrose diagram.

1. ZEO means the assumption that space-time surfaces for a particular sub-WCW ("world of classical worlds") are contained inside given $CD$ identifiable as a the correlate for the "spotlight of consciousness" in TGD inspired theory of consciousness. The space-time surface has ends at the upper and lower light-like boundaries of $CD$. The 3-surfaces at the the ends define space-time correlates for the initial and final states in positive energy ordinary ontology. In ZEO they carry opposite total quantum numbers.

2. General coordinate invariance (GCI) requires that once the 3-D ends are known, space-time surface connecting the ends is fixed (there is not path integral since it simply fails). This reduces ordinary holography to GCI and makes classical physics defined by preferred extremals an exact part of quantum theory, actually a key element in the definition of Kähler geometry of WCW.

   Strong form of GCI is also possible. One can require that 3-D light-like orbits of wormhole throats at which the induced metric changes its signature, and space-like 3-surfaces at the ends of $CD$ givee equivalent descriptions. This implies that quantum physics is coded by the their intersections which I call partonic 2-surfaces - wormhole throats - plus the 4-D tangent spaces of $X^4$ associated with them. One has strong form of holography. Physics is almost 2-D but not quite: 4-D tangent space data is needed.

3. The study of the modified Dirac equation [K67] leads to further results. The mere conservation of electromagnetic charge defined group theoretically for the induced spinors of $M^4 \times CP_2$ carrying spin and electroweak quantum numbers implies that for all other fermion states except right handed neutrino (, which does not couple at all all to electroweak fields), are localized at 2-D string world sheets and partonic 2-surfaces.

   String world sheets intersect the light-like orbits of wormhole throats along 1-D curves having interpretation as timelike braid strands (a convenient metaphor: braiding in time direction si created by dancers in the parquette).

   One can say that dynamics automatically implies effective discretization: the ends of time like braid strands at partonic 2-surfaces at the ends of CD define a collection of discrete points to each of which one can assign fermionic quantum numbers.

4. Both throats of the wormhole contact can carry many fermion state and known fermions correspond to states for which either throat carries single braid strand. Known bosons correspond to states for which throats carry fermion and antifermion number.

5. Partonic 2-surface is replaced with discrete set of points effectively. The interpretation is in terms of a space-time correlate for finite measurement resolution. Quantum correlate would be the inclusion of hyperfinite factors of type $II_1$.

This interpretation brings in even more topology!

1. String world sheets - present both in Euclidian and Minkowskian regions - intersect the 3-surfaces at the ends of $CD$ along curves - one could speak of strings. These strings give rise to the closed curves that I discussed above. These strings can be homologically non-trivial - in string models this corresponds to wrapping of branes.

2. For known bosons one has two closed loop but these loops could fuse to single. Space-like 2-braiding (including linking) becomes possible besides knotting.

3. When the partonic 2-surface contains several fermionic braid ends one obtains even more complex situation than above when one has only single braid end. The loops associated with the

braid ends and going around the monopole flux tube can form space-like N-braids. The states containing several braid ends at either throat correspond to exotic particles not identifiable as ordinary elementary particles.

### 7.8.3 How elementary particles interact as knots?

Elementary particles could reveal their knotted and even braided character via the topological interactions of knots. There are two basic interactions.

1. The basic interaction for single string is by self-touching and this can give to a local connected sum or a reconnection. In both cases the knot invariants can change and it is possible to achieve knotting or unknotting of the string by this mechanism. String can also split into two pieces but this might well be excluded in the recent case.

   The space-time dynamics for these interactions is that of closed string model with 4-D target space. The first guess would be topological string model describing only the dynamics of knots. Note that string world sheets define 2-knots and braids.

2. The basic interaction vertex for generalized Feynman diagrams (lines are 4-D space-time regions with Euclidian signature) is join along 3-D boundaries for the three particles involved: this is just like ordinary 3-vertex for Feynman diagrams and is not encountered in string models. The ends of lines must have same genus $g$. In this interaction vertex the homology charges in $Z^{2g+1}$ is conserved so that these charges are analogous to U(1) gauge charges. The strings associated with the two particles can touch each other and connected sum or reconnection is the outcome.

Consider now in more detail connected sum and reconnection vertices responsible for knotting and un-knotting.

1. The first interaction is connected sum of knots [A4]. A little mental exercise demonstrates that a local connected sum for the pieces of knot for which planar projections cross, can lead to a change in knotted-ness. Local connected sum is actually used to un-knot the knot in the construction of knot invariants.

   In dimension 3 knots form a module with respect to the connected sum. One can identify unique prime knots and construct all knots as products of prime knots with product defined as a connected sum of knots. In particular, one cannot have a situation on which a product of two non-trivial knots is un-knot so that one could speak about the inverse of a knot (indeed, the inverse of ordinary prime is not an integer!). For higher-dimensional knots the situation changes (string world sheets at space-time surface could form 2-knots but instead of linking they intersect at discrete points).

   Connected sum in the vertex of generalized Feynman graph (as described above) can lead to a decay of particle to two particles, which correspond to the summands in the connected sum as knots. Could one consider a situation in which un-knotted particle decomposes via the time inverse of the connected sum to a pair of knotted particles such that the knots are inverses of each other? This is not possible since knots do not have inverse.

2. Touching knots can also reconnect. For braids the strands $A \to B$ and $C \to D$ touch and one obtains strands $A \to D$ and $C \to B$. If this reaction takes place for strands whose planar projections cross, it can also change the character of the knot. One one can transform knot to un-knot by repeatedly applying connected sum and reconnection for crossing strands (the Alexandrian way).

3. In the evolution of knots as string world sheets these two vertices corresponds to closed string vertices. These vertices can lead to topological mixing of knots leading to a quantum superposition of different knots for a given elementary particle. This mixing would be analogous to CKM mixing understood to result from the topological mixing of fermion genera in TGD framework. It could also imply that knotted particles decay rapidly to un-knots and make the un-knot the only long-lived state.

A naive application of Uncertainty Principle suggests that the size scale of string determines the life time of particular knot configuration. The dependence on the length scale would however suggest that purely topological string theory cannot be in question. Zero energy ontology suggests that the size scale of the causal diamond assignable to elementary particle determines the time scale for the rates as secondary p-adic time scale: in the case of electron the time scale would be .1 seconds corresponding to Mersenne prime $M_{127} = 2^{127} - 1$ so that knotting and unknotting would be very slow processes. For electron the estimate for the scale of mass differences between different knotted states would be about $10^{-19} m_e$: electron mass is known for certain for 9 decimals so that there is no hope of detecting these mass differences. The pessimistic estimate generalizes to all other elementary particles: for weak bosons characterized by $M_{89}$ the mass difference would be of order $10^{-13} m_W$.

4. A natural guess is that p-adic thermodynamics can be applied to the knotting. In p-adic thermodynamics Boltzmann weights in are of form $p^{H/T}$ (p-adic number) and the allowed values of the Hamiltonian $H$ are non-negative integer powers of $p$. Clearly, $H$ representing a contribution to p-adic valued mass squared must be a non-negative integer valued invariant additive under connected sum. This guarantees extremely rapid convergence of the partition function and mass squared expectation value as the number of prime knots in the decomposition increases.

   An example of an knot invariant [A15] additive under connected sum is knot genus [A14] defined as the minimal genus of 2-surface having the knot as boundary (Seifert surface). For trefoil and figure eight knot one has $g = 1$. For torus knot $(p, q) \equiv (q, p)$ one has $g = (p-1)(q-1)/2$. Genus vanishes for un-knot so that it gives the dominating contribution to the partition function but a vanishing contribution to the p-adic mass squared.

   p-Adic mass scale could be assumed to correspond to the primary p-adic mass scale just as in the ordinary p-adic mass calculations. If the p-adic temperature is $T = 1$ in natural units (highest possible), and if one has $H = 2g$, the lowest order contribution corresponds to the value $H = 2$ of the knot Hamiltonian, and is obtained for trefoil and figure eight knot so that the lowest order contribution to the mass would indeed be about $10^{-19} m_e$ for electron. An equivalent interpretation is that $H = g$ and $T = 1/2$ as assumed for gauge bosons in p-adic mass calculations.

   There is a slight technical complication involved. When the string has a non-trivial homology in $X_g^2 \times S^1$ (it always has by construction), it does not allow Seifert surface in the ordinary sense. One can however modify the definition of Seifert surface so that it isolates knottedness from homology. One can express the string as connected sum of homologically non-trivial unknot carrying all the homology and of homologically trivial knot carrying all knottedness and in accordance with the additivity of genus define the genus of the original knot as that for the homologically trivial knot.

5. If the knots assigned with the elementary particles have large enough size, both connected sum and reconnection could take place for the knots associated with different elementary particles and make the many particle system a single connected structure. TGD based model for quantum biology is indeed based on this kind of picture. In this case the braid strands are magnetic flux tubes and connect bio-molecules to single coherent whole. Could electrons form this kind of stable connected structures in condensed matter systems? Could this relate to super-conductivity and Cooper pairs somehow? If one takes p-adic thermodynamics for knots seriously then knotted and braided magnetic flux tubes are more attractive alternative in this respect.

What if the thermalization of knot degrees of freedom does not take place? One can also consider the possibility that knotting contributes only to the vacuum conformal weight and thus to the mass squared but that no thermalization of ground states takes place. If the increment $\Delta m$ of inertial mass squared associated with knotting is of from $kgp^2$, where $k$ is positive integer and $g$ the above described knot genus, one would have $\Delta m/m \simeq 1/p$. This is of order $M_{127}^{-1} \simeq 10^{-38}$ for electron.

Could the knotting and linking of elementary particles allow topological quantum computation at elementary particle level? The huge number of different knottings would give electron a huge ground state degeneracy making possible negentropic entanglement. For negentropic entanglement probabilities must belong to an algebraic extension of rationals: this would be the case in the intersection of p-adic and real worlds and there is a temptation to assign living matter to this intersection.

Negentropy Maximization Principle could stabilize negentropic entanglement and therefore allow to circumvent the problems due to the fact that the energies involved are extremely tiny and far below thus thermal energy. In this situation bit would generalize to "nit" corresponding to $N$ different ground states of particle differing by knotting.

A very naive dimensional analysis using Uncertainty Principle would suggest that the number changes of electron state identifiable as quantum computation acting on q-nits is of order $1/\Delta t = \Delta m/hbar$ . More concretely, the minimum duration of the quantum computation would be of order $\Delta t = \hbar/\Delta m$. Single quantum computation would take an immense amount time: for electron single operation would take time of order $10^{17}$ s, which is of the order of the recent age of the Universe. Therefore this quantum computation would be of rather limited practical value!

# Chapter 8

# Miscellaneous topics

## 8.1 Introduction

As the title tells, this chapter contains topics which do not fit naturally under any umbrella, but which I feel might be of some relevance. Basically TGD inspired comments to the work of the people not terribly relevant to quantum TGD itself are in question. For few years ago Witten's approach to 3-D quantum gravitation raised a considerable interest and this inspired the comparison of this approach with quantum TGD in which light-like 3-surfaces are in a key role. Few years later the entropic gravity of Verlinde stimulated a lot of fuss in blogs and it is interesting to point out how the formal thermodynamical structure (or actually its "square root") emerges in the fundamental formulation of TGD. T-duality relating the physics in long and short length scales to each other is one of the basic dualities of string theory, and a natural question is whether it could have a counterpart in TGD. The proposal leads to a rather unexpected suggestion that biology might have counterpart at the level of particle physics. There are indeed several strong indications that in TGD Universe biology might not be at all something totally separate from the physics in $CP_2$ scale. Lisi's $E_8$ theory was a further blog favorite and some comments about its failures and possible manners to cure them are discussed. It is also shown ho how $E_8$ can be seed as being replaced with the Kac-Moody algebra associated standard model symmetry group in TGD framework.

## 8.2 Light-like 3-surfaces as vacuum solutions of 3-D vacuum Einstein equations and Witten's approach to quantum gravitation

There is an interesting relationship to the recent yet unpublished work of Witten related to 3-D quantum blackholes [B34] , which allows to get additional perspective.

1. The motivation of Witten is to find an exact quantum theory for blackholes in 3-D case. Witten proposes that the quantum theory for 3-D $AdS_3$ blackhole with a negative cosmological constant can be reduced by $AdS_3/CFT_2$ correspondence to a 2-D conformal field theory at the 2-D boundary of $AdS_3$ analogous to blackhole horizon. This conformal field theory would be a Chern-Simons theory associated with the isometry group $SO(1,2) \times SO(1,2)$ of $AdS_3$. Witten restricts the consideration to $\Lambda < 0$ solutions because $\Lambda = 0$ does not allow black-hole solutions and Witten believes that $\Lambda > 0$ solutions are non-perturbatively unstable.

2. This conformal theory would have the so called monster group [B34, B7] as the group of its discrete hidden symmetries. The primary fields of the corresponding conformal field theory would form representations of this group. The existence of this kind of conformal theory has been demonstrated already [B26] . In particular, it has been shown that this theory does not allow massless states. On the other hand, for the 3-D vacuum Einstein equations the vanishing of the Einstein tensor requires the vanishing of curvature tensor, which means that gravitational radiation is not possible. Hence $AdS_3$ theory in Witten's sense might define this conformal field theory.

### 8.2.1 Similarities with TGD

Witten's construction has obviously a strong structural similarity to TGD.

1. Chern-Simons action for the induced Kähler form - or equivalently, for the induced classical color gauge field proportional to Kähler form and having Abelian holonomy - corresponds to the Chern-Simons action in Witten's theory. Note however that in the recent formulation of Quantum TGD Kähler action and corresponding instanton density $J \wedge J$ define real and imaginary parts of complexified Kähler action. The imaginary part of the complexified Kähler function does not contribute to the configuration space metric but gives first principle description of anyons and purely topological degrees of freedom.

2. Light-like 3-surfaces can be regard as 3-D solutions of vacuum Einstein equations. Due to the effective 2-dimensionality of the induced metric Einstein tensor vanishes identically and vacuum Einstein equations are satisfied for $\Lambda = 0$. One can say that light-like partonic 3-surfaces correspond to empty space solutions of Einstein equations. Even more, partonic 3-surfaces are very much analogous to 3-D black-holes if one identifies the counterpart of black-hole horizon with the intersection of $\delta M^4_\pm \times CP_2$ with the partonic 2-surface.

3. For light-like 3-surfaces curvature tensor is non-vanishing which raises the question whether one obtains gravitons in this case. The fact that time direction does not contribute to the metric means that propagating waves are not possible so that no 3-D gravitational radiation is obtained. There is analog for this result at quantum level. If partonic fermions are assumed to be free fields as is done in the recent formulation of quantum TGD, gravitons can be obtained only as parton-antiparton bound states connected by flux tubes and are therefore genuinely stringy objects. Hence it is not possible to speak about 3-D gravitons as single parton states.

4. Vacuum Einstein equations can be regarded as gauge fixing allowing to eliminate partially the gauge degeneracy due to the general coordinate invariance. Additional super conformal symmetries are however present and have an identification in terms of additional symmetries related to the fact that space-time surfaces correspond to preferred extremals of Kähler action whose existence was concluded before the discovery of the formulation in terms of light-like 3-surfaces.

### 8.2.2 Differences from TGD

There are also interesting differences.

1. According to Witten, his theory has no obvious generalization to 4-D black-holes whereas 3-D light-like determinants define the generalization of blackhole horizons which are also light-like 3-surfaces in the induced metric. In particular, light-like 3-surfaces define a 4-D quantum holography.

2. Partonic 3-surfaces are dynamical unlike $AdS_3$ and the analog of Witten's theory results by freezing the vibrational degrees of freedom in TGD framework.

3. The very notion of light-likeness involves the induced metric implying that the theory is almost-topological but not quite. This small but important distinction indeed guarantees that the theory is physically interesting.

4. In Witten's theory the gauge group corresponds to the isometry group $SO(1,2) \times SO(1,2)$ of $AdS_3$. The group of isometries of light-like 3-surface is something much much mightier. It corresponds to the conformal transformations of 2-dimensional section of the 3-surfaces made local with respect to the radial light-like coordinate in such a manner that radial scaling compensates the conformal scaling of the metric produced by the conformal transformation.

   The direct TGD counterpart of the Witten's gauge group would be thus infinite-dimensional and essentially same as the group of 2-D conformal transformations. Presumably this can be interpreted in terms of the extension of conformal invariance implied by the presence of ordinary conformal symmetries associated with 2-D cross section plus "conformal" symmetries with respect to the radial light-like coordinate. This raises the question about the possibility to formulate quantum TGD as something analogous to string field theory using using Chern-Simons action for this infinite-dimensional group.

5. Monster group does not have any special role in TGD framework. However, all finite groups and - as it seems - also compact groups can appear as groups of dynamical symmetries at the partonic level in the general framework provided by the inclusions of hyper-finite factors of type $II_1$ [K12, K18] . Compact groups and their quantum counterparts would closely relate to a hierarchy of Jones inclusions associated with the TGD based quantum measurement theory with finite measurement resolution defined by inclusion as well as to the generalization of the imbedding space related to the hierarchy of Planck constants [K18] . Discrete groups would correspond to the number theoretical braids providing representations of Galois groups for extensions of rationals realized as braidings [K26] .

6. To make it clear, I am not suggesting that $AdS_3/CFT_2$ correspondence should have a TGD counterpart. If it had, a reduction of TGD to a closed string theory would take place. The almost-topological QFT character of TGD excludes this on general grounds. More concretely, the dynamics would be effectively 2-dimensional if the radial superconformal algebras associated with the light-like coordinate would act as pure gauge symmetries. Concrete manifestations of the genuine 3-D character are following.

    (a) Generalized super-conformal representations decompose into infinite direct sums of stringy super-conformal representations.

    (b) In p-adic thermodynamics explaining successfully particle massivation radial conformal symmetries act as dynamical symmetries crucial for the particle massivation interpreted as a generation of a thermal conformal weight.

    (c) The maxima of Kähler function defining Kähler geometry in the world of classical worlds correspond to special light-like 3-surfaces analogous to bottoms of valleys in spin glass energy landscape meaning that there is infinite number of different 3-D light-like surfaces associated with given 2-D partonic configuration each giving rise to different background affecting the dynamics in quantum fluctuating degrees of freedom. This is the analogy of landscape in TGD framework but with a direct physical interpretation in say living matter.

As noticed, Witten's theory is essentially for 2-D fundamental objects. It is good to sum up what is needed to get a theory for 3-D fundamental objects in TGD framework from an approach similar to Witten's in many respects. This connection is obtained if one brings in 4-D holography, replaces 3-metrics with light-like 3-surfaces (light-likeness constraint is possible by 4-D general coordinate invariance), and accepts the new view about S-matrix implied by the zero energy ontology [K12] .

1. Light-like 3-surfaces can be regarded as solutions vacuum Einstein equations with vanishing cosmological constant (Witten considers solutions with non-vanishing cosmological constant). The effective 2-D character of the induced metric is what makes this possible.

2. Zero energy ontology is also an essential element: quantum states of 3-D theory in zero energy ontology correspond to generalized S-matrices [K12] : **Matrix** or M-matrix might be a proper term. **Matrix** is a "complex square root" of density matrix -matrix valued generalization of Schrodinger amplitude - defining time like entanglement coefficients. Its "phase" is unitary matrix and might be rather universal. **Matrix** is a functor from the category of Feyman cobordisms and matrices have groupoid like structure [K12] . Without this generalization theory would reduce to a theory for 2-D fundamental objects.

3. Theory becomes genuinely 4-D because S-matrix is not universal anymore but characterizes zero energy states.

4. 4-D holography is obtained via the Kähler metric of the world of classical worlds assigning to light-like 3-surface a preferred extremal of Kähler action as the analog of Bohr orbit containing 3-D light-like surfaces as sub-manifolds (analogs of black hole horizons and light-like boundaries). Interiors of 4-D space-time sheets corresponds to zero modes of the metric and to the classical variables of quantum measurement theory (quantum classical correspondence). The conjecture is that Dirac determinant for the modified Dirac action associated with partonic 3-surfaces defines the vacuum functional as the exponent of Kähler function with Kähler coupling strength fixed completely as the analog of critical temperature so that everything reduces to almost topological QFT [K9] .

5. The counterpart of the ordinary unitary S-matrix in mathematical sense is between zero energy states. I call it U-matrix [K12] . As such it would have nothing to do with particle reactions but it is quite possible and also natural that $M$-matrix would serve as building block of $U$-matrix so that also $U$-matrix would be experimentally measurable quantity. It is crucial for understanding consciousness via *moment of consciousness as quantum jump* identification.

## 8.3 Entropic gravity and TGD

Eric Verlinde has posted an interesting eprint titled *On the Origin of Gravity and the Laws of Newton* to arXiv.org [B41] . What Linde heuristically derives is Newton's $F = ma$ and gravitational force $F = GMm/R^2$ from thermodynamical considerations plus something else which I try to clarify (at least to myself!) in the following.

### 8.3.1 Verlinde's argument for $F = ma$

The idea is to deduce Newton's $F = ma$ and gravitational force from thermodynamics by assuming that space-time emerges in some sense. There are however various assumptions involved which more or less impy that both special and general relativity has been feeded in besides quantum theory and thermodynamics.

1. Time translation invariance is required in order to have the notions of conserved energy and thermodynamics. This assumption requires not only require time but also symmetry with respect to time translations. This is quite a powerful assumption and time translation symmetry not hold true in General Relativity- this was actually the basic motivation for quantum TGD.

2. Holography is assumed. Information stored on surfaces, or screens and discretization is assumed. Again this means in practice the assumption of space-time since otherwise the notion of holography does not make sense. One could of course say that one considers the situation in the already emerged region of space-time but this idea does not look very convincing to me.

   **Comment**: In TGD framework holography is an essential piece of theory: light-like 3-surfaces code for the physics and space-time sheets are analogous to Bohr orbits fixed by the light-like 3-surfaces defining the generalized Feynman diagrams.

3. The first law of thermodynamics in the form

$$dE = TdS - Fdx \quad .$$

   Here $F$ denotes generalized force and x some coordinate variable. In usual thermodynamics pressure P would appear in the role of $F$ and volume $V$ in the role of $x$. Also chemical potential and particle number form a similar pair. If energy is conserved for the motion one has

$$Fdx = TdS \quad .$$

   This equation is basic thermodynamics and is used to deduce Newton's equations.

After this some quantum tricks -a rather standard game with Uncertainty Principle and quantization when nothing concrete is available- are needed to obtain F=ma which as such does not involve $\hbar$ nor Boltzmann constant $k_B$. What is needed are thermal expression for acceleration and force and identifying these one obtains F=ma.

1. The condition $\Delta S = 2\pi k_B$ states that entropy is quantized with a unit of $2\pi$ appearing as a unit. log(2) would be more natural unit if bit is the unit of information.

2. The identification $\Delta x = \hbar/mc$ involves Uncertainty principle for momentum and position. The presence of light velocity c in the formula means that Minkowski space and Special Relativity creeps in. At this stage I would not speak about emergence of space-time anymore.

   This gives

$$F = T\frac{\Delta S}{\Delta x} = T\frac{2\pi mck_B}{\hbar} \quad .$$

$F$ has been exressed in terms of thermal parameters and mass.

3. Next one feeds in something from General Relativity to obtain expression for acceleration in terms of thermal parameters. Unruh effect means that in an accelerted motion system measures temperate proportional to acceleration:

$$k_BT = \frac{\hbar a}{2\pi} \quad .$$

This quantum effect is known as Unruh effect. This temperature is extremely low for accelerations encountered in everyday life - something like $10^{-16}$ K for free fall near Earth's surface.

Using this expression for $T$ in previous equation one obtains the desired $F = ma$, which would thus have a thermodynamical interpretation. At this stage I have even less motivations for talking about emergence of space-time. Essentially the basic conceptual framework of Special and General Relativities, of wave mechanics and of thermodynamics are introduced by the formulas containing the basic parameters involved.

### 8.3.2 Verlinde's argument for $F = GMm/R^2$

The next challenge is to derive gravitational force from thermodynamic consideration. Now holography with a very specially chosen screen is needed.

**Comment**: In TGD framework light-like 3-surfaces (or equivalently their space-like duals) represent the holographic screens and in principle there is a slicing of space-time surface by equivalent screens. Also Verlinde introduces a slicing of space-time surfaces by holographic screens identified as surfaces for which gravitational potential is constant. Also I have considered this kind of identification.

1. The number of bits for the information represented on the holographic screen is assumed to be proportional to area.

$$N = \frac{A}{G\hbar} \quad .$$

This means bringing in blackhole thermodynamics and general relativity since the notion of area requires geometry.

**Comment**: In TGD framework the counterpart for the finite number of bits is finite measurement resolution meaning that the 2-dimensional partonic surface is effectively replaced with a set of points carrying fermion or antifermion number or possibly purely bosonic symmetry generator. The orbits of these points define braid giving a connection with topological QFTs for knots, links and braids and also with topological quantum computation.

2. It is assumed that the area of horizon corresponds to the area $A = 4\pi R^2$ for the sphere with radius which $R$ which is the distance between the masses. This means a very special choice of the holographic screen. Entropy obviously depends very sensitively on $R$.

**Comment**: In TGD framework the counterpart of the area would be the symplectic area of partonic 2-surfaces. This is invariant under symplectic transformations of light-cone boundary. These "partonic" 2-surfaces can have macroscopic size and the counterpart for blackhole horizon is one example of this kind of surface. Anyonic phases are second example of a phase assigned with a macroscopic partonic 2-surface.

3. Special relativity is brought in via the bomb formula

$$E = mc^2 \quad .$$

One introduces also other expression for the rest energy. Thermodynamics gives for non-relativistic thermal energy the expression

$$E = \frac{1}{2} N k_B T \ .$$

This thermal energy is identified with the rest mass. This identification looks to me completely ad hoc and I think that kind of holographic duality is assumed to justify it. The interpretation is that the points/bits on the holographic screen behave as particles in thermodynamical equilibrium and represent the mass inside the spherical screen. What are these particles on the screen? Do they correspond to gravitational flux?

**Comment**: In TGD framework p-adic thermodynamics replaces Higgs mechanism and identify particle's mass squared as thermal conformal weight. In this sense inertia has thermal origin in TGD framework. Gravitational flux is mediated by flux tubes with gigantic value of gravitational Planck constant and the intersections of the flux tubes with sphere could be TGD counterparts for the points of the screen in TGD. These 2-D intersections of flux tubes should be in thermal equilibrium at Unruh temperature. The light-like 3-surfaces indeed contain the particles so that the matter at this surface represents the system. Since all light-like 3-surfaces in the slicing are equivalent means that one can choose the reresentation of the system rather freely .

4. Eliminating the rest energy E from these two formulas one obtains $NT = 2mc^2$ and using the expression for $N$ in terms of area identified as that of a sphere with radius equal to the distance $R$ between the two masses, one obtains the standard form for gravitational force.

It is difficult to say whether the outcome is something genuinely new or just something resulting unavoidably by feeding in basic formulas from general thermodynamics, special relativity, and general relativity and using holography principle in highly questionable and ad hoc manner.

### 8.3.3 In TGD quantum classical correspondence predicts that thermodynamics has space-time correlates

From TGD point of view entropic gravity is a misconception. On basis of quantum classical correspondence - the basic guiding principle of quantum TGD - one expects that all quantal notions have space-time correlates. If thermodynamics is a genuine part of quantum theory, also temperature and entropy should have the space-time correlates and the analog of Verlinde's formula could exist. Even more, the generalization of this formula is expected to make sense for all interactions.

Zero energy ontology makes thermodynamics an integral part of quantum theory.

1. In zero energy ontology quantum states become zero energy states consisting of pairs of the positive and negative energy states with opposite conserved quantum numbers and interpreted in the usual ontology as physical events. These states are located at opposite light-like boundaries of causal diamond ($CD$) defined as the intersection of future and past directed light-cones. There is a fractal hierarchy of them. M-matrix generalizing S-matrix defines time-like entanglement coefficients between positive and negative energy states. M-matrix is essentially a "complex" square root of density matrix expressible as positive square root of diagonalized density matrix and unitary S-matrix. Thermodynamics reduces to quantum physics and should have correlate at the level of space-time geometry. The failure of the classical determinism in standard sense of the word makes this possible in quantum TGD (special properties of Kähler action (Maxwell action for induced Kahler form of $CP_2$) due to its vacuum degeneracy analogous to gauge degeneracy). Zero energy ontology allows also to speak about coherent states of bosons, say of Cooper pairs of fermions- without problems with conservation laws and the undeniable existence of these states supports zero energy ontology.

2. Quantum classical correspondence is very strong requirement. For instance, it requires also that electrons traveling via several routes in double slit experiment have classical correlates. They have. The light-like 3-surfaces describing electrons can branch and the induced spinor fields at them "branch" also and interfere again. Same branching occurs also for photons so that electrodynamics has hydrodynamical aspect too emphasize in recent empirical report about knotted light beams. This picture explains the findings of Afshar challenging the Copenhagen interpretation.

These diagrams could be seen as generalizations of stringy diagrams but do not describe particle decays in TGD framework. In TGD framework stringy diagrams are replaced with a direct generalization of Feynman diagrams in which the ends of 3-D lightlike lines meet along 2-D partonic surfaces at their ends. The mathematical description of vertices becomes much simpler since the 2-D manifolds describing vertices are not singular unlike the 1-D manifolds associated with string diagrams ("eyeglass" in fusion of closed strings).

3. If entropy has a space-time correlate then also first and second law should have such and Verlinde's argument that gravitational force attraction follows from first law assuming energy correlation might identify this correlate. This of course applies only to the classical gravitation. Also other classical forces should allow analogous interpretation as space-time correlates for something quantal.

### 8.3.4 The simplest identification of thermodynamical correlates in TGD framework

The first questions that pop up are following. Inertial mass emerges from p-adic thermodynamics as thermal conformal weight. Could the first law for p-adic thermodynamics, which allows to calculate particle masses in terms of thermal conformal weights, allow to deduce also other classical forces? One could think that by adding to the Hamiltonian defining partition function chemical potential terms characterizing charge conservation it might be possible to obtain also other forces.

In fact, the situation might be much simpler. The basic structure of quantum TGD allows a very natural thermodynamical interpretation.

1. The basic structure of quantum TGD suggests a thermodynamic interpretation. The basic observation is that the vacuum functional identified as the exponent of Kähler function is analogous to a square root of partition function and Kähler coupling strength is analogous to critical temperature. Kähler function identified as Kähler action for a preferred extremal appears in the role of Hamiltonian. Preferred extremal property realizes holography identifying space-time surface as analog of Bohr orbit. One can interpret the exponent of Kähler function as the density of states in the world of classical worlds so that Kähler function would be analogous to entropy density. Ensemble entropy is average of Kähler function involving functional integral over the world of classical worlds. This exponent is the counterpart for the quantity $\Omega$ appearing in Verlinde's basic formula.

2. The addition of a measurement interaction term to the modified Dirac action gives rise to a coupling to conserved charges. Vacuum functional is identified as Dirac determinant and this addition is visible as an addition of an interaction term to Kähler function. The interaction gives rise to forces coupling to various charges at classical level for quantum states with fixed quantum numbers for positive energy part of the state. These terms are analogous to chemical potential terms in thermodynamics fixing the average values of various charges or particle numbers. In ordinary non-relativistic thermodynamics energy is in a special role. In the recent case there is a complete quantum number democracy very natural in a framework with coordinate invariance and with four-momentum assigned with the isometries of the 8-D imbedding space. In Verlinde's formula there is exponential factor $exp(-E/T - Fx)$ analogous to the measurement interaction term. In TGD however conserved charges multiplied by chemical potentials defining generalized forces appear in the exponent.

3. This gives an analog of thermodynamics in the world of classical worlds (WCW) for fixed values of quantum numbers of the positive energy part of state. For zero energy states one however has also additional thermodynamics- or rather its square root. This thermodynamics is for the conserved quantum numbers whose averages are fixed. For general zero energy states one has sum over state pairs labelled by momenta and various other quantum numbers labelling the positive energy part of the state. The coefficients of the conserved quantities of the measurement interaction term linear in conserved quantum numbers define the analogs of temperature and various chemical potentials. The field equations defined by Kähler function and chemical potential terms have thermodynamical interpretation and give coupling to conserved charges and also to their thermal averages. What is important is that temperature and various chemical potentials assigned to

positive and negative energy parts of the state allow a complete geometrization in a general coordinate invariant manner and allow explicit expressions in terms of functions expressible in terms of the induced geometry.

4. The explicit expressions must be deduced from Dirac determinant defining exponent of Kähler function plus measurement interaction term, in which the conserved isometry charges of Cartan algebra (necessarily!) appearing in the exponent are contracted with the analogs of chemical potentials. One make two rather detailed educated guesses for the chemical potentials. For modified Dirac action the measurement interaction term is 4-dimensional. For the Kähler action one can imagine two candidates for the measurement interaction term. For the first option the term is 4-dimensional and for the second one 3-dimensional.

### 8.3.5 Some details related to the measurement interaction term

As noticed, one can imagine two options for the measurement interaction term defining the chemical potentials in terms of the space-time geometry.

1. For both options the $M^4$ part of the interaction term is proportional to $n(M^4)G/R$ and $CP_2$ part to a dimensionless constant $n(CP_2)$, and the condition that there is no dependence of $\hbar$ excludes the dependence on the dimensionless constant $G\hbar/R^2$.

2. One can consider two different forms of the measurement interaction part in Kähler function. For the first option the conserved Kähler current replaces fermion current in the modified Dirac action and defines a 4-dimensional interaction term highly analogous to that assigned with the modified Dirac action. The source term induced to the field equations corresponds to the variation of

$$[\frac{G}{R} \times n(M^4)p_{q,A}g^{AB}(M^4)j_{A\alpha} + n(CP_2)Q_{q,A}g^{AB}J_{A\alpha}(CP_2)]J^\alpha \quad .$$

   Here $J^\alpha$ is Kähler current.

3. For the second option the measurement interaction term in Kähler action is sum over contractions of quantum Cartan charges with corresponding classical Noether charges giving the sum of the term

$$[\frac{G}{R} \times n(M^4)p_{q,A}p^{cl,A} + n(CP_2)Q_{q,A}Q^{cl,A}]$$

   from both ends of the space-time sheet. For a general space-time sheet the classical charges are different at its ends so that the variation gives non-trivial boundary conditions equating the normal (time-like) component of the canonical momentum current with the contraction of the variation of classical Noether charges contracted with quantum charges. By the extremal property the measurement interaction terms at the ends of the space-time sheet cancel each other so that the effect on Kähler function is only via the boundary conditions in accordance with zero energy ontology. For this option the thermodynamics for conserved charges is visible at space-time level only via the appearence of the average quantal charges and universal chemical potentials.

4. The vanishing of Kähler gauge current *resp.* classical Noether charges for the first *resp.* second option would suggest an interpretation in terms of infinite temperature limit. The fact that momenta and color charges are in completely symmetric position suggests however the vanishing of chemical potentials. One can in fact fix the value of the temperature to say $T = R/G$ without loss of information and code thermodynamics in terms of the chemical potentials alone.

   The vanishing of the measurement interaction term occurs for the vacuum extremals. For $CP_2$ type vacuum extemals with Euclidian signature of the induced metric interpretation in terms of vanishing chemical potentials is more natural. For vacuum extremals with Minkowskian signature of the induced metric fluctuations and consequently classical non-determinism are maximal so that the interpretation in terms of high temperature phase cannot be excluded. One must however notice that $CP_2$ projection for vacuum extremals is 2-dimensional whereas high temperature limit would suggest 4-D projection so that the interpretation in terms of vanishing chemical potentials is more natural also now.

To sum up, TGD suggests two thermodynamical interpretations. p-Adic thermodynamics gives inertial mass squared as thermal conformal weight and also the basic formulation of quantum TGD allows thermodynamical interpretation. The thermodynamical structure of quantum TGD has of course been guiding principle for two decades. In particular, quantum criticality as the counterpart of thermal criticality has been extremely useful guide line and led to a breakthrough in the understanding of the modified Dirac equation during the last year. Also p-adic thermodynamics has been in the scene for more than 15 years and makes TGD a theory able to make precise quantitative predictions. Some conclusions drawn from Verlinde's argument is that gravitation is entropic interaction, that gravitons do not exist, and that string models and theories introducing higher-dimensional space-time are a failure. TGD view is different. Only a generalization of string model allowing to realize space-time as surface is needed and this requires fixed 8-D imbedding space. Gravitons also exist and only classical gravitation as well as other classical interactions code for thermodynamical information by quantum classical correspondence. In any case, it is encouraging that also colleagues might be finally beginning to get on the right track although the path from Verlinde's arguments to quantum TGD as it is now will be desperately long and tortuous if colleagues continually refuse to receive the helping hand.

## 8.4 What could be the counterpart of T-duality in TGD framework?

Stephen Crowley sent me a book of Michel Lapidus [A55] about zeros of Riemann zeta and also about his own ideas in this respect. The book has been written in avery lucid manner and looks very interesting. The big idea is that the T-duality of string models could correspond to the functional equation for Riemann zeta relating the values of zeta at different sides of the critical line. T-duality [A26] is formulated for strings in space $M^d \times S^1$ or its generalization replacing $S^1$ with higher-dimensional torus and generalized to fractal strings. Duality states that the transformation $R \rightarrow 1/R$ with suitable unit for $R$ defined by string tension is a duality: the physics for these different values of $R$ is same. Intuitively this is due to the fact that the contributions of the string modes representing n-fold winding and those representing vibrations labelled by integer $n$ are transformed to each other in the transformation $R \rightarrow 1/R$.

Lapidus is a mathematician and mathematicians often do not care too much about the physical meaning of the numbers. For a physicist like me it is extremely painful to type the equation $R \rightarrow 1/R$ without explicitly explaining that it should actually read as $R \rightarrow R_0^2/R$, where $R_0$ is length unit, which must represent fundamental length scale remaining invariant under the duality transformation. Only after this physicist could reluctantly put $R_0 = 1$ but still would feel himself guilty of unforgivable sloppiness. $R_0 = 1$ simplifies the formulas but one must not forget that there are three scales involved rather than only two. The question inspired by this nitpicking is how the physics in the length scales $R_1$ and $R$ relates to the physics in length scale $R$. Are dualities - or perhaps holography like relations in question- so that T-duality would follow from these dualities?

### 8.4.1 Could one replace winding number with magnetic charge and T-duality with canonical identification?

How could one generalize T-duality to TGD framework? One should identify the counterpart of the winding number, the three fundamental scales, and say something about the duality transformation itself.

1. In TGD Universe partonic 2-surfaces are the basic object. Partonic 2-surface is not strings and the only reasonable generalization for winding number is as Kähler magnetic charge representing the analog of winding of the partonic 2-surface around magnetically charged 2-sphere of $CP_2$. Magnetic charge tells how many times partonic 2-surface wraps around the homologically non-trivial geodesic sphere with unit magnetic charge. If the generalization of T-duality holds true,

one would expect that the contributions of the oscillations and windings of the partonic two-surface to ground state energy must be transformable to each other by the counterpart of the transformation $R \to R_0^2/R$ - or something akin to that. Also less concrete and more general interpretations are possible, and below the most plausible interpretation will be considered.

2. The duality $R \to R_0^2/R = R_1$ gives $R_0$ as a geometric mean $R_0 = \sqrt{RR_1}$ of the scales $R$ and $R_1$. What are these three length scales in TGD Universe? The obvious candidate for $R$ is $CP_2$ size scale. p-Adic mass calculations [K29] imply that the primary p-adic length scale $L_{p,1} = \sqrt{p}R$ is of order of Compton length of the elementary particle characterized by the p-adic prime $p$. The secondary p-adic length scale $L_{p,2} = pR$ in turn defines the size scale of causal diamond ($CD$) assignable to the magnetic body of the elementary particle characterized by prime $p$. For instance, for electron this scale corresponds to .1 seconds, a fundamental biological time scale.

   One indeed has $L_{p,1} = \sqrt{L_{p,2}R}$, and $CP_2$ scale and $CD$ length scale are dual to each other if T-duality holds true. Therefore the duality would relate physics at $CP_2$ scale - counterpart of Planck length in TGD framework - and in biological scales and would have direct relevance to quantum biology. One has an infinite hierarchy of p-adic length scales and each of them would give rise to one particular instance of the T-duality. Adeles [K66] would provide appropriate formulation of T-duality in TGD framework. The corresponding mass scales would be $\hbar/R, \hbar/\sqrt{p}R$ and $\hbar/pR$. The third scale corresponds to a scale, which for electron corresponds to the 10 Hz frequency in the case of photons. The duality would suggest that the physics associated with the frequencies in EEG scale related to the communications from the biological body to magnetic body is dual to the physics in $CP_2$ scale.

   Note that one cannot exclude alternative variants of T-duality. In particular, Planck scale and $CP_2$ length scale as candidates $R_1$ and $R$ could be considered.

3. What is the interpretation of these three length scales? $CP_2$ length scale corresponds naturally to the size scale of wormhole contacts. They are Euclidian regions of space-time surface and represent lines of generalized Feynman graphs. Both general arguments and the construction of elementary bosons forces [K32] to assign to these regions braid strands playing a role of Euclidian strings. Parallel translation along the strands is essential in the construction of fermionic bilinears as invariant under general coordinate transformations and gauge transformations [K32]. The ends of these strands carry fermion and anti-fermion numbers. The counterpart of string tension involved appearing in stringy mass formula implied by super-conformal invariance is indeed determined by $R$ and p-adic thermodynamics [K29] leads to a detailed and successful calculations for elementary particle masses using only p-adic thermodynamics, super-conformal invariance, and p-adic length scale hypothesis as basic assumptions.

4. The wormhole throats carrying fermion number are Kähler magnetic monopoles and the wormhole must be accompanied by a second wormhole throat carrying opposite magnetic charge and also a neutrino pair neutralizing the weak isospin so that weak massivation takes place. The end of the flux tube containing the neutrino pair is virtually non-existent at low energies. The length scale for this string must correspond to Compton length for elementary particle given essentially by primary p-adic length scale $L_{p,1}$. The more restrictive assumption that this length scale corresponds to the Compton length of weak bosons looks un-necessarily restrictive and looks also un-natural.

5. The excitations with mass scale $\hbar/pR$ would correspond to excitations assignable to entire $CD$, maybe assignable to the flux tubes of the magnetic bodies of elementary particles defining also string like objects but in macroscopic scales. For electron the scale is of order of the circumference of Earth. This dynamics would naturally correspond to the dynamics in Minkowskian space-time regions. The dynamics at intermediate length scale would be intermediate between the Euclidian and Minkowskian dynamics and reduce to that for light-like orbits of partonic 2-surfaces with metric intermediate between Minkowskian and Euclidian.

6. A natural interpretation for T-duality in this sense is in terms of strong form of holography. The interior dynamics at length scale $R$ *resp.* $pR$ assigned to Euclidian *resp.* Minkowskian regions of space-time surface corresponds by holography to the dynamics of light-like orbits of partonic 2-surfaces identified as wormhole throats. Therefore the dynamics in Euclidian and

Minkowskian regions are dual to each other. Therefore T-duality in TGD sense would follow from the possibility of having both Euclidian and Minkowskian holography. Strong form of holography in turn reduces to strong form of General Coordinate Invariance, which has turned out to be extremely powerful principle in TGD framework.

### 8.4.2 Is the physics of life dual to the physics in $CP_2$ scale?

The duality of life with elementary particle physics at $CP_2$ length scale - the TGD counterpart of Planck scale - looks rather far-fetched idea. There is however already earlier support for this idea.

1. p-Adic physics is physics of cognition, and one can say that living systems are in the algebraic intersection of real and p-adic worlds: the intersection of cognition and matter. Canonical identification maps p-adic physics to real physics. This map takes p-adic integers which are small in p-adic sense to larger integers in real sense and thus maps long real scales to short real scales. Clearly this map is jhighly analogous to the T-duality. p-Adic length scales are indeed explicitly related with the above identification of the T-duality so that canonical identification might be involved with T-duality.

   If this interpretation is correct, cognitive p-adic representations in long real length scales would give representations for the physics in short length scales. EEG range of frequencies allowing communication to the magnetic bodies is absolutely essential for brain function. $CD$s would correspond to the real physics scale associated with the cognitive representations. These cognitive representations are indeed exactly what our science is building so that T-duality would make also scientist as a part of the big vision!

2. The model for dark nucleons as three quark states led to one of the greatest surprises of my professional life [L2, K23]. Under rather general conditions the three quark states for nucleon are in one-one correspondence with the DNA, RNA, tRNA codons, and aminoacids for vertebrate genetic code and there is natural physical correspondence between DNA triplets and aminoacids. This suggests that genetic code is realized at the level of hadrons and that living matter is a kind of emulation for it, or that living matter is representation for matter at hadron level. This leads to rather far reaching speculations about biological evolution - not as random process - but a process analogous R&D applied in industry [K23]. New genes would be continually tested at the level of dark matter and the modifications of genome could be carried out if there is a transcription process transforming dark DNA to ordinary DNA.

3. The secondary p-adic mass scale of electron corresponds to the 10 Hz frequency, which defines a fundamental biorhythm. Also to current quark masses, which are actually not so well-known but are in MeV range, one can assign biologically interesting time scales in millisecond range. This suggests that all elementary particles induce physics in macroscopic time scales via their $CD$:s containing their magnetic bodies.

The unavoidable and completely crazy looking question raised by T-duality is whether there is intelligent life in the Euclidian realm below the $CP_2$ length scale - inside the lines of generalized Feynman graphs. This kind of possibility cannot be avoided if one takes holography absolutely seriously. In purely mathematical sense TGD suggests even stronger form of holography based on the notion of infinite primes [K52]. In this holography the number theoretic anatomy of given space-time point is infinitely complex and evolves. The notion of quantum mathematics replacing numbers by Hilbert spaces representing ordinary arithmetics in terms of direct sum and tensor product suggest the same [K66]. Space-time point would be in this picture its own infinitely complex Universe - the Platonia.

### 8.4.3 Could one get expression for Kähler coupling strength from restricted form of modular invariance?

The contributions to the exponent of the vacuum functional, which is proportional to Kähler action for preferred extremal, are real *resp.* imaginary in Euclidian *resp.* Minkowskian regions. Under rather general assumptions (weak form of electric-magnetic duality defining boundary conditions at wormhole throats plus additional intuitively plausible assumption) these contributions are proportional to the same Chern-Simons term but with possibly different constant of proportionality [K12].

These terms sum up to a Chern-Simons term with a coefficient analogous to the complex inverse gauge coupling

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_K^2} \quad .$$

The real part would correspond to Kähler function coming from Euclidian regions defining the lines of generalized Feynman diagrams and imaginary part to Minkowskian regions. There are could arguments suggesting that With the conventions that I have used $\theta/2\pi$ is counterpart for $1/\alpha_K$ and there are good arguments that it corresponds to finte structure constant in electron length scale. Furthermore, T-duality would suggest that $\tau$ is proportional to $1+i$ so that one would have

$$\frac{\theta}{2\pi} = \frac{4\pi}{g_K^2} \quad .$$

This condition would fit nicely with the fact that Chern-Simons contributions from Minkowskian and Euclidian regions are identical. If this equation holds true the modular transformations must reduce to those leaving this relationship invariant and can only permute the complex and real parts and thus leave $\tau$ invariant. One could also interpret this value of $\tau$ as physically especially interesting representation and assign to all values of $\tau$ related by modular transformation an isotropy group leaving it fixed. All other physically equivalent values would be obtained as SL(2,Z) orbit of this value.

The counterpart of T-duality should somehow relate dynamics in Minkowskian and Euclidian regions and this raises the question whether it corresponds to $\tau \rightarrow i\tau$ and is represented by some duality transformation

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad ,$$

where $(a, b; c, d)$ defines a unimodular matrix $(ad - bc = 1)$ with integer elements, that is in $SL(2, Z)$. The electric-magnetic duality $\tau \rightarrow -1/\tau$ [B8] and the shift $\tau \rightarrow \tau + 1$ are the generators of this group. It is not however quite clear whether they can be regarded as gauge symmetries in TGD framework. If they are gauge symmetries, then the critical values of Kähler coupling strength defined as fixed points of coupling constant evolution must form an orbit of $SL(2, Z)$. It could be also that modular symmetry is broken to a subgroup of $SL(2, Z)$ and this subgroup leaves $\tau$ invariant in the case of minimal symmetry.

1. $\tau \rightarrow i\tau$ would permute Euclidian and Minkowskian regions with each other and is therefore a candidate for the T-duality. This condition cannot be satisfied in generic case but one can ask whether for some special choices of $\tau$ these transformations could generate a non-trivial sub-group of modular transformations. This subgroup

   To see whether this is the case let us write explicitly the condition $\tau \rightarrow i\tau$:

   $$\frac{a\tau + b}{c\tau + d} = \frac{\theta}{2\pi} + i\frac{1}{\alpha_K} \quad , \quad \alpha_K = \frac{g_K^2}{4\pi} .$$

   The condition allows to solve $\tau$ as

   $$\tau = \frac{a - id}{2c}\left[1 + \epsilon_1\sqrt{1 + \frac{4ibc}{(d - ia)^2})}\right] \quad , \quad \epsilon_1 = \pm 1 \quad .$$

2. For

   $$d = \epsilon a \quad , \quad \epsilon = \pm 1$$

   implying $a^2 - bc = 1$, the solution simplifies since the argument of square root is real. One has

$$\tau = \frac{a}{2c}(1-\epsilon i)\left[1+\epsilon_1\sqrt{1-\epsilon\frac{\sqrt{a^2-1}}{a}}\right] \quad .$$

The imaginary and real parts of $\tau$ are identical: this might allow an interpretation in terms of the fact that Chern-Simons terms from two regions are identical (normal derivatives are however discontinuous at wormhole throat). Certainly this is a rather strong prediction.

3. Does this mean that SL(2,C) is broken down to the 4-element isotropy group generated by this transformation? If so, a the condition just deduced could allow to deduce additional constraints on the value of Kähler coupling strength, which is in principle fixed by the criticality condition to have only finite number of values? By the earlier arguments - related to p-adic mass calculations and the heuristic formula for the gravitational constant - the value of Kähler coupling strength is in a good approximation equal to fine structure constant at electron length scale:

$$\alpha_K = \frac{g_K^2}{4\pi} \simeq \alpha \ , \quad \frac{1}{\alpha} \simeq 137.035999084 \quad .$$

4. One obtains the following estimate for $a/2c$ from the estimate for $\alpha_K$ by considering the imaginary part of $\tau$:

$$\frac{a}{2c}\left[1+\epsilon_1\sqrt{1-\epsilon\frac{\sqrt{a^2-1}}{a}}\right] = \frac{1}{\alpha_K} \quad .$$

At the limit $a \to \infty$ one has

$$\frac{a}{2c}\left[1+\epsilon_1\sqrt{1-\epsilon}\right] = \frac{1}{\alpha_K} \quad .$$

The simplest option at this limit corresponds to $\epsilon = 1$ giving

$$\frac{a}{2c} \simeq 137.035999084 \quad .$$

Note that $a/2c = 137$ is not allowed by determinant condition so that the deviation of $\alpha_K$ from $1/137$ is predicted. One must have $a > 137 \times 2c \geq 2 \times 137$. This implies

$$1+\epsilon_1\sqrt{1-\frac{\sqrt{a^2-1}}{a}} = 1 + .0026\epsilon_1 \quad .$$

By expanding the square root in first order to Taylor series one obtains the condition

$$\frac{a}{2c}\left(1+\epsilon_1\frac{1}{2^{3/2}\times 137c}\right) \simeq \frac{1}{\alpha_K} \quad .$$

For large enough values of $a$ and $c$ it is possible to have arbitrary good approximation to fine structure constant. Note that the integers $a$ and $c$ cannot have common factors since this together with determinant condition $a^2 - bc = 1$ would lead to contradiction.

## 8.5 $E_8$ theory of Garrett Lisi and TGD

Recently (towards end of the year 2007) there has been a lot of fuss about the $E_8$ theory proposed by Garrett Lisi [B30] in physics blogs, in media, and even New Scientist [B5] wrote about the topic. There are serious objections against Lisi's theory and it is interesting to find whether the theory could be modified so that it would survive the basic objections. Although it seems that Lisi's theory cannot be saved, one achieves further insights about HO-H duality. Number theoretical spontaneous compactification can be formulated in terms of the Kac-Moody algebra assignable to Poincare group and standard model gauge group having also rank 8. The representation can be constructed in standard manner using quantized $M^8$ coordinates at partonic 2-surfaces. Also $E_8$ representations are in principle possible and the question concerns their physical interpretation.

### 8.5.1 Objections against Lisi's theory

The basic claim of Lisi is that one can understand the particle spectrum of standard model in terms of the adjoint representation of a noncompact version $E_8$ group [B3] . There are several objections against $E_8$ gauge theory interpretation of Lisi.

1. Statistics does not allow to put fermions and bosons in the same gauge multiplet. Also the identification of graviton as a part of a gauge multiplet seems very strange if not wrong since there are no roots corresponding to a spin 2 two state.

2. Gauge couplings come out wrong for fermions and one must replace YM action with an ad hoc action.

3. Poincare invariance is a problem. There is no clear relationship with the space-time geometry so that the interpretation of spin as $E_8$ quantum numbers is not really justified.

4. Finite-dimensional representations of non-compact $E_8$ are non-unitary. Non-compact gauge groups are however not possible since one would need unitary infinite-dimensional representations which would change the physical interpretation completely. Note that also Lorentz group has only infinite-D unitary representations and only the extension to Poincare group allows to have fields transforming according to finite-D representations.

5. The prediction of three fermion families is nice but one can question the whole idea of putting particles with mass scales differing by a factor of order $10^{12}$ (top and neutrinos) into same multiplet. For some reason colleagues stubbornly continue to see fundamental gauge symmetries where there seems to be no such symmetry. Accepting the existence of a hierarchy of mass scales seems to be impossible for a theoretical physicist in main main stream although fractals have been here for decades.

6. Also some exotic particles not present in standard model are predicted: these carry weak hyper charge and color (6-plet representation) and are arranged in three families.

### 8.5.2 Three attempts to save Lisi's theory

To my opinion, the shortcomings of $E_8$ theory as a gauge theory are fatal but the possibility to put gauge bosons and fermions of the standard model to $E_8$ multiplets is intriguing and motivatse the question whether the model could be somehow saved by replacing gauge theory with a theory based on extended fundamental objects possessing conformal invariance.

1. In TGD framework H-HO duality allows to consider Super-Kac Moody algebra with rank 8 with Cartan algebra assigned with the quantized coordinates of partonic 2-surface in 8-D Minkowski space $M^8$ (identifiable as hyper-octonions HO). The standard construction for the representations of simply laced Kac-Moody algebras allows quite a number of possibilities concerning the choice of Kac-Moody algebra and the non-compact $E_8$ would be the maximal choice.

2. The first attempt to rescue the situation would be the identification of the weird spin 1/2 bosons in terms of supersymmetry involving addition of righthanded neutrino to the state giving it spin 1. This options does not seem to work.

3. The construction of representations of non-simply laced Kac-Moody algebras (performed by Goddard and Olive at eighties [A62] ) leads naturally to the introduction of fermionic fields for algebras of type B, C, and F: I do not know whether the construction has been made for $G_2$. $E_6$, $E_7$, and $E_8$ are however simply laced Lie groups with single root length 2 so that one does not obtain fermions in this manner.

4. The third resuscitation attempt is based on fractional statistics. Since the partonic 2-surfaces are 2-dimensional and because one has a hierarchy of Planck constants, one can have also fractional statistics. Spin 1/2 gauge bosons could perhaps be interpreted as anyonic gauge bosons meaning that particle exchange as permutation is replaced with braiding homotopy. If so, $E_8$ would not describe standard model particles and the possibility of states transforming according to its representations would reflect the ability of TGD to emulate any gauge or Kac-Moody symmetry.

The standard construction for simply laced Kac-Moody algebras might be generalized considerably to allow also more general algebras and fractionization of spin and other quantum numbers would suggest fractionization of roots. In stringy picture the symmetry group would be reduced considerably since longitudinal degrees of freedom (time and one spatial direction) are non-physical. This would suggest a symmetry breaking to $SO(1,1) \times E_6$ representations with ground states created by tachyonic Lie allebra generators and carrying mass squared 2 in suitable units. In TGD framework the tachyonic conformal weight can be compensated by super-symplectic conformal weight so that massless states getting their masses via Higgs mechanism and p-adic thermodynamics would be obtained.

### 8.5.3 Could super-symmetry rescue the situation?

$E_8$ is unique among Lie algebras in that its adjoint rather than fundamental representation has the smallest dimension. One can decompose the 240 roots of $E_8$ to 112 roots for which two components of SO(7,1) root vector are $\pm$ 1 and to 128 vectors for which all components are $\pm$ 1/2 such that the sum of components is even. The latter roots Lisi assigns to fermionic states. This is not consistent with spin and statistics although SO(3,1) spin is half-integer in $M^8$ picture.

The first idea which comes in mind is that these states correspond to super-partners of the ordinary fermions. In TGD framework they might be obtained by just adding covariantly constant right-handed neutrino or antineutrino state to a given particle state. The simplest option is that fermionic super-partners are complex scalar fields and sbosons are spin 1/2 fermions. It however seems that the super-conformal symmetries associated with the right-handed neutrino are strictly local in the sense that global super-generators vanish. This would mean that super-conformal super-symmetries change the color and angular momentum quantum numbers of states. This is a pity if indeed true since super-symmetry could be broken by different p-adic mass scale for super partners so that no explicit breaking would be needed.

### 8.5.4 Could Kac Moody variant of $E_8$ make sense in TGD?

One can leave gauge theory framework and consider stringy picture and its generalization in TGD framework obtained by replacing string orbits with 3-D light-like surfaces allowing a generalization of conformal symmetries.

H-HO duality is one of the speculative aspects of TGD. The duality states that one can either regard imbedding space as $H = M^4 \times CP_2$ or as 8-D Minkowski space $M^8$ identifiable as the space HO of hyper-octonions which is a subspace of complexified octonions. Spontaneous compactification for $M^8$ described as a phenomenon occurring at the level of Kac-Moody algebra would relate HO-picture to H-picture which is definitely the fundamental picture. For instance, standard model symmetries have purely number theoretic meaning in the resulting picture.

The question is whether the non-compact $E_8$ could be replaced with the corresponding Kac Moody algebra and act as a stringy symmetry. Note that this would be by no means anything new. The Kac-Moody analogs of $E_{10}$ and $E_{11}$ algebras appear in M-theory speculations. Very little is known about these algebras. Already $E < sub > n < /sub >$, $n > 8$ is infinite-dimensional as an analog of Lie algebra. The following argument shows that $E_8$ representations do not work in TGD context unless one allows anyonic statistics.

1. In TGD framework space-time dimension is D=8. The speculative hypothesis of HO-H duality inspired by string model dualities states that the descriptions based on the two choices of imbedding space are dual. One can start from 8-D Cartan algebra defined by quantized $M^8$ coordinates regarded as fields at string orbit just as in string model. A natural constraint is that the symmetries act as isometries or holonomies of the effectively compactified $M^8$. The article "The Octonions" [A32] of John Baez discusses exceptional Lie groups and shows that compact form of $E_8$ appears as isometry group of 16-dimensional octo-octonionic projective plane $E_8/(Spin(16)/Z_2)$: the analog of $CP_2$ for complexified octonions. There is no 8-D space allowing $E_8$ as an isometry group. Only SO(1,7) can be realized as the maximal Lorentz group with 8-D translational invariance.

2. In HO picture some Kac Moody algebra with rank 8 acting on quantized $M^8$ coordinates defining stringy fields is natural. The charged generators of this algebra are constructible using the standard recipe involving operators creating coherent states and their conjugates obtained as

operator counterparts of plane waves with momenta replaced by roots of the simply laced algebra in question and by normal ordering.

3. Poincare group has 4-D maximal Cartan algebra and this means that only 4 Euclidian dimensions remain. Lorentz generators can be constructed in standard manner in terms of Kac-Moody generators as Noether currents.

4. The natural Kac-Moody counterpart for spontaneous compactification to $CP_2$ would be that these dimensions give rise to the generators of electro-weak gauge group identifiable as a product of isometry and holonomy groups of $CP_2$ in the dual H-picture based on $M^4 \times CP_2$. Note that in this picture electro-weak symmetries would act geometrically in $E^4$ whereas in $CP_2$ picture they would act only as holonomies.

Could one weaken the assumption that Kac-Moody generators act as symmetries and that spin-statistics relation would be satisfied?

1. The hierarchy of Planck constants relying on the generalization of the notion of imbedding space breaks Poincare symmetry to Lorentz symmetry for a given sector of the world of classical worlds for which one considers light-like 3-surfaces inside future and past directed light cones. Translational invariance is obtained from the wave function for the position of the tip of the light cone in $M^4$. In this kind of situation one could consider even $E_8$ symmetry as a dynamical symmetry.

2. The hierarchy of Planck constants involves a hierarchy of groups and fractional statistics at the partonic 2-surface with rotations interpreted as braiding homotopies. The fractionization of spin allows anyonic statistics and could allow bosons with anyonic half-odd integer spin. Also more general fractional spins are possible so that one can consider also more general algebras than Kac-Moody algebras by allowing roots to have more general values. Quantum versions of Kac-Moody algebras would be in question. This picture would be consistent with the view that TGD can emulate any gauge algebra with 8-D Cartan algebra and Kac-Moody algebra dynamically. This vision was originally inspired by the study of the inclusions of hyper-finite factors of type II¡sub¿1¡/sub¿. Even higher dimensional Kac-Moody algebras are predicted to be possible.

3. It must be emphasized that these considerations relate in TGD framework to Super-Kac Moody algebra only. The so called super-symplectic algebra is the second quintessential part of the story. In particular, color is not spin-like quantum number for quarks and quark color corresponds to color partial waves in the world of classical worlds or more concretely, to the rotational degrees of freedom in $CP_2$ analogous to ordinary rotational degrees of freedom of rigid body. Arbitrarily high color partial waves are possible and also leptons can move in triality zero color partial waves and there is a considerable experimental evidence for color octet excitations of electron and muon but put under the rug.

### 8.5.5 Can one interpret three fermion families in terms of $E_8$ in TGD framework?

The prediction of three fermion generations by $E_8$ picture must be taken very seriously. In TGD three fermion generations correspond to three lowest genera $g = 0, 1, 2$ (handle number) for which all 2-surfaces have $Z_2$ as global conformal symmetry (hyper-ellipticity [K11, K29] ). One can assign to the three genera a dynamical SU(3) symmetry. Theye are related by SU(3) triality which brings in mind the triality symmetry acting on fermion generations in $E_8$ model. SU(3) octet and singlet bosons correspond to pairs of light-like 3-surfaces defining the throats of a wormhole contact and since their genera can be different one has color singlet and octet bosons. Singlet corresponds to ordinary bosons. Color octet bosons must be heavy since they define neutral currents between fermion families.

The three $E_8$ anyonic boson families cannot represent family replication since these symmetries are not local conformal symmetries: it obviously does not make sense to assign a handle number to a given point of partonic 2-surface! Also bosonic octet would be missing in $E_8$ picture.

One could of course say that in $E_8$ picture based on fractional statistics, anyonic gauge bosons can mimic the dynamical symmetry associated with the family replication. This is in spirit with the idea that TGD Universe is able to emulate practically any gauge - or Kac-Moody symmetry and that TGD Universe is busily mimicking also itself.

To sum up, the rank 8 Kac-Moody algebra - emerging naturally if one takes HO-H duality seriously - corresponds very naturally to Kac-Moody representations in terms of free stringy fields for Poincare-, color-, and electro-weak symmetries. One can however consider the possibility of anyonic symmetries and the emergence of non-compact version of $E_8$ as a dynamical symmetry, and TGD suggests much more general dynamical symmetries if TGD Universe is able to act as the physics analog of the Universal Turing machine.

# Chapter 1

# Appendix

## A-1 Basic properties of $CP_2$ and elementary facts about p-adic numbers

### A-1.1 $CP_2$ as a manifold

$CP_2$, the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space $C^3$ under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{A-1.1}$$

Here $\lambda$ is any non-zero complex number. Note that $CP_2$ can be also regarded as the coset space $SU(3)/U(2)$. The pair $z^i/z^j$ for fixed $j$ and $z^i \neq 0$ defines a complex coordinate chart for $CP_2$. As $j$ runs from 1 to 3 one obtains an atlas of three oordinate charts covering $CP_2$, the charts being holomorphically related to each other (e.g. $CP_2$ is a complex manifold). The points $z^3 \neq 0$ form a subset of $CP_2$ homoeomorphic to $R^4$ and the points with $z^3 = 0$ a set homeomorphic to $S^2$. Therefore $CP_2$ is obtained by "adding the 2-sphere at infinity to $R^4$".

Besides the standard complex coordinates $\xi^i = z^i/z^3$ , $i = 1, 2$ the coordinates of Eguchi and Freund [A69] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \ , \\ \xi^2 &= x + iy \ . \end{aligned} \tag{A-1.2}$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= rexp(i\frac{(\Psi + \Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= rexp(i\frac{(\Psi - \Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{A-1.3}$$

The ranges of the variables $r, \Theta, \Phi, \Psi$ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold $CP_2$ is compact and simply connected, with Euler number Euler number 3, Pontryagin number 3 and second $b = 1$.

### A-1.2 Metric and Kähler structure of $CP_2$

In order to obtain a natural metric for $CP_2$, observe that $CP_2$ can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere $S^5$: $\sum z^i \bar{z}^i = R^2$. The metric of $CP_2$ is obtained by projecting the metric of $S^5$ orthogonally to the orbits of the isometries. Therefore the distance between the points of $CP_2$ is that between the representative orbits on $S^5$.

The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-1.4}$$

where the Hermitian, in fact Kähler metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K \ , \tag{A-1.5}$$

where the function $K$, Kähler function, is defined as

$$\begin{aligned} K &= log(F) \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-1.6}$$

The Kähler function for $S^2$ has the same form. It gives the $S^2$ metric $dzd\bar{z}/(1+r^2)^2$ related to its standard form in spherical coordinates by the coordinate transformation $(r, \phi) = (tan(\theta/2), \phi)$.

The representation of the $CP_2$ metric is deducible from $S^5$ metric is obtained by putting the angle coordinate of a geodesic sphere constant in it and is given

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} \ , \tag{A-1.7}$$

where the quantities $\sigma_i$ are defined as

$$\begin{aligned} r^2\sigma_1 &= Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_2 &= -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_3 &= -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \end{aligned} \tag{A-1.8}$$

$R$ denotes the radius of the geodesic circle of $CP_2$. The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A \ , \tag{A-1.9}$$

are given by

$$\begin{aligned} e^0 &= \tfrac{dr}{F} \ , \quad & e^1 &= \tfrac{r\sigma_1}{\sqrt{F}} \ , \\ e^2 &= \tfrac{r\sigma_2}{\sqrt{F}} \ , \quad & e^3 &= \tfrac{r\sigma_3}{F} \ . \end{aligned} \tag{A-1.10}$$

The explicit representations of vierbein vectors are given by

$$\begin{aligned} e^0 &= \tfrac{dr}{F} \ , & e^1 &= \tfrac{r(sin\Theta cos\Psi d\Phi + sin\Psi d\Theta)}{2\sqrt{F}} \ , \\ e^2 &= \tfrac{r(sin\Theta sin\Psi d\Phi - cos\Psi d\Theta)}{2\sqrt{F}} \ , & e^3 &= \tfrac{r(d\Psi + cos\Theta d\Phi)}{2F} \ . \end{aligned} \tag{A-1.11}$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 = \frac{dr^2}{F^2} + \frac{r^2}{4F^2}(d\Psi + cos\Theta d\Phi)^2 + \frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2) \ . \tag{A-1.12}$$

The vierbein connection satisfying the defining relation

$$de^A = -V^A_B \wedge e^B \ , \tag{A-1.13}$$

is given by

$$\begin{array}{lllllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r-\frac{1}{r})e^3 \ , & V_{12} & = & (2r+\frac{1}{r})e^3 \ . \end{array} \tag{A-1.14}$$

The representation of the covariantly constant curvature tensor is given by

$$\begin{array}{lllllll} R_{01} & = & e^0\wedge e^1 - e^2\wedge e^3 \ , & R_{23} & = & e^0\wedge e^1 - e^2\wedge e^3 \ , \\ R_{02} & = & e^0\wedge e^2 - e^3\wedge e^1 \ , & R_{31} & = & -e^0\wedge e^2 + e^3\wedge e^1 \ , \\ R_{03} & = & 4e^0\wedge e^3 + 2e^1\wedge e^2 \ , & R_{12} & = & 2e^0\wedge e^3 + 4e^1\wedge e^2 \ . \end{array} \tag{A-1.15}$$

Metric defines a real, covariantly constant, and therefore closed 2-form $J$

$$J = -ig_{a\bar{b}}d\xi^a d\bar{\xi}^b \ , \tag{A-1.16}$$

the so called Kähler form. Kähler form $J$ defines in $CP_2$ a symplectic structure because it satisfies the condition

$$J^k_{\ r}J^{rl} = -s^{kl} \ . \tag{A-1.17}$$

The form $J$ is integer valued and by its covariant constancy satisfies free Maxwell equations. Hence it can be regarded as a curvature form of a $U(1)$ gauge potential $B$ carrying a magnetic charge of unit $1/2g$ ($g$ denotes the gauge coupling). Locally one has therefore

$$J = dB \ , \tag{A-1.18}$$

where $B$ is the so called Kähler potential, which is not defined globally since $J$ describes homological magnetic monopole.

It should be noticed that the magnetic flux of $J$ through a 2-surface in $CP_2$ is proportional to its homology equivalence class, which is integer valued. The explicit representations of $J$ and $B$ are given by

$$\begin{array}{lll} B & = & 2re^3 \ , \\ J & = & 2(e^0\wedge e^3 + e^1\wedge e^2) = \frac{r}{F^2}dr\wedge(d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F}sin\Theta d\Theta d\Phi \ . \end{array} \tag{A-1.19}$$

The vierbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1,1).

Useful coordinates for $CP_2$ are the so called canonical coordinates in which Kähler potential and Kähler form have very simple expressions

$$\begin{array}{lll} B & = & \sum_{k=1,2} P_k dQ_k \ , \\ J & = & \sum_{k=1,2} dP_k\wedge dQ_k \ . \end{array} \tag{A-1.20}$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$
\begin{aligned}
P_1 &= -\frac{1}{1+r^2} \ , \\
P_2 &= \frac{r^2 cos\Theta}{2(1+r^2)} \ , \\
Q_1 &= \Psi \ , \\
Q_2 &= \Phi \ .
\end{aligned} \tag{A-1.21}
$$

### A-1.3 Spinors in $CP_2$

$CP_2$ doesn't allow spinor structure in the conventional sense [A63] . However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of $CP_2$ play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space $M$. The parallel propagation around a closed curve with a base point $x$ leads to a rotated vierbein at $x$: $e^A = R^A_B e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0,1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere $S^2$ in M and the element $R^A_B(v)$ defines a closed path in $SO(4)$. When the sphere $S^2$ is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface $S^2$ the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallel propagate also spinors and by the above construction associate a closed path of Spin(4) to the surface $S^2$. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the non-allowed $-1$- factor associated with the parallel transport of the spinor around the sphere $S^2$ by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating $-1$-factor. For a $U(1)$ gauge potential this factor is given by the exponential $exp(i2\Phi)$ , where $\Phi$ is the magnetic flux through the surface. This factor has the value $-1$ provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of $CP_2$ the required gauge potential is half odd multiple of the Kähler potential $B$ defined previously. In the case of $M^4 \times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

### A-1.4 Geodesic sub-manifolds of $CP_2$

Geodesic sub-manifolds are defined as sub-manifolds having common geodesic lines with the imbedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors $h^k_\alpha$ (understood as vectors of $H$) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to $H$ and $X^4$.

In [A48] a general characterization of the geodesic sub-manifolds for an arbitrary symmetric space $G/H$ is given. Geodesic sub-manifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra $g$ of the group $G$. The Lie triple system $t$ is defined as a subspace of $g$ characterized by the closedness property with respect to double commutation

$$
[X,[Y,Z]] \in t \ \text{ for } \ X,Y,Z \in t \ . \tag{A-1.22}
$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal $3\times 3$ matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic sub-manifold of $CP_2$.

Standard representatives for the geodesic spheres of $CP_2$ are given by the equations

$$S^2_I \ : \ \xi^1 = \bar{\xi}^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Psi = 0) \ ,$$

$$S^2_{II} \ : \ \xi^1 = \xi^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Phi = 0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in $CP_2$. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for $S^2_I$. $S^2_{II}$ is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

# A-2 $CP_2$ geometry and standard model symmetries

## A-2.1 Identification of the electro-weak couplings

The delicacies of the spinor structure of $CP_2$ make it a unique candidate for space $S$. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different $H$-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [B23] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space $H$ allows to define three different chiralities for spinors. Spinors with fixed $H$-chirality $e=\pm 1$, $CP_2$-chirality $l,r$ and $M^4$-chirality $L,R$ are defined by the condition

$$\begin{aligned} \Gamma\Psi &= e\Psi \ , \\ e &= \pm 1 \ , \end{aligned} \qquad \text{(A-2.1)}$$

where $\Gamma$ denotes the matrix $\Gamma_9 = \gamma_5\times\gamma_5$, $1\times\gamma_5$ and $\gamma_5\times 1$ respectively. Clearly, for a fixed $H$-chirality $CP_2$- and $M^4$-chiralities are correlated.

The spinors with $H$-chirality $e=\pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite $H$-chirality one can identify the vielbein group of $CP_2$ as the electro-weak group: $SO(4) = SU(2)_L\times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ . \qquad \text{(A-2.2)}$$

Here $V$ and $B$ denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor $H$-chirality $+(-)$. The integers $n_\pm$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection $V$ and of $B$ are given by the equations

$$\begin{aligned} V_{01} &= -\frac{e^1}{r} \ , & V_{23} &= \frac{e^1}{r} \ , \\ V_{02} &= -\frac{e^2}{r} \ , & V_{31} &= \frac{e^2}{r} \ , \\ V_{03} &= (r-\frac{1}{r})e^3 \ , & V_{12} &= (2r+\frac{1}{r})e^3 \ , \end{aligned} \qquad \text{(A-2.3)}$$

and

$$B = 2re^3 \ , \qquad \text{(A-2.4)}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying $\Sigma_3^0$ and $\Sigma_2^1$ as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23}I_L^1 + 2V_{13}I_L^2 \ , \tag{A-2.5}$$

where one have defined

$$\begin{aligned} I_L^1 &= \frac{(\Sigma_{01} - \Sigma_{23})}{2} \ , \\ I_L^2 &= \frac{(\Sigma_{02} - \Sigma_{13})}{2} \ . \end{aligned} \tag{A-2.6}$$

$A_{ch}$ is clearly left handed so that one can perform the identification

$$W^{\pm} = \frac{2(e^1 \pm ie^2)}{r} \ , \tag{A-2.7}$$

where $W^{\pm}$ denotes the charged intermediate vector boson.

Consider next the identification of the neutral gauge bosons $\gamma$ and $Z^0$ as appropriate linear combinations of the two functionally independent quantities

$$\begin{aligned} X &= re^3 \ , \\ Y &= \frac{e^3}{r} \ , \end{aligned} \tag{A-2.8}$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$\begin{aligned} \bar{\gamma} &= aX + bY \ , \\ \bar{Z}^0 &= cX + dY \ , \end{aligned} \tag{A-2.9}$$

where the normalization condition

$$ad - bc = 1 \ ,$$

is satisfied. The physical fields $\gamma$ and $Z^0$ are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$\begin{aligned} A_{nc} &= [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma} \\ &+ [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 \ . \end{aligned} \tag{A-2.10}$$

Identifying $\Sigma_{12}$ and $\Sigma_{03} = 1 \times \gamma_5 \Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that $\gamma$ couples vectorially leads to the condition

$$c = -d \ . \tag{A-2.11}$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$A_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-2.12}$$

Here the electromagnetic charge $Q_{em}$ and the weak isospin are defined by

$$\begin{aligned} Q_{em} &= \Sigma^{12} + \frac{(n_+ 1_+ + n_- 1_-)}{6} \ , \\ I_L^3 &= \frac{(\Sigma^{12} - \Sigma^{03})}{2} \ . \end{aligned} \tag{A-2.13}$$

The fields $\gamma$ and $Z^0$ are defined via the relations

$$\begin{aligned} \gamma &= 6d\bar{\gamma} = \frac{6}{(a+b)}(aX + bY) \ , \\ Z^0 &= 4(a+b)\bar{Z}^0 = 4(X - Y) \ . \end{aligned} \tag{A-2.14}$$

The value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{3b}{2(a+b)} \ , \tag{A-2.15}$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of Weinberg angle is a dynamical problem. The angle is completely fixed once the YM action is fixed by requiring that action contains no cross term of type $\gamma Z^0$. Pure symmetry non-broken electro-weak YM action leads to a definite value for the Weinberg angle. One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle.

To evaluate the value of the Weinberg angle one can express the neutral part $F_{nc}$ of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+ 1_+ + n_- 1_-) \ , \tag{A-2.16}$$

where one has

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \end{aligned} \tag{A-2.17}$$

in terms of the fields $\gamma$ and $Z^0$ (photon and $Z$- boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-2.18}$$

Evaluating the expressions above one obtains for $\gamma$ and $Z^0$ the expressions

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{03} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \tag{A-2.19}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) \ . \tag{A-2.20}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned} L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} \ , \\ L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) \ , \end{aligned} \tag{A-2.21}$$

where the trace is taken in spinor representation, in terms of $\gamma$ and $Z^0$ one obtains for the coefficient $X$ of the $\gamma Z^0$ cross term (this coefficient must vanish) the expression

$$\begin{aligned} X &= -\frac{K}{2g^2} + \frac{fp}{18} \ , \\ K &= Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] \ , \end{aligned} \tag{A-2.22}$$

In the general case the value of the coefficient $K$ is given by

$$K = \sum_i \left[-\frac{(18+2n_i^2)sin^2\theta_W}{9}\right] \ , \tag{A-2.23}$$

where the sum is over the spinor chiralities, which appear as elementary fermions and $n_i$ is the integer describing the coupling of the spinor field to the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i(18+n_i^2))} \ . \tag{A-2.24}$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} \ . \tag{A-2.25}$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is quite close to the typical value 9/24 of GUTs [B44] .

## A-2.2 Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:
a) Symmetries must be realized as purely geometric transformations.
b) Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [B14] .

The action of the reflection $P$ on spinors of is given by

$$\Psi \rightarrow P\Psi = \gamma^0 \otimes \gamma^0 \Psi \ . \tag{A-2.26}$$

in the representation of the gamma matrices for which $\gamma^0$ is diagonal. It should be noticed that $W$ and $Z^0$ bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in $CP_2$ is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) \ , \\ \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \gamma^1\gamma^3 \otimes 1\Psi \ . \end{aligned} \tag{A-2.27}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in $CP_2$:

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \Psi^\dagger \gamma^2 \gamma^0 \otimes 1 \ . \end{aligned} \tag{A-2.28}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

# A-3 Basic facts about induced gauge fields

Since the classical gauge fields are closely related in TGD framework, it is not possible to have space-time sheets carrying only single kind of gauge field. For instance, em fields are accompanied by $Z^0$ fields for extremals of Kähler action. Weak forces is however absent unless the space-time sheets contains topologically condensed exotic weakly charged particles responding to this force. Same applies to classical color forces. The fact that these long range fields are present forces to assume that there exists a hierarchy of scaled up variants of standard model physics identifiable in terms of dark matter.

Classical em fields are always accompanied by $Z^0$ field and some components of color gauge field. For extremals having homologically non-trivial sphere as a $CP_2$ projection em and $Z^0$ fields are the only non-vanishing electroweak gauge fields. For homologically trivial sphere only $W$ fields are non-vanishing. Color rotations does not affect the situation.

For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy by 2-dimensionality of the $CP_2$ projection. Color gauge field has $U(1)$ holonomy for all space-time surfaces and quantum classical correspondence suggest a weak form of color confinement meaning that physical states correspond to color neutral members of color multiplets.

## A-3.1 Induced gauge fields for space-times for which CP$_2$ projection is a geodesic sphere

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP$_2$ projection, only vacuum extremals and space-time surfaces for which CP$_2$ projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing $W$ fields and homologically non-trivial sphere to non-vanishing $W$ fields but vanishing $\gamma$ and $Z^0$. This can be verified by explicit examples.

$r=\infty$ surface gives rise to a homologically non-trivial geodesic sphere for which $e_0$ and $e_3$ vanish imply the vanishing of $W$ field. For space-time sheets for which CP$_2$ projection is $r=\infty$ homologically non-trivial geodesic sphere of $CP_2$ one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \ .$$

The induced $W$ fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

$Im(\xi^1) = Im(\xi^2) = 0$ corresponds to homologically trivial geodesic sphere. A more general representative is obtained by using for the phase angles of standard complex $CP_2$ coordinates constant values. In this case $e^1$ and $e^3$ vanish so that the induced em, $Z^0$, and Kähler fields vanish but induced $W$ fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP$_2$ projection color rotations and weak symmetries commute.

## A-3.2 Space-time surfaces with vanishing em, $Z^0$, or Kähler fields

In the following the induced gauge fields are studied for general space-time surface without assuming the extremal property. In fact, extremal property reduces the study to the study of vacuum extremals and surfaces having geodesic sphere as a $CP_2$ projection and in this sense the following arguments are somewhat obsolete in their generality.

### Space-times with vanishing em, $Z^0$, or Kähler fields

The following considerations apply to a more general situation in which the homologically trivial geodesic sphere and extremal property are not assumed. It must be emphasized that this case is possible in TGD framework only for a vanishing Kähler field.

Using spherical coordinates $(r,\Theta,\Psi,\Phi)$ for $CP_2$, the expression of Kähler form reads as

$$\begin{aligned} J &= \frac{r}{F^2}dr\wedge(d\Psi+cos(\Theta)d\Phi)+\frac{r^2}{2F}sin(\Theta)d\Theta\wedge d\Phi \ , \\ F &= 1+r^2 \ . \end{aligned} \tag{A-3.1}$$

The general expression of electromagnetic field reads as

$$\begin{aligned} F_{em} &= (3+2p)\frac{r}{F^2}dr\wedge(d\Psi+cos(\Theta)d\Phi)+(3+p)\frac{r^2}{2F}sin(\Theta)d\Theta\wedge d\Phi \ , \\ p &= sin^2(\Theta_W) \ , \end{aligned} \tag{A-3.2}$$

where $\Theta_W$ denotes Weinberg angle.

a) The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\begin{aligned} \Psi &= k\Phi \ , \\ (3+2p)\frac{1}{r^2F}(d(r^2)/d\Theta)(k+cos(\Theta)) &+ (3+p)sin(\Theta)=0 \ , \end{aligned} \tag{A-3.3}$$

hold true. The conditions imply that $CP_2$ projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$\begin{aligned} r &= \sqrt{\frac{X}{1-X}} \ , \\ X &= D\left[|\frac{(k+u}{C}|\right]^{\epsilon} \ , \\ u &\equiv cos(\Theta) \ , \ C=k+cos(\Theta_0) \ , \ D=\frac{r_0^2}{1+r_0^2} \ , \ \epsilon=\frac{3+p}{3+2p} \ , \end{aligned} \tag{A-3.4}$$

where $C$ and $D$ are integration constants. $0\leq X\leq 1$ is required by the reality of $r$. $r=0$ would correspond to $X=0$ giving $u=-k$ achieved only for $|k|\leq 1$ and $r=\infty$ to $X=1$ giving $|u+k|=[(1+r_0^2)/r_0^2)]^{(3+2p)/(3+p)}$ achieved only for

$$sign(u+k)\times[\frac{1+r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}}\leq k+1 \ ,$$

where $sign(x)$ denotes the sign of $x$.

The expressions for Kähler form and $Z^0$ field are given by

$$\begin{aligned} J &= -\frac{p}{3+2p}Xdu\wedge d\Phi \ , \\ Z^0 &= -\frac{6}{p}J \ . \end{aligned} \tag{A-3.5}$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range $Z^0$ vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

b) The vanishing of $Z^0$ fields is achieved by the replacement of the parameter $\epsilon$ with $\epsilon=1/2$ as becomes clear by considering the condition stating that $Z^0$ field vanishes identically. Also the relationship $F_{em}=3J=-\frac{3}{4}\frac{r^2}{F}du\wedge d\Phi$ is useful.

c) The vanishing Kähler field corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and $Z^0$ fields are proportional to each other:

$$\begin{aligned}
Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k+u)\frac{\partial r}{\partial u} du \wedge d\Phi = (k+u) du \wedge d\Phi \ , \\
r &= \sqrt{\frac{X}{1-X}} \ , \ X = D|k+u| \ , \\
\gamma &= -\frac{p}{2}Z^0 \ .
\end{aligned} \tag{A-3.6}$$

For a vanishing value of Weinberg angle ($p = 0$) em field vanishes and only $Z^0$ field remains as a long range gauge field. Vacuum extremals for which long range $Z^0$ field vanishes but em field is non-vanishing are not possible.

### The effective form of $CP_2$ metric for surfaces with 2-dimensional $CP_2$ projection

The effective form of the $CP_2$ metric for a space-time having vanishing em,$Z^0$, or Kähler field is of practical value in the case of vacuum extremals and is given by

$$\begin{aligned}
ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta})d\Theta^2 + (s_{\Phi\Phi} + 2ks_{\Phi\Psi})d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta}d\Theta^2 + s^{eff}_{\Phi\Phi}d\Phi^2] \ , \\
s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X\right] \ , \\
s^{eff}_{\Phi\Phi} &= X \times \left[(1-X)(k+u)^2 + 1 - u^2\right] \ ,
\end{aligned} \tag{A-3.7}$$

and is useful in the construction of vacuum imbedding of, say Schwartchild metric.

### Topological quantum numbers

Space-times for which either em, $Z^0$, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers ($\omega_1$ and $\omega_2$) are frequency type parameters, two ($k_1$ and $k_2$ ) are wave vector like quantum numbers, two of the quantum numbers ($n_1$ and $n_2$) are integers. The parameters $\omega_i$ and $n_i$ will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of $CP_2$ coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

Under rather general conditions the coordinates $\Psi$ and $\Phi$ can be written in the form

$$\begin{aligned}
\Psi &= \omega_2 m^0 + k_2 m^3 + n_2\phi + \text{Fourier expansion} \ , \\
\Phi &= \omega_1 m^0 + k_1 m^3 + n_1\phi + \text{Fourier expansion} \ .
\end{aligned} \tag{A-3.8}$$

$m^0$,$m^3$ and $\phi$ denote the coordinate variables of the cylindrical $M^4$ coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters $\omega_i$,$k_i$ and $n_i$ and $m$ and $C$ are bounded by the surfaces at which space-time surface becomes ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters $r_0$ and $\Theta_0$. At $r = \infty$ surfaces $n_2$,$\omega_2$ and $m$ can change since all values of $\Psi$ correspond to the same point of $CP_2$: at $r = 0$ surfaces also $n_1$ and $\omega_1$ can change since all values of $\Phi$ correspond to same point of $CP_2$, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global imbedding for, say a constant magnetic field. Although global imbedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate $u$ in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteing the absence of edges are satisfied.

For instance, the vanishing of the electromagnetic fields implies that the condition

$$\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ , \tag{A-3.9}$$

is satisfied. In particular, the ratio $\omega_2/\omega_1$ is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter $n_1$ and $n_2$ ($\omega_1$ and $\omega_2$) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

# A-4 p-Adic numbers and TGD

## A-4.1 p-Adic number fields

p-Adic numbers ($p$ is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A37] . p-Adic numbers are representable as power expansion of the prime number $p$ of form:

$$x = \sum_{k \geq k_0} x(k)p^k, \ x(k) = 0, ...., p-1 \ . \tag{A-4.1}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \ . \tag{A-4.2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \ , \tag{A-4.3}$$

where $\varepsilon(x) = k + ....$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x, z) \leq max\{d(x, y), d(y, z)\} \ . \tag{A-4.4}$$

The properties of the distance function make it possible to decompose $R_p$ into a union of disjoint sets using the criterion that $x$ and $y$ belong to same class if the distance between $x$ and $y$ satisfies the condition

$$d(x, y) \leq D \ . \tag{A-4.5}$$

This division of the metric space into classes has following properties:

a) Distances between the members of two different classes $X$ and $Y$ do not depend on the choice of points $x$ and $y$ inside classes. One can therefore speak about distance function between classes.

b) Distances of points $x$ and $y$ inside single class are smaller than distances between different classes.

c) Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B36] . The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

## A-4.2 Canonical correspondence between p-adic and real numbers

The basic challenge encountered by p-adic physicist is how to map the predictions of the p-adic physics to real numbers. p-Adic probabilities provide a basic example in this respect. Identification via common rationals and canonical identification and its variants have turned out to play a key role in this respect.

### Basic form of canonical identification

There exists a natural continuous map $I : R_p \to R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \to x = \sum_{k<N} y_k p^{-k} \quad , \\ y_k &\in \{0, 1, .., p-1\} \quad . \end{aligned} \tag{A-4.6}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique $(1 = 0.999...)$ for the real numbers $x$, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \quad , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \quad . \end{aligned} \tag{A-4.7}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^k \quad , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k \\ &= y_1 + (x_N - 1)p^N - p^{N+1} \quad , \end{aligned} \tag{A-4.8}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

### The topology induced by canonical identification

The topology induced by the canonical identification in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. A-4.2) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of $p$ is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

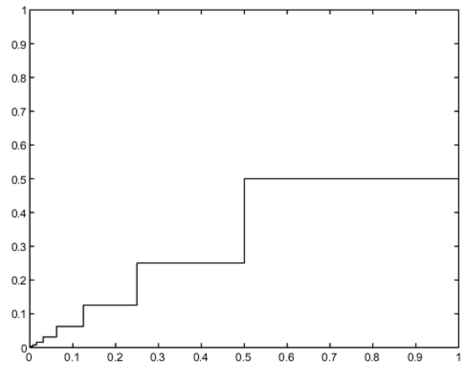


Figure 1: The real norm induced by canonical identification from 2-adic norm.

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of $p$. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative $-1_p$ associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number $x$. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

These results suggest that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$
\begin{aligned}
(x+y)_R &\leq x_R + y_R \ , \\
|x|_p |y|_R \leq (xy)_R &\leq x_R y_R \ ,
\end{aligned}
\tag{A-4.9}
$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$
\begin{aligned}
(x+y)_R &\leq x_R + y_R \ , \\
|\lambda|_p |y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ ,
\end{aligned}
\tag{A-4.10}
$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{A-4.11}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of $p$.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some non-linear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

### Modified form of the canonical identification

The original form of the canonical identification is continuous but does not respect symmetries even approximately. This led to a search of variants which would do better in this respect. The modification of the canonical identification applying to rationals only and given by

$$I_Q(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{A-4.12}$$

is uniquely defined for rationals, maps rationals to rationals, has also a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$. It has turned out that it is this map which most naturally appears in the applications. The map is obviously continuous locally since p-adically small modifications of $r$ and $s$ mean small modifications of the real counterparts.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. The predictions for the light elementary particle masses are within extreme accuracy same for $I$ and $I_Q$ but $I_Q$ is theoretically preferred since the real probabilities obtained from p-adic ones by $I_Q$ sum up to one in p-adic thermodynamics.

### Generalization of number concept and notion of imbedding space

TGD forces an extension of number concept: roughly a fusion of reals and various p-adic number fields along common rationals is in question. This induces a similar fusion of real and p-adic imbedding spaces. Since finite p-adic numbers correspond always to non-negative reals $n$-dimensional space $R^n$ must be covered by $2^n$ copies of the p-adic variant $R^n_p$ of $R^n$ each of which projects to a copy of $R^n_+$ (four quadrants in the case of plane). The common points of p-adic and real imbedding spaces are rational points and most p-adic points are at real infinity.

For a given p-adic space-time sheet most points are literally infinite as real points and the projection to the real imbedding space consists of a discrete set of rational points: the interpretation in terms of the unavoidable discreteness of the physical representations of cognition is natural. Purely local p-adic physics implies real p-adic fractality and thus long range correlations for the real space-time surfaces having enough common points with this projection.

p-Adic fractality means that $M^4$ projections for the rational points of space-time surface $X^4$ are related by a direct identification whereas $CP_2$ coordinates of $X^4$ at these points are related by $I$, $I_Q$ or some of its variants implying long range correlates for $CP_2$ coordinates. Since only a discrete set of points are related in this manner, both real and p-adic field equations can be satisfied and there are no problems with symmetries. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs. Also multi-p-fractality is possible.

## REFERENCES

# Mathematics

[A1] Analytization trick. http://en.wikipedia.org/wiki/Analytization_trick.

[A2] Bracket Polynomial. http://en.wikipedia.org/wiki/Bracket_polynomial.

[A3] Catastrophe Theory. http://en.wikipedia.org/wiki/Catastrophe_theory#Cusp_catastrophe.

[A4] Connected sum. http://en.wikipedia.org/wiki/Connected_sum.

[A5] Contact geometry. http://en.wikipedia.org/wiki/Contact_geometry.

[A6] Donaldson theorem. http://en.wikipedia.org/wiki/Donaldson%27s_theorem.

[A7] Einstein manifold. http://en.wikipedia.org/wiki/Einstein_manifold.

[A8] Gerbe. http://en.wikipedia.org/wiki/Gerbe.

[A9] Homepage of Louis Kauffman. http://homepages.math.uic.edu/~kauffman/.

[A10] Hyper-Kähler manifold. http://en.wikipedia.org/wiki/Hyper-Khler_manifold.

[A11] Hyperbolic manifold. http://en.wikipedia.org/wiki/Hyperbolic_manifold.

[A12] Jones Polynomial. http://en.wikipedia.org/wiki/Jones_polynomial.

[A13] Kac-Moody algebra. http://en.wikipedia.org/wiki/KacMoody_algebra.

[A14] Knot genus. http://en.wikipedia.org/wiki/Knot_genus.

[A15] Knot invariants. http://en.wikipedia.org/wiki/Knot_invariants.

[A16] Malcev algebra. http://en.wikipedia.org/wiki/Malcev_algebra.

[A17] Mandelbrot set. http://en.wikipedia.org/wiki/Mandelbrot_set.

[A18] Mostow Rigidity Theorem. http://en.wikipedia.org/wiki/Mostow_rigidity_theorem.

[A19] Octonions. http://en.wikipedia.org/wiki/Octonions.

[A20] Operad theory. http://en.wikipedia.org/wiki/Operad.

[A21] Quaternions. http://en.wikipedia.org/wiki/Quaternion.

[A22] Ricci flow. http://en.wikipedia.org/wiki/Ricci_flow.

[A23] Scale invariance vs. conformal invariance. http://en.wikipedia.org/wiki/Conformal_field_theory#Scale_invariance_vs._conformal_invariance.

[A24] Seven-dimensional cross product. http://en.wikipedia.org/wiki/Seven-dimensional_cross_product.

[A25] Super Virasoro algebra. http://en.wikipedia.org/wiki/Super_Virasoro_algebra.

[A26] T-duality. http://en.wikipedia.org/wiki/T-duality.

[A27] The Riemann zeta function interpreted as partition function. http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin//zeta/physics2.htm.

[A28] Thurston's theorem. http://en.wikipedia.org/wiki/Thurston's_theorem.

[A29] Video about Edward Witten's talk relating to his work with knot invariants. http://video.ias.edu/webfm_send/1787.

[A30] Yangian symmetry. http://en.wikipedia.org/wiki/Yangian.

[A31] M. Atyiah and N. Hitschin. *The Geometry and Dynamics of Magnetic Monopoles.* Princeton University Press, 1988.

[A32] John C. Baez. The Octonions. *Bull. Amer. Math. Soc.*. *http://math.ucr.edu/home/baez/Octonions/octonions.html*, 39(2002), 2001.

[A33] D. Bar-Natan. Khovanovs homology for tangles and cobordisms. http://www.math.toronto.edu/~drorbn/papers/Cobordism/Cobordism.pdf, 2005.

[A34] M. Barnsley. *Fractals Everywhere.* Academic Press, 1988.

[A35] A. L. Besse. *Einstein Manifolds.* Springer Verlag, 1987.

[A36] Z. I. Borevich and I. R. Shafarevich. *Number Theory.* Academic Press, 1966.

[A37] Z. I. Borevich and I. R. Shafarevich. *Number Theory.* Academic Press, 1966.

[A38] G. Sierra C. Gomez, M. Ruiz-Altaba. *Quantum Groups and Two-Dimensional Physics.* Cambridge University Press, Cambridge, 1996.

[A39] M. L. Ge C. N. Yang. *Braid Group, Knot Theory, and Statistical Mechanics.* World Scientific, 1989.

[A40] J. Scott Carter. A Survey of Quandle Ideas. http://arxiv.org/pdf/1002.4429v2, 2010.

[A41] J. Dixmier. *Von Neumann Algebras.* North-Holland, Amsterdam, 1981.

[A42] J. Duistermaat, J. and J. Heckmann, G. *Inv. Math.*, 69, 1982.

[A43] Eisenhart. *Riemannian Geometry.* Princeton University Press, 1964.

[A44] D. S. Freed. The Geometry of Loop Groups , 1985.

[A45] C. N. Pope G. W. Gibbons. $CP_2$ as gravitational instanton. *Comm. Math. Phys.*, 55, 1977.

[A46] P. Goddard and D. Olive. Kac-Moody and Virasoro algebras in relation to Quantum Physics, 1986.

[A47] W. Hawking, S. and N. Pope, C. Generalized Spin Structures in Quantum Gravity. *Phys. Lett.*, (1), 1978.

[A48] S. Helgason. *Differential Geometry and Symmetric Spaces.* Academic Press, New York, 1962.

[A49] N. Hitchin. What is a Gerbe? *Notices of the AMS. http://www.ams.org/notices/200302/what-is.pdf*, (2), 2003.

[A50] R. A. Minklos I. M. Gelfand and Z. Ya. Shapiro. *Representations of the rotation and Lorentz groups and their applications.* Pergamon Press, 1963.

[A51] F. R. Jones. *Braid groups, Hecke algebras and type $II_1$ factors.* 1983.

[A52] C. Kassel. *Quantum Groups.* Springer Verlag, 1995.

[A53] L. Kauffman. Virtual knot theory. *European J. Combin.*, 20:663–690, 1999.

[A54] M. Kontsevich. Operads and Motives in Deformation Quantization. http://arxiv.org/abs/math/9904055, 1999.

[A55] M. L. Lapidus. *In Search of the Riemann Zeros: Strings, Fractal Membrans and Noncommutative Spacetimes.* American Mathematical Society, 2008.

[A56] B. Mandelbrot. *The Fractal Geometry of Nature.* Freeman, New York, 1977.

[A57] J. Mickelson. *Current Algebras and Groups.* Plenum, New York, 1989.

[A58] J. Mickelson. Gerbes, (Twisted) K-Theory, and the Supersymmetric WZW Model. http://arxiv.org/abs/hep-th/0206139, 2002.

[A59] J. Milnor. *Topology form Differential Point of View.* The University Press of Virginia, Virginia, 1965.

[A60] S. Matsuura N. Drukker, J. Gomis. Probing N = 4 SYM With Surface Operators. http://arxiv.org/pdf/0805.4199v1, 2006.

[A61] S. Nelson. The Combinatorial Revolution in Knot Theory. *Notices of the AMS.http://www.lqp.uni-goettingen.de/papers/99/04/99042900.html*, pages 1553–1561, 2011.

[A62] A. Kent P. Goddard and D. Olive. Unitary representations of the Virasoro and super-Virasoro algebras. *Comm. Math. Phys.*, 103(1), 1986.

[A63] N. Pope, C. Eigenfunctions and $Spin^c$ Structures on $CP_2$, 1980.

[A64] C. Vafa S. Gukov, A. Schwartz. Khovanov-Rozansky Homology and Topological Strings. http://arxiv.org/pdf/hep-th/0412243v3, 2005.

[A65] S. Salamon. Quaternionic Kähler manifolds. *Invent. Math.*, 67:143, 1982.

[A66] S. Sawin. Links, Quantum Groups, and TQFT's. http://arxiv.org/abs/q-alg/9506002, 1995.

[A67] M. Spivak. *Differential Geometry I,II,III,IV.* Publish or Perish, Boston, 1970.

[A68] H. Sugawara. A field theory of currents. *Phys. Rev.*, 176, 1968.

[A69] J. Hanson T. Eguchi, B. Gilkey. *Phys. Rep.*, 66:1980, 1980.

[A70] R. Thom. *Commentarii Math. Helvet.*, 28, 1954.

[A71] Wallace. *Differential Topology.* W. A. Benjamin, New York, 1968.

[A72] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121:351–399, 1989.

[A73] E. Witten. Fivebranes and Knots. http://arxiv.org/pdf/1101.3216v1, 2010.

[A74] E. C. Zeeman. *Catastrophe Theory.* Addison-Wessley Publishing Company, 1977.

# Theoretical Physics

[B1] Chern-Simons theory. http://en.wikipedia.org/wiki/ChernSimons_theory.

[B2] Duff, string theory, entanglement and hyperdeterminants. http://blog.vixra.org/2010/09/02/duff-string-theory-entanglement-and-hyperdeterminants/.

[B3] $E_8$. http://en.wikipedia.org/wiki/E8_(mathematics).

[B4] Hyperdeterminant. http://en.wikipedia.org/wiki/Hyperdeterminant.

[B5] Is mathematical pattern the theory of everything? *New Scientist.* *http://www.newscientist.com/channel/fundamentals/mg19626303.900-is-mathematical-pattern-the-theory-of-everything.html*, 2630.

[B6] M-theory lesson 350. http://pseudomonad.blogspot.com/2010/09/m-theory-lesson-350.html.

[B7] Monster group. http://en.wikipedia.org/wiki/Monster_group.

[B8] Montonen Olive Duality. http://en.wikipedia.org/wiki/Montonen-Olive_duality.

[B9] No-cloning theorem. http://en.wikipedia.org/wiki/No_cloning_theorem.

[B10] Self-Organized Criticality. http://en.wikipedia.org/wiki/Self-organized_criticality.

[B11] Topological quantum field theory. http://en.wikipedia.org/wiki/Topological_quantum_field_theory.

[B12] L. Alvarez-Gaume and Z. Freedman, D. Geometrical Structure and Ultraviolet Finiteness in the Super-symmetric $\sigma$-Model. *Comm. Math. Phys.*, 80, 1981.

[B13] Nima Arkani-Hamed. Scattering amplitides and the positive Grassmannian. LMU Lectures.https://cast.itunes.uni-muenchen.de/vod/clips/ifNaivs8Pf/flash.html.

[B14] J. Björken and S. Drell. *Relativistic Quantum Fields.* Mc Graw-Hill, New York, 1965.

[B15] M. M. Bowick and S. G. Rajeev. The holomorphic geometry of closed bosonic string theory and $Diff(S^1)/S^1$. *Nucl. Phys. B*, 293, 1987.

[B16] J.-B. Zuber C. Itzykson. *Quantum Field Theory.* Mc Graw-Hill, New York, 1980.

[B17] S. M. Christensen. *Quantum Theory of Gravity.* Adam Hilger Ltd, 1984.

[B18] R. E. Cutkosky. *J. Math. Phys.*, (1):429–433, 1960.

[B19] S. de Haro Olle. Quantum Gravity and the Holographic Principle. http://arxiv.org/abs/hep-th/0107032, 2001.

[B20] Nima Arkani-Hamed et al. Scattering amplitides and the positive Grassmannian. http://arxiv.org/pdf/1212.5605v1.pdf.

[B21] R. W. Gray. Another type of Structure and Dynamics: Trefoil Knot. http://www.rwgrayprojects.com/Lynn/Presentation20070926/p008.html, 2011.

[B22] B. Schwartz J. H. Green, M. and E. Witten. *Superstring Theory.* Cambridge University Press, Cambridge, 1987.

[B23] K. Huang. *Quarks,Leptons* & *Gauge Fields.* World Scientific, 1982.

[B24] R. Jackiw. In *Gauge Theories of Eighties, Conference Proceedings, Äkäslompolo, Finland* , Lecture Notes in Physics. Springer Verlag, 1983.

[B25] Lindström U. Rocek M. Karlhede, A. Hyper Kähler Metrics and Super Symmmetry. *Comm. Math. Phys.*, 108(4), 1987.

[B26] P. Ginsparg L. Dixon and J. Harvey. Beauty and the Beast; Superconformal Symmetry in a Monster Module. http://ccdb4fs.kek.jp/cgi-bin/img_index?8806247, 1988.

[B27] E. Witten L. Dolan, C. R. Nappi. Yangian Symmetry in $D = 4$ superconformal Yang-Mills theory. http://arxiv.org/abs/hep-th/0401243, 2004.

[B28] J. Maldacena L. F. Alday. Comments on gluon scattering amplitudes via AdS/CFT. http://arxiv.org/pdf/0710.1060, 2011.

[B29] A. Lakthakia. *Beltrami Fields in Chiral Media*, volume 2. World Scientific, Singapore, 1994.

[B30] G. Lisi. An exceptionally simple theory of everything. http://www.arxiv.org/pdf/0711.0770, 2007.

[B31] H. Larsen M. Freedman and Z. Wang. A modular functor which is universal for quantum computation. *Comm. Math. Phys.. http://arxiv.org/abs/quant-ph/0001108*, 1(2):605–622, 2002.

[B32] S. Seki M. Girodano, R. Peschanski. $\mathcal{N} = 4$ SYM Regge Amplitudes and Minimal Surfaces in AdS/CFT Correspondences. http://arxiv.org/pdf/1110.3680, 2011.

[B33] J. M. Maldacena. The Large N Limit of Superconformal Field Theories and Supergravity. http://arxiv.org/abs/hep-th/9711200, 1997.

[B34] L. Motl. Monstrous symmetry of black holes. http://motls.blogspot.com/2007/05/monstrous-symmetry-of-black-holes.html, 2007.

[B35] C. Cheung J. Kaplan N. Arkani-Hamed, F. Cachazo. A duality for the S-matrix. http://arxiv.org/abs/0907.5418, 2009.

[B36] G. Parisi. *Field Theory, Disorder and Simulations.* World Scientific, 1992.

[B37] R. Penrose. The Central Programme of Twistor Theory. *Chaos, Solitons and Fractals*, 10, 1999.

[B38] C. N. Pope and K. S. Stelle. The Witten index for the supermembrane , 1988.

[B39] L. Kauffman S. Bilson-Thompson, J. Hackett. Emergent Braided Matter of Quantum Geometry. http://arxiv.org/pdf/1109.0080v2.pdf, 2011.

[B40] J. H. Schwartz. *Super strings. The first 15 years of Superstring Theory.* World Scientific, 1985.

[B41] A. Verlinde. On the Origin of Gravity and the Laws of Newton. http://arxiv.org/abs/1001.0785, 2010.

[B42] E. Witten. Coadjoint orbits of the Virasoro Group. PUPT-1061 preprint, 1987.

[B43] E. Witten. Perturbative Gauge Theory As a String Theory In Twistor Space. http://arxiv.org/abs/hep-th/0312171, 2003.

[B44] A. Zee. *The Unity of Forces in the Universe.* World Science Press, Singapore, 1982.

# Particle and Nuclear Physics

[C1] How many neutrinos in the sky? http://resonaances.blogspot.fi/2013/01/how-many-neutrinos-in-sky.html.

[C2] Alice Collaboration. Charged-particle multiplicity density at mid-rapidity in central Pb-Pb collisions at $\sqrt{s_{NN}}$= 2.76 TeV. http://arxiv.org/abs/1011.3916, 2010.

[C3] G. Hinshaw et al. Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results. http://arxiv.org/abs/1212.5226.

# Condensed Matter Physics


[D1] Viscosity. http://en.wikipedia.org/wiki/Viscosity.

[D2] C. Cao et al. Universal Quantum Viscosity in a Unitary Fermi Gas. http://www.sciencemag.org/content/early/2010/12/08/science.1195219, 2010.

[D3] D. J. P. Morris et al. Dirac Strings and Magnetic Monopoles in Spin Ice Dy2Ti2O7. *Physics World*, 326(5951):411–414, 2009.

[D4] P. K. Kotvun et al. Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics. http://arxiv.org/abs/hep-th/0405231, 2010.

[D5] S. Sachdev. Quantum phase transitions (summary). *Physics World*, pages 33–38, April 1999.


# Cosmology and Astro-Physics


[E1] D. Da Roacha and L. Nottale. Gravitational Structure Formation in Scale Relativity. http://arxiv.org/abs/astro-ph/0310036, 2003.


# Books related to TGD


[K1] M. Pitkänen. 1983.

[K2] M. Pitkänen. Basic Properties of $CP_2$ and Elementary Facts about p-Adic Numbers. In *Towards M-matrix*. Onlinebook. http://tgdtheory.com/public_html/pdfpool/append.pdf, 2006.

[K3] M. Pitkänen. Is it Possible to Understand Coupling Constant Evolution at Space-Time Level? In *Towards M-Matrix*. Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#rgflow, 2006.

[K4] M. Pitkänen. About Nature of Time. In *TGD Inspired Theory of Consciousness*. Onlinebook. http://tgdtheory.com/public_html/tgdconsc/tgdconsc.html#timenature, 2006.

[K5] M. Pitkänen. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html#class, 2006.

[K6] M. Pitkänen. *Bio-Systems as Conscious Holograms.* Onlinebook. http://tgdtheory.com/public_html/hologram/hologram.html, 2006.

[K7] M. Pitkänen. *Bio-Systems as Self-Organizing Quantum Systems.* Onlinebook. http://tgdtheory.com/public_html/bioselforg/bioselforg.html, 2006.

[K8] M. Pitkänen. Category Theory and Quantum TGD. In *Towards M-Matrix.* Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#categorynew, 2006.

[K9] M. Pitkänen. Configuration Space Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry.* Onlinebook. http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html#cspin, 2006.

[K10] M. Pitkänen. Construction of Configuration Space Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry.* Onlinebook. http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html#compl1, 2006.

[K11] M. Pitkänen. Construction of elementary particle vacuum functionals. In *p-Adic length Scale Hypothesis and Dark Matter Hierarchy.* Onlinebook. http://tgdtheory.com/public_html/paddark/paddark.html#elvafu, 2006.

[K12] M. Pitkänen. Construction of Quantum Theory: M-matrix. In *Towards M-Matrix.* Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#towards, 2006.

[K13] M. Pitkänen. Construction of Quantum Theory: Symmetries. In *Towards M-Matrix.* Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#quthe, 2006.

[K14] M. Pitkänen. Cosmic Strings. In *Physics in Many-Sheeted Space-Time.* Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html#cstrings, 2006.

[K15] M. Pitkänen. Dark Forces and Living Matter. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy.* Onlinebook. http://tgdtheory.com/public_html/paddark/paddark.html#darkforces, 2006.

[K16] M. Pitkänen. Dark Matter Hierarchy and Hierarchy of EEGs. In *TGD and EEG.* Onlinebook. http://tgdtheory.com/public_html/tgdeeg/tgdeeg.html#eegdark, 2006.

[K17] M. Pitkänen. DNA as Topological Quantum Computer. In *Genes and Memes.* Onlinebook. http://tgdtheory.com/public_html/genememe/genememe.html#dnatqc, 2006.

[K18] M. Pitkänen. Does TGD Predict the Spectrum of Planck Constants? In *Towards M-Matrix.* Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#Planck, 2006.

[K19] M. Pitkänen. Does the Modified Dirac Equation Define the Fundamental Action Principle? In *Quantum Physics as Infinite-Dimensional Geometry.* Onlinebook. http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html#Dirac, 2006.

[K20] M. Pitkänen. Does the QFT Limit of TGD Have Space-Time Super-Symmetry? In *Towards M-Matrix.* Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#susy, 2006.

[K21] M. Pitkänen. Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory. In *TGD as a Generalized Number Theory.* Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#mblocks, 2006.

[K22] M. Pitkänen. *Genes and Memes.* Onlinebook. http://tgdtheory.com/public_html/genememe/genememe.html, 2006.

[K23] M. Pitkänen. Homeopathy in Many-Sheeted Space-Time. In *Bio-Systems as Conscious Holograms.* Onlinebook. http://tgdtheory.com/public_html/hologram/hologram.html#homeoc, 2006.

[K24] M. Pitkänen. Identification of the Configuration Space Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html#kahler, 2006.

[K25] M. Pitkänen. Knots and TGD. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html#knotstgd, 2006.

[K26] M. Pitkänen. Langlands Program and TGD. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdeeg/tgdnumber.html#Langlandia, 2006.

[K27] M. Pitkänen. *Magnetospheric Consciousness*. Onlinebook. http://tgdtheory.com/public_html/magnconsc/magnconsc.html, 2006.

[K28] M. Pitkänen. Many-Sheeted DNA. In *Genes and Memes*. Onlinebook. http://tgdtheory.com/public_html/genememe/genememe.html#genecodec, 2006.

[K29] M. Pitkänen. Massless states and particle massivation. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.com/public_html/paddark/paddark.html#mless, 2006.

[K30] M. Pitkänen. *Mathematical Aspects of Consciousness Theory*. Onlinebook. http://tgdtheory.com/public_html/mathconsc/mathconsc.html, 2006.

[K31] M. Pitkänen. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness*. Onlinebook. http://tgdtheory.com/public_html/tgdconsc/tgdconsc.html#nmpc, 2006.

[K32] M. Pitkänen. New Particle Physics Predicted by TGD: Part I. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.com/public_html/paddark/paddark.html#mass4, 2006.

[K33] M. Pitkänen. Nuclear String Hypothesis. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.com/public_html/paddark/paddark.html#nuclstring, 2006.

[K34] M. Pitkänen. *p-Adic length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.com/public_html/paddark/paddark.html, 2006.

[K35] M. Pitkänen. p-Adic Particle Massivation: Hadron Masses. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.com/public_html/paddark/paddark.html#mass3, 2006.

[K36] M. Pitkänen. p-Adic Physics: Physical Ideas. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#phblocks, 2006.

[K37] M. Pitkänen. *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html, 2006.

[K38] M. Pitkänen. Quantum Astrophysics. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html#qastro, 2006.

[K39] M. Pitkänen. Quantum Field Theory Limit of TGD from Bosonic Emergence. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#emergence, 2006.

[K40] M. Pitkänen. Quantum Hall effect and Hierarchy of Planck Constants. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.com/public_html/tpaddark/paddark.html#anyontgd, 2006.

[K41] M. Pitkänen. *Quantum Hardware of Living Matter*. Onlinebook. http://tgdtheory.com/public_html/bioware/bioware.html, 2006.

[K42] M. Pitkänen. Quantum Model for Nerve Pulse. In *TGD and EEG*. Onlinebook. http://tgdtheory.com/public_html//tgdeeg/tgdeeg/tgdeeg.html#pulse, 2006.

[K43] M. Pitkänen. *Quantum Physics as Infinite-Dimensional Geometry.* Onlinebook.http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html, 2006.

[K44] M. Pitkänen. *Quantum TGD.* Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html, 2006.

[K45] M. Pitkänen. Quantum Theory of Self-Organization. In *Bio-Systems as Self-Organizing Quantum Systems.* Onlinebook. http://tgdtheory.com/public_html/bioselforg/bioselforg.html#selforgac, 2006.

[K46] M. Pitkänen. Riemann Hypothesis and Physics. In *TGD as a Generalized Number Theory.* Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#riema, 2006.

[K47] M. Pitkänen. TGD and Astrophysics. In *Physics in Many-Sheeted Space-Time.* Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html#astro, 2006.

[K48] M. Pitkänen. TGD and Cosmology. In *Physics in Many-Sheeted Space-Time.* Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html#cosmo, 2006.

[K49] M. Pitkänen. *TGD and EEG.* Onlinebook. http://tgdtheory.com/public_html/tgdeeg/tgdeeg.html, 2006.

[K50] M. Pitkänen. *TGD and Fringe Physics.* Onlinebook. http://tgdtheory.com/public_html/freenergy/freenergy.html, 2006.

[K51] M. Pitkänen. *TGD as a Generalized Number Theory.* Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html, 2006.

[K52] M. Pitkänen. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory.* Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#visionc, 2006.

[K53] M. Pitkänen. TGD as a Generalized Number Theory: p-Adicization Program. In *TGD as a Generalized Number Theory.* Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#visiona, 2006.

[K54] M. Pitkänen. TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts. In *TGD as a Generalized Number Theory.* Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#visionb, 2006.

[K55] M. Pitkänen. *TGD Inspired Theory of Consciousness.* Onlinebook. http://tgdtheory.com/public_html/tgdconsc/tgdconsc.html, 2006.

[K56] M. Pitkänen. The Notion of Wave-Genome and DNA as Topological Quantum Computer. In *Genes and Memes.* Onlinebook. http://tgdtheory.com/public_html/genememe/genememe.html#gari, 2006.

[K57] M. Pitkänen. The Recent Status of Lepto-hadron Hypothesis. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy.* Onlinebook. http://tgdtheory.com/public_html/paddark/paddark.html#leptc, 2006.

[K58] M. Pitkänen. The Relationship Between TGD and GRT. In *Physics in Many-Sheeted Space-Time.* Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html#tgdgrt, 2006.

[K59] M. Pitkänen. Three new physics realizations of the genetic code and the role of dark matter in bio-systems. In *Genes and Memes.* Onlinebook. http://tgdtheory.com/public_html/genememe/genememe.html#dnatqccodes, 2006.

[K60] M. Pitkänen. *Topological Geometrodynamics: an Overview.* Onlinebook.http://tgdtheory.com/public_html/tgdview/tgdview.html, 2006.

[K61] M. Pitkänen. Twistors, N=4 Super-Conformal Symmetry, and Quantum TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#twistor, 2006.

[K62] M. Pitkänen. Was von Neumann Right After All. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#vNeumann, 2006.

[K63] M. Pitkänen. Yangian Symmetry, Twistors, and TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.com/public_html/tgdquant/tgdquant.html#Yangian, 2006.

[K64] M. Pitkänen. Motives and Infinite Primes. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#infmotives, 2011.

[K65] M. Pitkänen. Quantum Arithmetics and the Relationship between Real and p-Adic Physics. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#qarithmetics, 2011.

[K66] M. Pitkänen. Quantum Adeles. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#galois, 2012.

[K67] M. Pitkänen. The Recent Vision About Preferred Extremals and Solutions of the Modified Dirac Equation. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html#dirasvira, 2012.

# Articles about TGD

[L1] M. Pitkänen. Basic Properties of $CP_2$ and Elementary Facts about p-Adic Numbers. http://tgdtheory.com/pdfpool/append.pdf, 2006.

[L2] M. Pitkänen. Further Progress in Nuclear String Hypothesis. http://tgdtheory.com/articles/nuclstring.pdf, 2007.

[L3] M. Pitkänen. Physics as Generalized Number Theory II: Classical Number Fields. https://www.createspace.com/3569411, July 2010.

[L4] M. Pitkänen. Physics as Infinite-dimensional Geometry I: Identification of the Configuration Space Kähler Function. https://www.createspace.com/3569411, July 2010.

[L5] M. Pitkänen. Physics as Infinite-dimensional Geometry II: Configuration Space Kähler Geometry from Symmetry Principles. https://www.createspace.com/3569411, July 2010.

[L6] M. Pitkänen. Physics as Generalized Number Theory I: p-Adic Physics and Number Theoretic Universality. https://www.createspace.com/3569411, July 2010.

[L7] M. Pitkänen. Physics as Generalized Number Theory III: Infinite Primes. https://www.createspace.com/3569411, July 2010.

[L8] M. Pitkänen. Physics as Infinite-dimensional Geometry III: Configuration Space Spinor Structure. https://www.createspace.com/3569411, July 2010.

[L9] M. Pitkänen. Physics as Infinite-dimensional Geometry IV: Weak Form of Electric-Magnetic Duality and Its Implications. https://www.createspace.com/3569411, July 2010.

[L10] M. Pitkänen. The Geometry of $CP_2$ and its Relationship to Standard Model. https://www.createspace.com/3569411, July 2010.

[L11] M. Pitkänen. QCD and TGD. http://tgdtheory.com/articles/qcdtgd.pdf, 2011.

[L12] M. Pitkänen. TGD inspired vision about entropic gravitation. http://tgdtheory.com/articles/egtgd.pdf, 2011.

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